

LECTURES ON SUPERCONDUCTIVITY

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LECTURE 9

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THE JOSEPHSON EFFECT

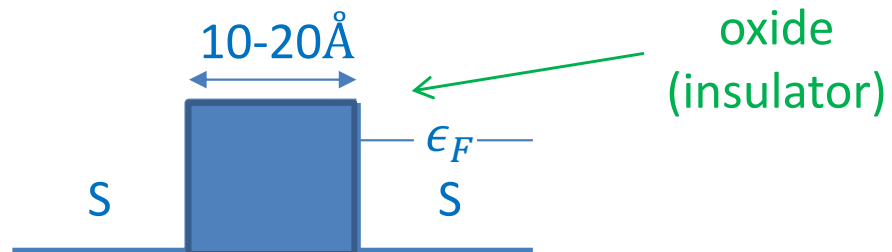


The Josephson Effect

Josephson effect occurs when 2 bulk superconductors connected by **weak link**, i.e. region which allows passage of electrons but with (much) increased difficulty.

Examples:

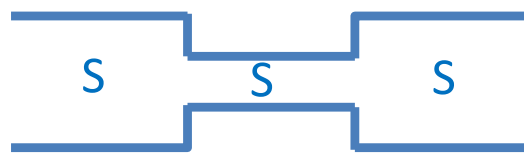
(1) Tunnel oxide (S-I-S) junction: schematically,



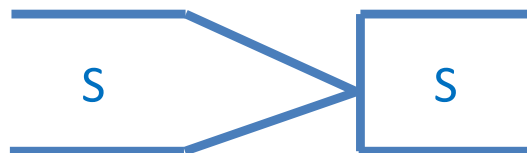
(2) Proximity (S-N-S) junction



(3) Constriction ('microbridge')



(4) Point contact



Original Josephson predictions made for case (1), often but not always valid for other cases.

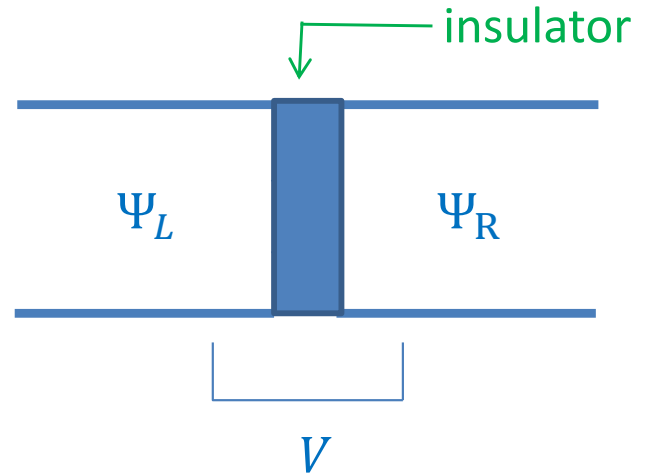


2 equations involving $\Delta\varphi$ (drop in phase of order parameter Ψ across junction, i.e. $\arg(\Psi_L^*\Psi_R)$):

1. $I = I_c \sin \Delta\varphi$ ← dissipationless supercurrent



critical current, $\sim 1 \text{ nA} - 1 \text{ mA}$



2. $\frac{d}{dt} \Delta\varphi = \frac{2eV}{\hbar}$ ← voltage (electrochemical potential) drop across junction

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- (a) dc Josephson effect (current bias):
 $\Delta\varphi = \sin^{-1}(I/I_c) (I \leq I_c)$
 - (b) ac Josephson effect (voltage bias):
 $I = I_c \sin\left(\frac{2eV}{\hbar}t\right)$

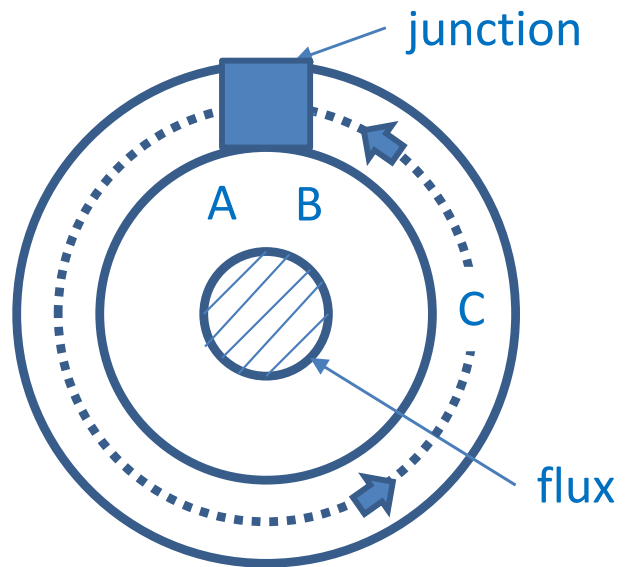
Fundamental significance of Josephson effect: critical current I_c corresponds (see below) to characteristic energy $I_c \Phi_0 / 2\pi \sim 20 \text{ mK} - 20,000 \text{ K}$, i.e. often $\ll k_B T_{\text{room}}$, yet this tiny energy controls aspects of the system (e.g. trapped flux in ring) which are by any reasonable definition **macroscopic!** (see below)



"Derivation" of second Josephson equation (Bloch):

Assume thickness of ring $\gg \lambda_L$ but self-inductance negligible (i.e. $LI_c \ll \Phi_0$) so for $I \leq I_c$ flux through ring is externally applied AB flux Φ .

If contour C is well inside penetration depth, the electric current $J(r)$ on $C = 0$. But $J(r) \propto (\nabla\varphi(r) - 2eA(r)/\hbar)$, so $\nabla\varphi(r) = 2eA(r)/\hbar$. We can integrate from A to B and, since thickness of junction \ll radius of ring, extend integral of RHS to full circle, whereupon it gives $2\pi\Phi/\Phi_0$. Hence, defining $\Delta\varphi$ as phase drop from A to B :



$\Delta\varphi = 2\pi\Phi/\Phi_0$ ← fundamental relation for Josephson circuits

Differentiation with respect to time gives

$\frac{d}{dt} \Delta\varphi = \frac{2\pi}{\Phi_0} \frac{d\Phi}{dt}$, voltage drop across junction

but by Faraday's law $d\Phi/dt = V_{\text{circ}}(t) = V(t)$, so

since bulk superconductor shorts out V_{circ}

so $\frac{d}{dt} \Delta\varphi = \frac{2\pi}{\Phi_0} V \equiv \frac{2eV(t)}{\hbar}$ i.e. 2nd Josephson equation

This equation is rather generally valid for any kind of weak link.



Derivation of 1st Josephson equation (also Bloch):

Consider dependence of free energy $F(\Phi; T)$ of system on AB flux Φ . By Byers-Yang theorem, F must be periodic in Φ with period $h/e \equiv \Phi^{sp} \equiv 2\Phi_0$. Also, by TRI.

↑
(time-reversal invariance)

$F(\Phi) = F(-\Phi)$. Hence

$F(\Phi, T) = \sum_n A_n(T) \cos(2\pi n\Phi/2\Phi_0)$ and since $\Delta\varphi = 2\pi\Phi/\Phi_0$

$$F(\Phi; T) = \sum_{n=0}^{\infty} A_n(T) \cos(n\Delta\varphi/2)$$

↑
quite generally valid, independently of nature of weak link.

Suppose t is the matrix element for a single electron (not a Cooper pair!) to traverse the barrier. Then by an extension of the argument used above to obtain $\Delta\varphi = 2n\Phi/\Phi_0$, we see that the term in n involves n single-particle traversals of the ring, and thus of the barrier, so that the amplitude for this process $\propto t^n$. Since the term in $n = 0$ is independent of Φ , the first "interesting" term is apparently $n=1$.

This term does occur in "mesoscopic" rings; however, it requires phase coherence of the single-electron wave functions around the ring, $\Rightarrow R \lesssim l_\varphi$.

↑
"phase breaking" length



Hence in practice for most systems involving Josephson junctions, the term $n=1$ (and more generally odd- n) is negligibly small. Then changing the notation so that $n/2 \rightarrow n$, we find

$$F(\Phi, T) = \sum_{n=0}^{\infty} A_n(T) \cos(n\Delta\varphi) \quad A_n(T) \propto |t|^{2n}$$



general for weak links with no single-electron phase coherence

Simplest case is $t \rightarrow 0$ (typical for tunnel oxide junctions, original Josephson case): then neglect $n > 1$ and find (- sign for convenience!)

$$F(\Phi, T) = -E_J \cos(\Delta\varphi)$$

However, quite generally we have for the current flowing in the ring (and thus through the junction)

$$I = \partial F / \partial \Phi = \frac{2\pi}{\Phi_0} \partial F / \partial (\Delta\varphi)$$

and so

$$I = I_c \sin \Delta\varphi \quad I_c = (2\pi/\Phi_0)E_J \equiv (2e/\hbar)E_J$$

i.e. the 1st Josephson equation.



In the more general case ($t \neq 0$) the free energy need not be a single-valued function of $\Delta\varphi \pmod{2\pi}$. Suppose by applying constant V we "crank up" $\Delta\varphi$. What happens at $\Delta\varphi = \pi$?

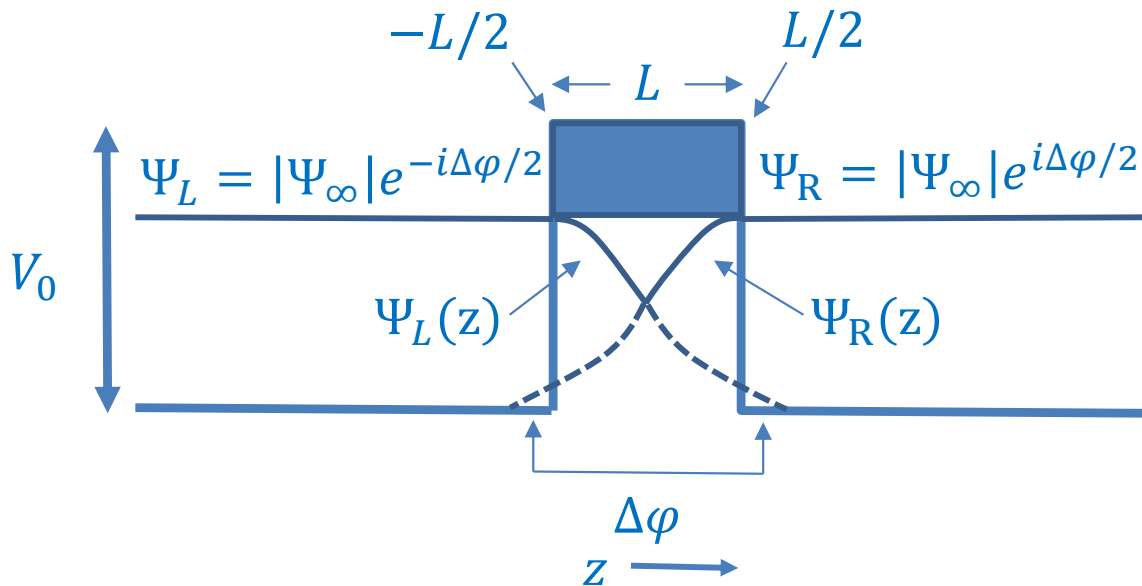
- (a) nonhysteretic case (original Josephson case): $|\Psi| \rightarrow 0$ in junction \Rightarrow phase drop $\Delta\varphi$ "slips" from π to $-\pi$
- (b) hysteretic case (typical e.g. for microbridge): $|\Psi| \neq 0$, so $\Delta\varphi$ increases beyond π with increase of KE. Eventually (cf. l. 4) $|\Psi|$ decreases and I reaches a max value I_c

What determines I_c (i.e. E_J)?

(a) "Toy" model of tunnel-oxide junction: extend GL description under barrier ($|z| < L/2$) but since $|\Psi(z)| \rightarrow 0$, neglect term in $|\Psi|^4$

\rightarrow Schrodinger problem, with mass $m_p \equiv 2m_{el}$, potential $V_p(z) \equiv 2V_{el}(z)$, energy eigenvalue $\mu_p \equiv 2\mu_{el}$





In limit $L \rightarrow \infty$, solution starting from LHS is

$$\Psi_L(z) = \Psi_L(\infty) \exp\left(-\int_{-L/2}^z \sqrt{2m_p(2V_0 - 2\mu)} dz\right)$$

$$\Psi_L(\infty) \equiv |\Psi_\infty| e^{-i\Delta\phi/2}$$

and that starting from RHS is

$$\Psi_R(z) = \Psi_R(\infty) \exp\left(-\int_z^{L/2} \sqrt{2m_p(2V_0 - 2\mu)} dz\right)$$

$$\Psi_R(\infty) \equiv |\Psi_\infty| e^{+i\Delta\phi/2}$$

For $L \rightarrow \infty$, must superpose:

$$\Psi(z) = \Psi_L(z) + \Psi_R(z) \equiv \Psi_{\Delta\phi}(z)$$



The dependence of the energy on $\Delta\varphi$ must come entirely from the cross-terms between Ψ_R and Ψ_L

$$E_J(\Delta\varphi) \sim \left\{ \int \Psi_L^*(z) \Psi_R(z) + \text{c. c.} \right\}$$

This expression is proportional to $\cos(\Delta\varphi)$ and to the WKB factor $\exp\left(-\int_{-L/2}^{L/2} \sqrt{2m_p(2V_0 - 2\mu)} dz\right)$: since the integrand is twice that for single-electron tunneling, we have $f_{\text{WKB}} \sim |t|^2$. Thus,

$$E_J(\Delta\varphi) = A|t|^2 \cdot \cos(\Delta\varphi)$$

But the constant A is tricky! At first sight $A > 0$, but this cannot be right, since the solution for $\Delta\varphi = 0$ (" Ψ_+ ") is nodeless, whereas that for $\Delta\varphi = \pi$ (" Ψ_- ") has a node, so that Ψ_- must lie higher ($E_- > E_+$) For general gap $\Delta\varphi$

$$\Psi_{\Delta\varphi}(z) \equiv \cos(\Delta\varphi/2)\Psi_+(z) + i\sin(\Delta\varphi/2)\Psi_-(z)$$

so that by linearity of the TISE ← time-independent Schrodinger equation

$$E(\Delta\varphi) = (\text{const.} +)(E_+ - E_-)\cos(\Delta\varphi) \equiv -E_J\cos(\Delta\varphi),$$

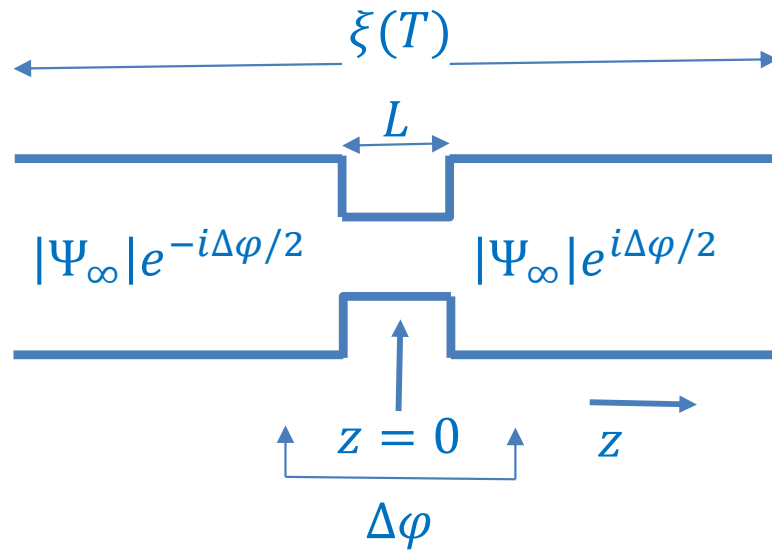
$$E_J \equiv E_- - E_+ > 0$$

A more detailed calculation confirms $E_J \propto |t|^2$: since the normal-state resistance R_N of the junction $\propto |t|^{-2}$ we have

$$E_J \propto R_N^{-1}$$



(b) More realistic example: "short" microbridge ($L \ll \xi(T)$)



In dimensionless form GL free energy is

$$F\{\Psi(z)\} = \mathcal{F}_0(T) \int \left\{ -|f|^2 + \frac{1}{2}|f|^4 + \frac{1}{2}\xi^2(T) \left| \frac{df}{dz} \right|^2 \right\} dz$$

↖ bulk free energy
 ↑ $\Psi(z)/\Psi_\infty$
 ↑ GL healing length

With boundary conditions $f(-\infty) = e^{-i\Delta\phi/2}$, $f(+\infty) = e^{+i\Delta\phi/2}$. If $\xi(T) \gg L$ and $\Delta\phi \neq 0$, bending term will dominate, so we minimize it $\Rightarrow \partial^2 f / \partial z^2 = 0$.

Solution:

$$\begin{aligned}
 f &= \frac{1}{2} \left\{ \left(1 + \frac{z}{L/2} \right) e^{\frac{i\Delta\phi}{2}} + \left(1 - \frac{z}{L/2} \right) e^{-\frac{i\Delta\phi}{2}} \right\} \\
 &\equiv \cos(\Delta\phi/2) + i(z/(L/2))\sin(\Delta\phi/2)
 \end{aligned}$$

I Free energy is dominated by bending term, *i.e.* by the $\sin \Delta\phi/2$ term:

$$\Delta F = \frac{2A}{L} \mathcal{F}_0(T) \xi^2(T) \sin^2 \Delta\varphi / 2 \quad (A = \text{cross-section})$$

or using $\mathcal{F}_0(T) \xi^2(T) = \hbar^2 / (2m) |\Psi_\infty(T)|^2$

$$\Delta F = \frac{\hbar^2 A}{2mL} |\Psi_\infty|^2 (1 - \cos \Delta\varphi)$$

so that

$$E_J = \frac{\hbar^2 A}{2mL} |\Psi_\infty|^2$$

Note again $E_J \propto R_N^{-1}$ (in sense that it $\propto A/L$)

(c) Realistic (Bardeen-Josephson) model of tunnel junction (result only)

(Ambegaokar-Baratoff): start from single-electron tunnelling, express matrix elements in terms of Bogoliubov quasiparticles, then coherence factors [not discussed in these lectures] give nontrivial dependence on $\Delta\varphi$. Result at $T = 0$

$$I_c = \frac{\pi\Delta}{eR_N} \quad \left(\equiv \frac{2\pi E_J}{\Phi_0} \right)$$

or at non-zero T

$$I_c = \left(\frac{\pi\Delta(T)}{eR_N} \right) \tanh(\Delta(T)/2T)$$

↑
AB formula

usually fairly well satisfied in junctions between "classic" superconductors



The dc SQUID

$$I = I_1 + I_2 = I_c(\sin\Delta\varphi_1 + \sin\Delta\varphi_2)$$

but, $\Delta\varphi_1$ and $\Delta\varphi_2$ are **not independent!** Analogously to above discussion of single ring,

$$\Delta\varphi_{AD} = \frac{2e}{\hbar} \int_D^A \mathbf{A} \cdot d\mathbf{l}$$

$$\Delta\varphi_{CB} = \frac{2e}{\hbar} \int_B^C \mathbf{A} \cdot d\mathbf{l}$$

so if contributions to \int from junctions themselves negligible,

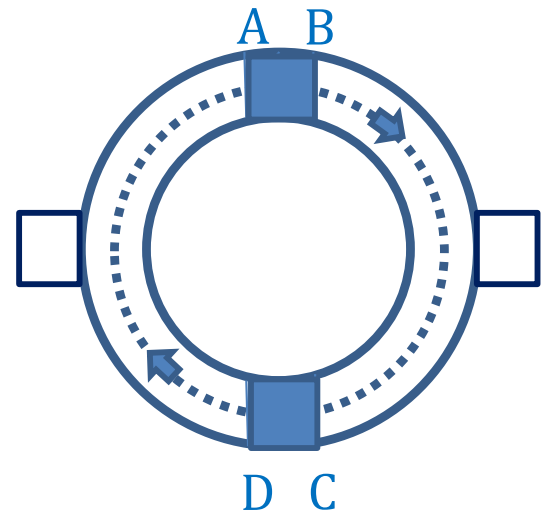
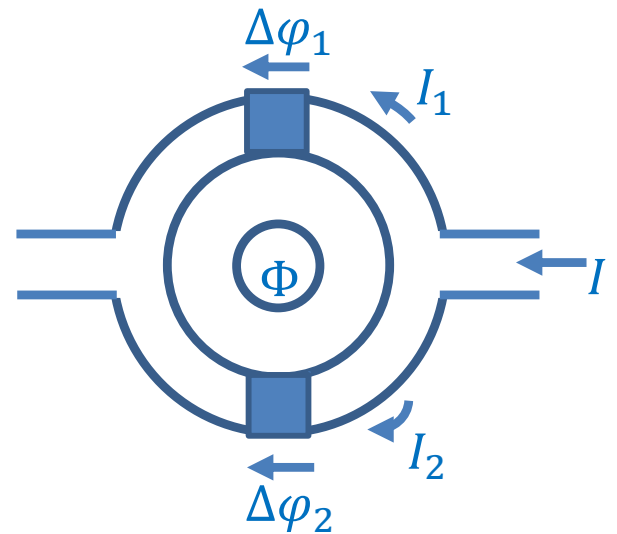
$$\Delta\varphi_1 - \Delta\varphi_2 = 2\pi \Phi / \Phi_0$$

Hence if $\xi \equiv \frac{1}{2}(\Delta\varphi_1 + \Delta\varphi_2)$,

$$I = I_c(\sin \Delta\varphi_1 + \sin \Delta\varphi_2) = 2I_c \sin \xi \cos (\pi\Phi/\Phi_0)$$

so total critical current of SQUID or $f(\Phi)$ (attained for $\xi = \pi/2$) is

$$I_c(\Phi) = 2I_c |\cos (\pi\Phi/\Phi_0)|$$



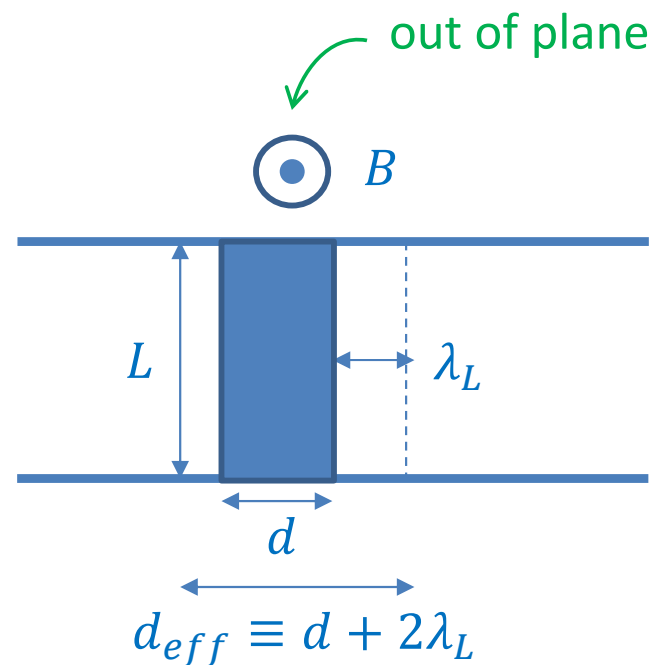
with $\max(2I_c)$ at $\Phi = n\Phi_0$ and \min (zero) at $\Phi = (n + 1/2)\Phi_0$.

An extension of the argument gives for a single junction subject to a parallel magnetic field

$$I_c(\Phi) = I_c \frac{|\sin(\pi\Phi/\Phi_0)|}{|\pi\Phi/\Phi_0|}$$

where

$$\Phi = BLd_{eff} \longleftarrow d + 2\lambda_L$$



Summary of lecture 9

Josephson effect generic in “weak links”, that is, regions between 2 bulk superconductors where $OP \Psi$ is depressed far below bulk value.

2 fundamental equations:

$$\frac{d}{dt}(\Delta\varphi(t)) = \frac{2eV(t)}{\hbar} \quad \leftarrow \text{ (“2nd Josephson equation”)} \quad \text{generally valid}$$

$$I(t) = I_c \sin \Delta\varphi(t) \quad \left(\text{“1st Josephson equation”} \right) \quad \begin{array}{l} \text{valid for tunnel-} \\ \text{oxide barriers and} \\ \text{some other} \\ \text{nonhysteretic ones} \end{array}$$

For hysteretic junctions, $\Delta\varphi$ can be $\gg 2\pi$
and then roughly $I \propto \Delta\varphi$ up to I_c

Fundamental relation for multiply connected Josephson circuits (SQUIDs etc.)

$$(\Sigma)\Delta\varphi = 2\pi \Phi / \Phi_0$$

