PHYS 1303 - Special Relativity I

Lee Kai Ming
CYMP Rm 415A
kmlee@lily.physics.hku.hk
2859 2370

http://www.physics.hku.hk/~phys1303/

January 2007

Contents

Chapter 1

Pre-Special Relativity Physics

We will first review some relevant topics before relativity was discovered. The most important section is the one about Galilean transformations. The readers should already know it well, although maybe do not know the name. We will find in later chapters that these concepts are very important, but not correct. In particular, in Galilean point of view, space and time are absolute, in a sense that will be explained in the text. However, this is no longer true in special relativity, which means that space and time are not absolute in nature.

1.1 Newtonian Mechanics

We will briefly review the Newton's laws in this section. The readers are assumed to know the material well. This section only serves as a reminder.

The first law of mechanics describes the resistance of matter to change in its state of motion: A body in motion will remain in motion, unless it is acted upon by some external force.

Newton's formulation of the second law is the familiar

$$\mathbf{F} = m\,\mathbf{a} = m\,\dot{\mathbf{v}} \tag{1.1}$$

where $\mathbf{F}$ is the force vector, m is the mass of an object and $\mathbf{a}$ is the acceleration vector. The mass in this equation is the **inertial mass**, which relates the response of the body to external force. The acceleration is the rate of change of the velocity. Velocity describes both the speed and the direction of the motion. Thus, sometime the acceleration is non-zero even if the speed of the body remains constant.

Eq. (1.1) is valid only in an inertial frame. This is a very important concept and we will discuss it much more in Subsection 1.2.1.

The third law states that whenever there is an action, there will be an equal in magnitude but opposite in direction reaction. For example, we feel the gravitational attraction of the Earth pulling us down, at the same time, there is a force of the same strength pulling the Earth "up."

Momentum, or linear momentum, of a particle is defined as the product

$$\mathbf{p} = m\mathbf{v} \ . \tag{1.2}$$

Hence, the Newton's second law can be re-stated as

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \ . \tag{1.3}$$

This is the form that will be generalized to special relativity. We found that in the absence of any external force, by Eq. (1.1), the total momentum of a system remains constant. This is the **conservation of momentum**.

The addition of velocities in classical mechanics is very simple. For example, if a train is moving with velocity $\mathbf{v}_t$ relative to the station and a ball is moving with velocity $\mathbf{v}_b$ relative to the train, then relative to the station, the ball is moving with velocity $\mathbf{v}_t + \mathbf{v}_b$.

The **kinetic energy** of a particle is given by

$$\text{K. E.} = \frac{1}{2}mv^2 = \frac{p^2}{2m} \tag{1.4}$$

where v and p are the magnitude of velocity and momentum respectively. If we want to change the velocity of the particle, a force must act on it. The change of kinetic energy is equal to the **work done** W, which, for constant force, is the dot product of the force vector $\mathbf{F}$ and the displacement vector $\mathbf{r}$ of the particle

$$W = \mathbf{F} \cdot \mathbf{r} \ . \tag{1.5}$$

If the force is not constant, consider the infinitesimal small displacement $\mathrm{d}\mathbf{r}$, which means, consider the particle moves from position $\mathbf{r}$ to $\mathbf{r} + \mathrm{d}\mathbf{r}$, the contribution to the work done is $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$. The total work done is to integrate

$$W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{1.6}$$

where $\mathbf{r}_0$ and $\mathbf{r}_1$ are the initial and final points.

Apart from the kinetic energy, there are other kinds of energies, like the potential energy, chemical energy or nuclear energy. If we sum up all kinds of energies in an isolated system, the total energy also remains constant. This is the principle of **conservation of energy**.

1.2 Galilean Transformations

We will discuss Galilean transformations after we talk about two important concepts.

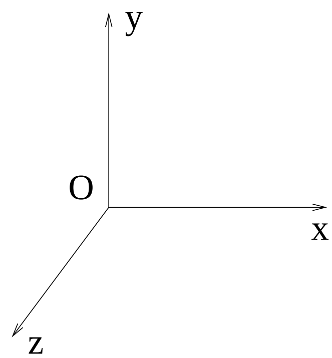

Figure 1.1: An inertial frame.

1.2.1 Inertial Frames

Eq. (1.1) is valid only in an inertial frame. What is an inertial frame? A reference frame or just a frame is a standard to which motion and other mechanical quantities can be measured. Usually, it means a coordinate system of space and time. Note that the coordinate system does not have to be "linear." For example, a useful coordinate system on the surface of a sphere is non-linear. A well known example is the longitude and latitude on the Earth.

In some textbooks, an inertial frame is defined as the frame on which Eq. (1.1) is valid. This is, of course, not a good definition. In practice, an inertial frame can be defined as any frame at rest or in constant velocity with respect to the fixed stars. (In contrast, for example, a rotating frame, like the surface of the earth, is not in constant velocity with respect to the fixed stars.) This means, we have to choose some initial time as $t = 0$, a point in space as the origin O, then three perpendicular directions as directions of the three axes, Fig. 1.1. The origin O is allowed to move with constant velocity with respect to the far away fixed stars, but the axes are not allowed to rotate with respect to the stars.

After we find one inertial frame, any Cartesian coordinate system moving with constant velocity and no rotation with respect to the inertial frame is also an inertial frame. If we are careful enough, we can find by experiments that the "rest" frame on the Earth is not an inertial frame.

In this course, we will only consider inertial frames. Hence, we usually will only speak about frames, instead of the more correct term inertial frames.

1.2.2 Events

In common language, an event is something happened at some place at some time. In special relativity, an event means exactly this: a point in space and time. Thus, we have to specify both the space coordinates and time

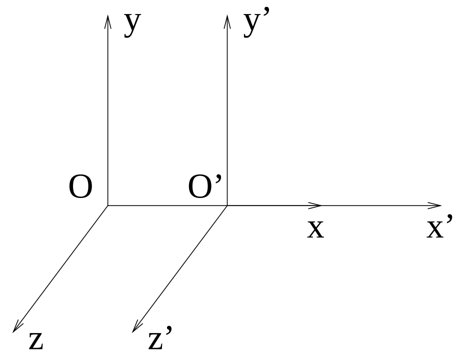

Figure 1.2: The relation between two frames.

coordinate to indicate an event. To visualize it, we usually speak of the explosion of a bomb or the moment when and where we turn on a flashlight. Note that the same event could have different coordinates with respect to different frames, while there is no disagreement on the nature of the event. For example, everyone will agree whether it is an explosion of a bomb or only the light-up of a flashlight.

We will talk about the Galilean transformations after we discuss a few more terms. An **observer** is a being that makes measurements of, among others, lengths and durations. In relativity, we have to be very careful about the motion of the observer. The **rest frame** (or co-moving frame) of a body or an observer is the frame where the body is at rest. The measurement made in this frame is usually described by the word "proper." For example, the **proper length** of a rod is the length of the rod as measured by a ruler at rest (or moving) with the rod. The **proper time** of a process is the time taken as measured by a clock at rest (or moving) with the process.

Now, suppose there are two frames S and S', one moving with constant velocity with respect to another. We can choose the coordinate systems such that the x and x'-axes are aligned with the velocity and their origins coincide, Fig. 1.2. Hence, for the origins, we have

$$\begin{aligned} t' &= t = 0 \\ x' &= x = 0 \\ y' &= y = 0 \\ z' &= z = 0 \ . \end{aligned} \tag{1.7}$$

How are the relations between the coordinates of other points?

$$t' = t \tag{1.8}$$
$$x' = x - vt \tag{1.9}$$
$$y' = y \tag{1.10}$$
$$z' = z \ . \tag{1.11}$$

These are the **Galilean transformations**. Note that these are the coordinates of the same event with respect to two different frames. Because time is the same for all frames, Eq. (1.8), we say that time is absolute in Galilean transformation. Also, suppose the two ends of a moving rod have coordinates $(x_1', y_1', z_1') = (0, 0, 0)$ and $(x_2', y_2', z_2') = (l, 0, 0)$. Their coordinates in S frame are $(x_1, y_1, z_1) = (vt, 0, 0)$ and $(x_2, y_2, z_2) = (l + vt, 0, 0)$. They depends on time, which simply means that they are moving. The length of the rod in the S frame is

$$x_2 - x_1 = (x_1' + vt) - (x_2' + vt) = l \ . \tag{1.12}$$

Length does not change. We say that space is also absolute.

A particle moving with constant velocity with respect to S is also moving with constant (different) velocity with respect to S'. Hence, the Newton's first law is invariant under Galilean transformations. The force is assumed to be invariant under the change of frame, and the acceleration is also invariant. Second law is also invariant. Forces are invariant, as a result, so is the third law. In summary, the laws of mechanics are Galilean invariant. A conclusion of these is no mechanical experiment within one inertial frame can tell if it is at rest or moving.

The last topics of this section is the conservation of momentum and energy under Galilean transformations. If the velocity of S' relative to S is $\mathbf{v}$, and the momentum of a particle of mass m relative to S is $\mathbf{p} = m\mathbf{u}$, then its momentum relative to S' is $\mathbf{p}' = m(\mathbf{u} - \mathbf{v})$, and its energy is $1/2m(\mathbf{u} - \mathbf{v})^2$. Let consider the collision of two particles of masses m_1 and m_2, with initial velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ and final velocities $\mathbf{w}_1$ and $\mathbf{w}_2$ relative to S. The conservation of momentum and energy are

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{w}_1 + m_2\mathbf{w}_2 \tag{1.13}$$

$$\frac{1}{2}m_1\mathbf{u}_1^2 + \frac{1}{2}m_2\mathbf{u}_2^2 = \frac{1}{2}m_1\mathbf{w}_1^2 + \frac{1}{2}m_2\mathbf{w}_2^2 \ . \tag{1.14}$$

In the frame S', we have

$$\begin{aligned} & m_1(\mathbf{u}_1 - \mathbf{v}) + m_2(\mathbf{u}_2 - \mathbf{v}) - m_1(\mathbf{w}_1 - \mathbf{v}) - m_2(\mathbf{w}_2 - \mathbf{v}) \\ = \ & m_1\mathbf{u}_1 + m_2\mathbf{u}_2 - m_1\mathbf{w}_1 - m_2\mathbf{w}_2 \\ = \ & 0 \ , \end{aligned} \tag{1.15}$$

using conservation of momentum in S. Momentum is also conserved in S'. For energy, we have

$$\begin{aligned} & \frac{1}{2}m_1(\mathbf{u}_1 - \mathbf{v})^2 + \frac{1}{2}m_2(\mathbf{u}_2 - \mathbf{v})^2 - \frac{1}{2}m_1(\mathbf{w}_1 - \mathbf{v})^2 - \frac{1}{2}m_2(\mathbf{w}_2 - \mathbf{v})^2 \\ = \ & \frac{1}{2}m_1\mathbf{u}_1^2 + \frac{1}{2}m_2\mathbf{u}_2^2 - \frac{1}{2}m_1\mathbf{w}_1^2 - \frac{1}{2}m_2\mathbf{w}_2^2 \\ & + (m_1\mathbf{u}_1 + m_2\mathbf{u}_2 - m_1\mathbf{w}_1 - m_2\mathbf{w}_2) \cdot \mathbf{v} \\ = \ & 0 \ , \end{aligned} \tag{1.16}$$

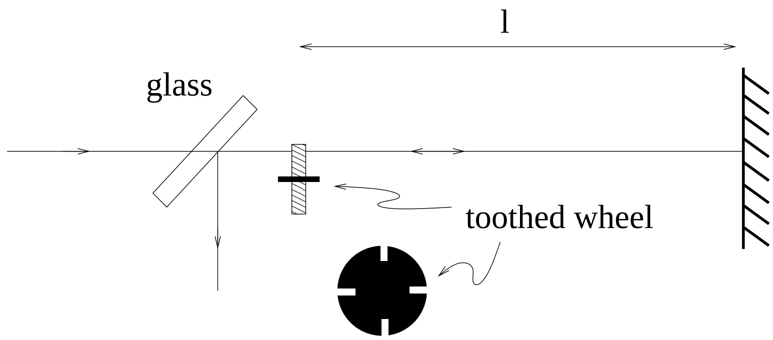

Figure 1.3: Measure the speed of light.

using conservation of momentum and energy in S. Hence, energy is also conserved in S'.

1.3 Light

Light is usually understood in terms of Maxwell's electromagnetic theory. It is a kind of wave, the oscillations of electric and magnetic fields. More precisely, light is electromagnetic waves within certain range of frequencies. Other electromagnetic waves include radio waves, X-rays, etc. They have all the properties of waves, like diffraction and interference.

One fundamental property of light is its speed, c. We found by experiments that all electromagnetic waves have the same speed. One of the first measurements of speed of light was done by Armand Fizeau in 1849. In Fig. 1.3, light is entering from the left, passes through the teeth of the wheel and travels a long distance l to a mirror and is reflected back. If the wheel is turning slowly, the reflected light will hit the wheel and it will be dark if we look up from below the glass. Only when the wheel is turning in the correct speed, we can see the reflected light. If the wheel has n teeth and is turning r turns per second, then the time taken for the wheel to turn for one tooth is equal to that of the round trip of the light,

$$\frac{1}{nr} = \frac{2l}{c} \,, \tag{1.17}$$

or $c = 2nrl$. For $l = 8$km, the wheel has to turn hundreds of times per second for us to see the reflected light.

Nowadays, the speed of light is a defined value, $c = 2.99792458 \times 10^8$m/s. We define the length of one meter by this value and the definition of time (which is defined by the properties of some atom). Gravitational wave is believed to have the same speed. It is, in fact, the ultimate speed of nature. Only because we discover it as speed of light that we call it the speed of light.

Since light is a kind of wave, scientists before Einstein would ask what the medium of the light wave is. To them, the medium of water waves is water, the medium of sound waves is air and the medium of light waves is **ether**, although they did not have direct evidence of its existence yet. Then, the rest frame of ether is very important. We, of course, now know that ether does not exist, but many famous scientists at that time supported this idea.

After the discovery of quantum electrodynamic in 1940s, we know that light also has some particle properties. The fundamental unit of light is called **photon**. We will only use the term, not the theory.

Chapter 2

Special Relativity

The theory of special relativity was proposed by Albert Einstein in 1905. It can naturally explain the experimental results at the time, mainly the constancy of speed of light or the absence of ether. It also predicts that length and time duration depend on the motion of observers. These predictions were revolutionary and were later experimentally confirmed.

2.1 Michelson-Morley Experiment

This is the most important experiment in the development of special relativity. It was designed to find out the rest frame of ether, but instead, it proves that the speed of light does not depend on the velocities of the source and observer.

The setup is relatively simple, Fig. 2.1. Light comes from the left, goes toward the semi-transparent glass. Half of the light is reflected and goes up, half goes through. They are reflected by the two mirrors and recombined. We observe the interference pattern upward from below the semi-transparent glass. Then, we slowly rotate the whole setup by 90 degrees, and observe if there is any shift in the interference pattern.

If ether exists, observers with different velocities relative to the rest frame of ether will find different speed of light, according to the addition of velocities in Galilean transformations. For the Michelson-Morley experiment, for simplicity, let us assume that in the beginning, the whole experiment is moving to the right with speed v relative to the ether. Then, the time taken for light to go one round trip along the l_1 leg is

$$t_1 = \frac{l_1}{c-v} + \frac{l_1}{c+v} = \frac{2l_1}{c}\frac{1}{1-v^2/c^2} \; . \tag{2.1}$$

For the l_2 leg, let the time taken be t_2. When the light travels from the glass to the mirror and back, the glass has already moved to the right by a

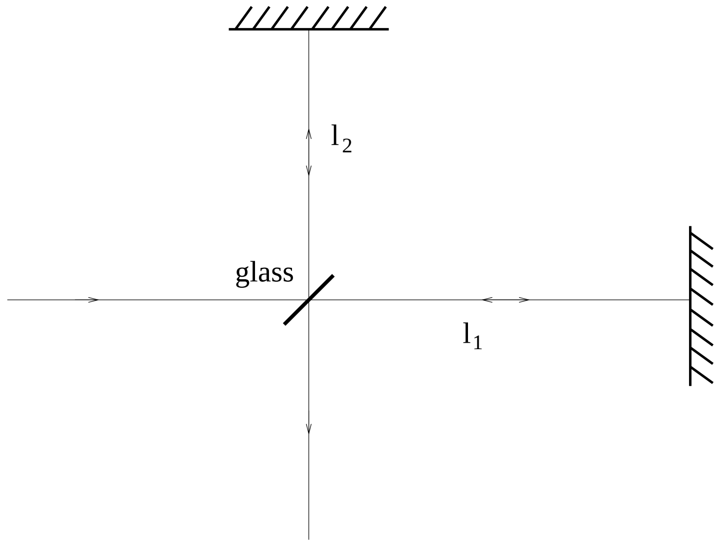

Figure 2.1: Michelson-Morley experiment.

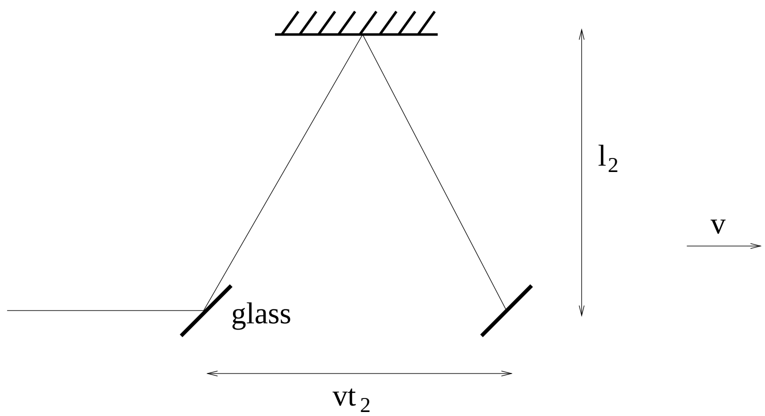

Figure 2.2: The whole experiment is moving to the right with speed v.

distance vt_2, Fig. 2.2. Hence, we have

$$\begin{aligned} 2\left(l_2^2 + (\frac{vt_2}{2})^2\right)^{1/2} &= ct_2 \\ t_2 &= \frac{2l_2}{c}\frac{1}{\sqrt{1 - v^2/c^2}} \ . \end{aligned} \tag{2.2}$$

Let $\beta = v/c$. The time difference between them is

$$\Delta t = t_2 - t_1 = \frac{2}{c}\left(\frac{l_2}{\sqrt{1-\beta^2}} - \frac{l_1}{1-\beta^2}\right) \ . \tag{2.3}$$

To simplify this expression, notice that for small β,

$$\frac{1}{\sqrt{1-\beta^2}} = 1 + \frac{\beta^2}{2} + \frac{3\beta^4}{8} + \cdots \tag{2.4}$$

$$\frac{1}{1-\beta^2} = 1+\beta^2+\beta^4+\cdots \,. \tag{2.5}$$

(These are the Taylor's expansions, if you know that.) The β^4 term and all higher order terms are very small. We can safely ignore them, we have

$$\Delta t = \frac{2}{c}\left(l_2(1+\frac{\beta^2}{2}) - l_1(1+\beta^2)\right) = \frac{2}{c}\left(l_2 - l_1 + (\frac{l_2}{2} - l_1)\beta^2\right) \,. \tag{2.6}$$

This time difference determines the interference pattern that we see. For example, if it is an integral multiple of the period of the light wave, then there will be constructive interference at the middle and we will see a bright fringe. If it is an half-integral, we will see a dark fringe.

After the rotation, the time taken along l_1 and l_2 are then

$$t_1' = \frac{2l_1}{c}\frac{1}{\sqrt{1-\beta^2}} \tag{2.7}$$

$$t_2' = \frac{2l_2}{c}\frac{1}{1-\beta^2} \,. \tag{2.8}$$

The time difference is

$$\Delta t' = t_2' - t_1' = \frac{2}{c}\left(l_2 - l_1 + (l_2 - \frac{l_1}{2})\beta^2\right) \,. \tag{2.9}$$

If the two time differences, Δt and $\Delta t'$, are the same, there will be no shift in the interference pattern. Hence, if λ is the wavelength of the light, the number of fringes shifted is

$$\Delta N = \frac{c}{\lambda}(\Delta t' - \Delta t) = \frac{(l_1+l_2)\beta^2}{\lambda} \,. \tag{2.10}$$

For the revolutionary speed of the Earth, $\beta \approx 10^{-4}$. If $l_1 = l_2 = 10$m and $\lambda = 5.5 \times 10^{-7}$m (yellow light), then $\Delta N = 0.4$. This is easily observable.

Michelson and Morley did not see the predicted shift. For one such experiment, it could just happen that the velocity of the Earth relative to ether is such that there is no shift. So, they repeated the experiment many times in a year, but still no shift was observed.

There were many proposed reasons for this phenomenon. The correct one is that speed of light is independent of the velocity of the source. This violates the Galilean law of addition of velocities. But this law is derived from mechanics. Light and electromagnetic experiments are much more accurate than mechanics. Thus, we should accept the constancy of speed of light and abandon the Galilean law.

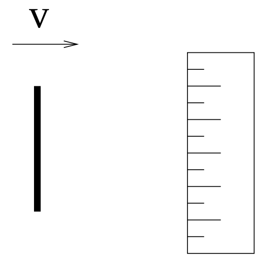

Figure 2.3: A rod of length l is moving to the right with speed v.

2.2 Basic Postulates

Newtonian mechanics does not work well when the object is moving very fast, especially, when its speed approaches the speed of light. We need special relativity. Daily experiences do not provide any good intuitions to understand relativity. We have to rely on hypothetical models.

The two basic premises of special relativity are:

- The physics on all inertial frames are the same.
- Speed of light is the same in all inertial frames, independent of the source and target of the light beam.

The first statement means that, for example, if we measure the lifetime of some radioactive substance, and if the whole laboratory is on a fast moving train, then the measured lifetime will be the same as what we got in a stationary laboratory. It also means that there is no absolute "rest," all translational motions are relative.

The second statement directly contradicts with our daily experiences. If we measure the speed of light coming from the head light of a car approaching us, we would think that the measured speed would be c plus the speed of the car. But no, we still get c. We must accept this no matter how "strange" it is, because this is what we find in Michelson-Morley experiment.

2.3 Transverse Distances

Let us re-examine the measurement of distances. Our first conclusion will be that distances transverse to the velocity do not change.

This is the first of many thought experiments. Let us be extremely careful. Suppose a rod of proper length l is moving to the right with speed v, and the length of the rod is transverse to its velocity, Fig. 2.3. Also suppose that there are two pencils at the two ends of the rod. The pencils will leave some marks on the stationary ruler, as arranged in the figure.

The marks on the ruler will tell us the "moving length" of the rod. If the proper length of the rod is doubled (for example, there are two identical

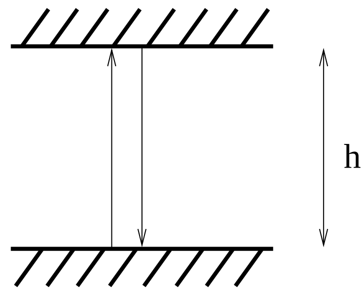

Figure 2.4: A stationary light clock.

rods aligned end to end), the moving length should also be doubled. Hence, the moving length is proportional to the proper length. Let it be αl, where the proportional constant does not depend on l, but in general depends on v. Just for easy visualization, suppose the length of our ruler is just αl. Then, the two ends of the rod touch the two ends of the ruler. All observers, no matter the velocities, will agree that the ends touch, although they may not agree on when and where they touch.

Now, in the rest frame of the rod, the ruler is moving to the left with speed v. On one hand, since their ends meet, the "moving length" of the ruler must be equal to the proper length of the rod, l. On the other hand, by the first basic premise of special relativity, the "moving length" of the ruler must be equal to α times its proper length, which is αl. We have

$$l = \alpha^2 l \ . \tag{2.11}$$

Since we are talking length, α must be positive, and $\alpha = 1$. Transverse length does not change.

2.4 Time Dilation

The next important result we discuss is the **time dilation**: a moving clock will run slower. We will illustrate this effect by the following example.

We consider a light clock as shown in Fig. 2.4. Two parallel mirrors are separated by a distance h. One unit of time is defined by the time taken for the light pulse to travel one round trip,

$$t = \frac{2h}{c} \ . \tag{2.12}$$

Now suppose the clock is moving to the right with speed v relative to a stationary observer, Fig. 2.5. Let the time taken for the pulse to go back to the lower mirror be t', as measured by the observer. Then, as shown in Fig. 2.5, since the mirrors have moved by a distance vt', the distance traveled

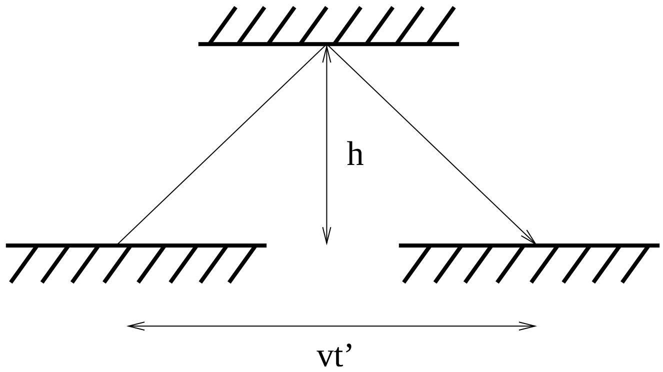

Figure 2.5: A moving light clock.

by the light pulse is $2\sqrt{h^2 + (vt'/2)^2}$. We have

$$2\sqrt{h^2 + (vt'/2)^2} = ct' \ . \tag{2.13}$$

Solving for t',

$$t' = \frac{2h/c}{\sqrt{1 - v^2/c^2}} = \frac{t}{\sqrt{1 - v^2/c^2}} \tag{2.14}$$

which is greater than t. Thus, a moving light clock is running slower. Notice that not only the light clock, any clock will run slower by the same factor, because, by the first premise, all physics is the same in any inertial frame.

2.5 Length Contraction

Length contraction is also called **Lorentz contraction**: the length of a moving object will be shorter along the direction of motion. If the object is of length l_0 at rest, then we will show that when it is moving with speed v, the length becomes

$$l = l_0\sqrt{1 - v^2/c^2} \ . \tag{2.15}$$

We try to illustrate the length contraction as follows.

If the proper length of a rod is l_0, and it is at rest relative to the observer, the time taken for a photon to go from one end to another and back is $t_0 = 2l_0/c$.

Now, assume that the rod is moving with a speed v to the right, Fig. 2.6. Again, we use the time of flight of a photon to measure its length l. In the first half (top of Fig. 2.6), when the photon reaches the other end, the rod has moved a distance vt', where t' is the time of flight of the first half. We have $l + vt' = ct'$. Similarly, in the second half (bottom of Fig. 2.6), while the photon is coming back, we have $l - v(t - t') = c(t - t')$, where t is the

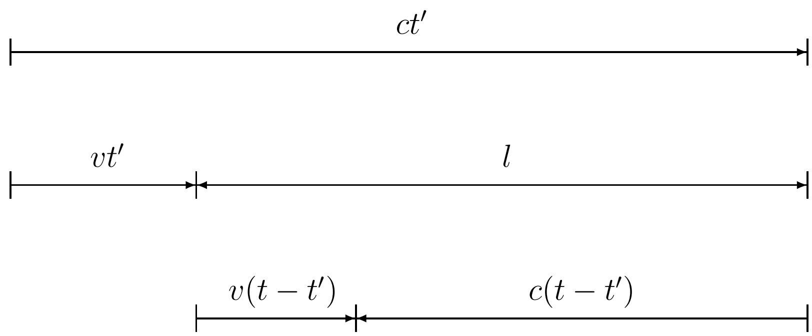

Figure 2.6: Length Contraction.

total time of flight. Eliminate t', we have

$$2l = (1 - v^2/c^2)ct \; . \tag{2.16}$$

The times of flight in the two reference frames, t_0 and t, are related by Eq. (2.14). Finally, we have

$$l = (1 - v^2/c^2)\, ct/2 = \sqrt{1 - v^2/c^2}\; ct_0/2 = \sqrt{1 - v^2/c^2}\; l_0 \; . \tag{2.17}$$

Notice that the transverse direction will not be contracted. Therefore, the shape of a fast moving object will be distorted.

Now, we will give an example that shows the various concepts in this chapter. When some cosmic particles enter our atmosphere and collide with the molecules, many other unstable particles will be created. Let us assume the lifetime of such an unstable particle be T, also assume for simplicity the thickness of the atmosphere be L. What will be their minimum speed if we detect those particles on the Earth surface?

There are two equivalent ways to solve this problem. Let the minimum speed of the particle be v. From the point of view of the particle, due to length contraction, the thickness of the atmosphere is shorten to

$$L\sqrt{1 - v^2/c^2} \; . \tag{2.18}$$

In order to reach the Earth surface, vT must be equal to or larger than this length. Hence, we have

$$\begin{aligned} vT &= L\sqrt{1 - v^2/c^2} \\ v &= \frac{c}{\sqrt{1 + (cT/L)^2}} \; . \end{aligned} \tag{2.19}$$

From the point of view of an observer on the Earth, due to time dilation, the lifetime of the particle is

$$T' = \frac{T}{\sqrt{1 - v^2/c^2}} \ . \tag{2.20}$$

Thus, to detect the particle on the Earth surface, vT' must be equal to L. Simple algebra leads us to the same expression, Eq. (2.19). Notice that the value of v is never greater than the speed of light.

Chapter 3

Lorentz Transformations

3.1 Lorentz Transformation

One event will have different coordinates on different coordinate systems. In Newtonian mechanics, if there is a coordinate system moving to the $+x$-direction relative to another coordinate, the Galilean transformation between the two are

$$x' = x - vt \tag{3.1}$$
$$t' = t \tag{3.2}$$

where v is the velocity and we have chosen the coordinate systems that their origins coincide, $(x', t') = (0, 0)$ if and only if $(x, t) = (0, 0)$. Let emphasize again that these are the coordinates of the same event, for example, a bomb explodes at (x, t) relative to a stationary observer and at (x', t') relative to a coordinate system fixed with a train moving to the right with speed v.

The coordinate transformation in special relativity is called the **Lorentz Transformation**. We have mentioned that the transverse directions do not change, we will only focus on the direction along the velocity. First, because two rulers should be two times the length of one ruler in any coordinate systems, the transformation between the coordinates must be linear,

$$x' = fx + gt \tag{3.3}$$
$$t' = hx + kt \tag{3.4}$$

where f, g, h and k are functions of the velocity v. Second, the origin of the moving system must have coordinates $x = vt$ and $x' = 0$, hence, by Eq. (3.3), for all t,

$$0 = fvt + gt$$
$$g = -vf \ . \tag{3.5}$$

Third, the origin of the stationary system must have $x = 0$ and $x' = -vt'$. By Eq. (3.3) and Eq. (3.4), for all t,

$$\begin{cases} x' &= -vft \\ t' &= kt \end{cases}$$
$$-vft = -vkt$$
$$k = f \ . \tag{3.6}$$

To summarize up to now, we have

$$x' = f(x - vt) \tag{3.7}$$
$$t' = hx + ft \ . \tag{3.8}$$

Fourth, let consider a photon with trajectory $x = ct$. Since it passes through the origin of the stationary coordinate system, it also passes through the origin of the moving coordinate system, $x' = t' = 0$. By the second basic premise of special relativity, it must also have velocity c. Hence, $x' = ct'$, and

$$\begin{aligned} f(x - vt) = x' &= ct' = c(hx + ft) \\ f(ct - vt) &= c(hct + ft) \\ fc - fv &= c^2 h + cf \\ h &= -fv/c^2 \ . \end{aligned} \tag{3.9}$$

Finally, consider the event that the photon strikes the lower mirror of the light clock in Section 2.4. It has the coordinates $(x, t) = (vt, t)$ and $(x', t') = (0, t\sqrt{1 - v^2/c^2})$. (Note that we used different notation in Section 2.4.) We have

$$\begin{aligned} t' &= f(t - vx/c^2) \\ t\sqrt{1 - v^2/c^2} &= f(1 - v^2/c^2)\, t \\ f &= \frac{1}{\sqrt{1 - v^2/c^2}} \ . \end{aligned} \tag{3.10}$$

To summarize, we have

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \tag{3.11}$$
$$y' = y \tag{3.12}$$
$$z' = z \tag{3.13}$$
$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \ . \tag{3.14}$$

These are the Lorentz transformation. Note that these two equations reduce to Eq. (3.1) and Eq. (3.2) if $c \to \infty$. One can easily check that the inverse transformation is

$$\begin{aligned} x &= \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} && (3.15) \\ y &= y' && (3.16) \\ z &= z' && (3.17) \\ t &= \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \, . && (3.18) \end{aligned}$$

Sometime, we write the Lorentz transformation in matrix form

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & -\frac{v}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{v/c^2}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \qquad (3.19)$$

where $\beta = v/c$.

We illustrate some of the effects that we have discussed by the Lorentz transformation. The time separation in the moving system between two events $(x', t') = (x', t'_1)$ and $(x', t') = (x', t'_2)$ is, of course, $t'_2 - t'_1$, if we assume that $t'_2 > t'_1$. The time separation between the two events in the stationary system is

$$\begin{aligned} t_2 - t_1 &= \frac{t'_2 + ux'/c^2}{\sqrt{1 - u^2/c^2}} - \frac{t'_1 + ux'/c^2}{\sqrt{1 - u^2/c^2}} \\ &= \frac{t'_2 - t'_1}{\sqrt{1 - u^2/c^2}} \\ &> t'_2 - t'_1 \, . && (3.20) \end{aligned}$$

This is time dilation. Similarly, the distance between two ends, (x'_1, t') and (x'_2, t'), of a ruler in the moving system is $x'_2 - x'_1$. While the coordinates of one end of the ruler in the stationary system is

$$\begin{aligned} x_1 &= \frac{x'_1 + vt'}{\sqrt{1 - v^2/c^2}} && (3.21) \\ t &= \frac{t' + vx'_1/c^2}{\sqrt{1 - v^2/c^2}} \, . && (3.22) \end{aligned}$$

Hence, we have $t' = t\sqrt{1 - v^2/c^2} - vx'_1/c^2$, and the trajectory of one end is

$$x_1 = \frac{1}{\sqrt{1 - v^2/c^2}} \left(x'_1 + vt\sqrt{1 - v^2/c^2} - v^2 x'_1/c^2\right) = x'_1\sqrt{1 - v^2/c^2} + vt \, . \qquad (3.23)$$

The trajectory of another end is $x_2 = x'_2\sqrt{1-v^2/c^2} + vt$. The distance in the stationary system is

$$\begin{aligned} x_2 - x_1 &= (x'_2 - x'_1)\sqrt{1-u^2/c^2} \\ &< x'_2 - x'_1 \ . \end{aligned} \tag{3.24}$$

This is length contraction.

In Newtonian mechanics, distance between two points does not change with observers. In special relativity, this is not true anymore. Let two events be labeled by subscripts 1 and 2, and $\Delta t' \equiv t'_2 - t'_1$, etc. Then,

$$\begin{aligned} & c^2(\Delta t')^2 - (\Delta x')^2 \\ = \ & c^2(t'_2 - t'_1)^2 - (x'_2 - x'_1)^2 \\ = \ & c^2\left(\frac{t_2 - t_1 - u/c^2(x_2 - x_1)}{\sqrt{1-u^2/c^2}}\right)^2 - \left(\frac{x_2 - x_1 - u(t_2 - t_1)}{\sqrt{1-u^2/c^2}}\right)^2 \\ = \ & \frac{c^2(\Delta t)^2 - 2u\Delta t\Delta x + u^2/c^2(\Delta x)^2 - ((\Delta x)^2 - 2u\Delta x\Delta t + u^2(\Delta t)^2)}{1-u^2/c^2} \\ = \ & \frac{(c^2-u^2)(\Delta t)^2 - (1-u^2/c^2)(\Delta x)^2}{1-u^2/c^2} \\ = \ & c^2(\Delta t)^2 - (\Delta x)^2 \ . \end{aligned} \tag{3.25}$$

This is called the **invariant interval** between two events, because the value does not change with observers.

- If $\Delta s^2 \equiv c^2(\Delta t)^2 - (\Delta x)^2 > 0$, the two events are **time-like** related;
- if $\Delta s^2 = 0$, they are **light-like** related;
- if $\Delta s^2 < 0$, they are **space-like** related.

(For three dimensional space, $\Delta s^2 \equiv c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$.)

Very often, we compare one event with the origin of the frame. Referring to Fig. 3.1, all events in the upper or lower wedges in the diagram are time-like related to the origin. All events in the left or right wedges are space-like related to the origin. All events on the two diagonal lines are light-like related to the origin. Their significance will be further discussed in the next section.

3.2 Simultaneity

How can we say that two events far apart from each other happen *at the same time*? Since the speed of light is constant, we can send light pulses from the middle point to the two events. If the two light pulses arrive at the same time with the events, they happen simultaneously, Fig. 3.2.

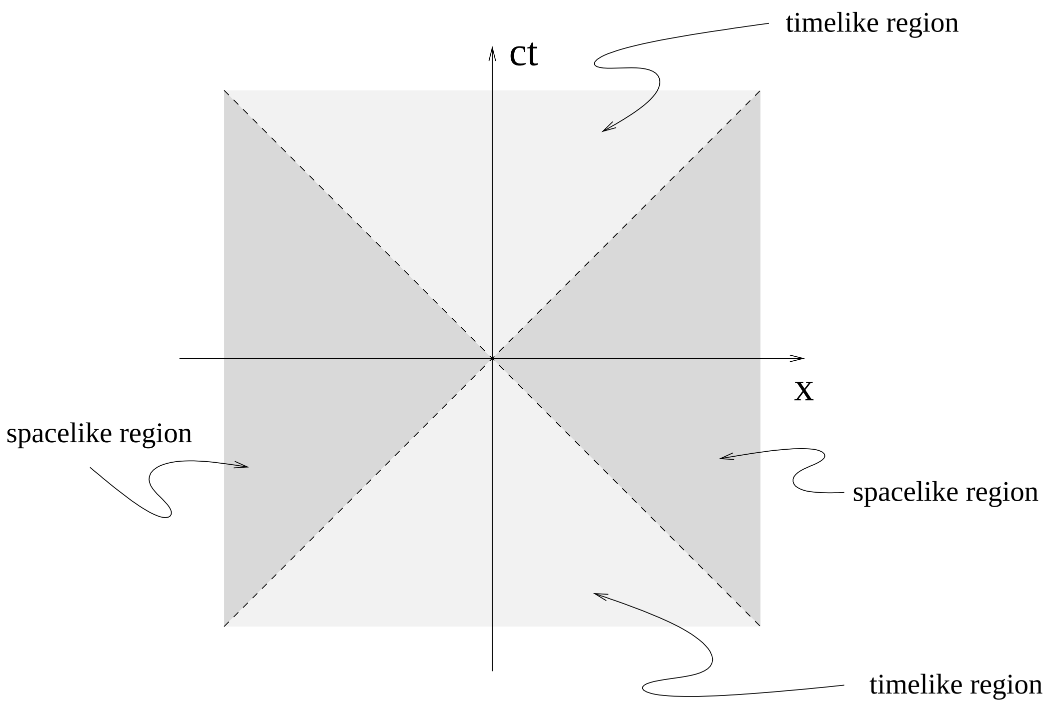

Figure 3.1: The spacetime diagram.

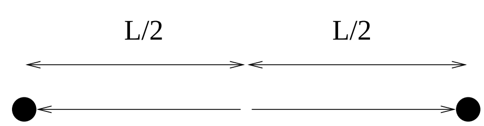

Figure 3.2: Check for simultaneity by two light pulses.

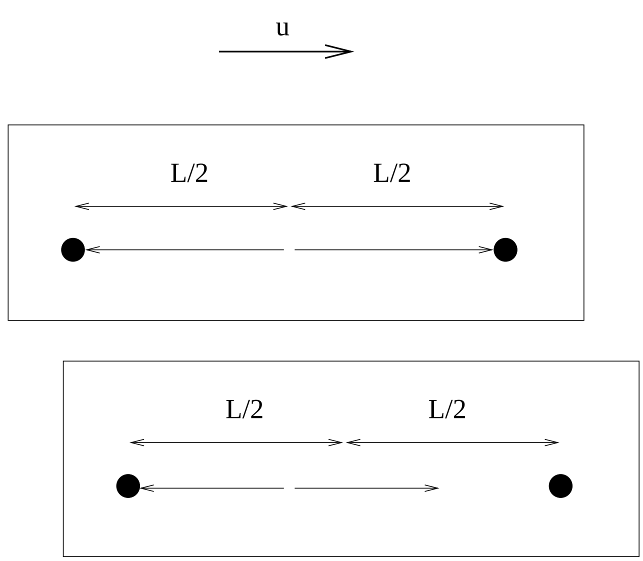

Figure 3.3: Check for simultaneity by two light pulses for moving events.

This is the best definition we have. However, once we accept this definition, simultaneity is no longer an absolute concept. Two events which are simultaneous for one observer may not be simultaneous for another observer, Fig. 3.3. If the events are moving to the right, by the time one light pulse arrives the left point, the light pulse on the right is still in its way. Thus, if something happens at the time of the light pulse arrived, event at the left will happen "earlier."

We now study simultaneity in details. Let us consider two time-like related events. Choose the coordinate system such that one is at the origin. Let the other be (x, t). Since they are time-like related, $c^2t^2 > x^2$, we have

$$\begin{aligned} |t| &> |x/c| \\ |t| &> |vx/c^2| \end{aligned} \tag{3.26}$$

for any $|v| < c$. Hence, t and $t - vx/c^2$ have the same sign. Look back to Eq. (3.14), it means that t and t' have the same sign, no matter the speed. If $t > 0$, then $t' > 0$ and the event (x, t) happens after the origin in any inertial frame. If $t < 0$, then the event happens before the origin in any inertial frame. In particular, if we choose

$$v = \frac{x}{t} , \tag{3.27}$$

which is smaller than c, then $x' = 0$ by Eq. (3.11). It means, in that frame, the two events happen at the same place.

If the two events are space-like related, let one be the origin and the other be (x, t), where $c^2t^2 < x^2$. Let

$$v_0 \equiv \frac{ct}{x} c . \tag{3.28}$$

Note that $|v_0| < c$, so it is a valid speed. Then, if $v > v_0$, $t' < 0$ by Eq. (3.14); if $v = v_0$, $t' = 0$; if $v < v_0$, $t' > 0$.

Thus, we have proved that

- if two events are time-like, then chronological order is preserved, which means that all observers agree which event is earlier than the other.

- If they are space-like, then some observers will find that one happened earlier than the other, some other observers will find that the other event is earlier and yet some other observers will find that they happened at the same time. In terms of Lorentz transformation, if $\Delta s^2 < 0$, then
  there are inertial frames that $\Delta s^2 = c^2(\Delta t)^2 - (\Delta x)^2$ and $\Delta t > 0$;
  there are inertial frames that $\Delta s^2 = c^2(\Delta t')^2 - (\Delta x')^2$ and $\Delta t' < 0$;
  and there is inertial frame that $\Delta s^2 = c^2(\Delta t'')^2 - (\Delta x'')^2$ and $\Delta t'' = 0$.

Hence, simultaneity depends on the observers. Recall that in Newtonian mechanics, all observers would agree that either one event happened earlier or the two events happened at the same time. (If the two events are light-like related, then all observers will find that they are light-like, and we usually do not discuss their chronological order.)

3.3 Addition of Velocities

If a bullet is moving with speed u relative to a train, and the train is moving with speed v relative to the ground in the same direction, then in Newtonian mechanics, the speed of the bullet relative to the ground is $u + v$. This is not correct when u or v or both are near c. Let us consider the case that the bullet is moving in general direction relative to the moving frame. Then, its coordinates are

$$x' = u'_x t' \quad (3.29)$$
$$y' = u'_y t' \quad (3.30)$$
$$z' = u'_z t' , \quad (3.31)$$

where $\mathbf{u}' \equiv (u'_x, u'_y, u'_z)$ is the velocity of the bullet in the moving frame. By Eq. (3.15) to Eq. (3.18), the coordinates of the bullet in the stationary frame are

$$\begin{cases} x &= \gamma_v(u'_x t' + vt') = \gamma_v(u'_x + v)\, t' \\ y &= u'_y t' \\ z &= u'_z t' \\ t &= \gamma_v(t' + vu'_x t'/c^2) = \gamma_v(1 + vu'_x/c^2)\, t' \end{cases}$$
$$\begin{cases} x &= \dfrac{u'_x + v}{1 + vu'_x/c^2}\, t \\ y &= \dfrac{u'_y}{\gamma_v(1 + vu'_x/c^2)}\, t \\ z &= \dfrac{u'_z}{\gamma_v(1 + vu'_x/c^2)}\, t \end{cases} \quad (3.32)$$

where $\gamma_v = 1/\sqrt{1 - v^2/c^2}$. Hence, the resulting velocity is

$$\mathbf{u} \equiv \left(\frac{u'_x + v}{1 + vu'_x/c^2}, \frac{u'_y}{\gamma_v(1 + vu'_x/c^2)}, \frac{u'_z}{\gamma_v(1 + vu'_x/c^2)} \right) . \quad (3.33)$$

This is called the **addition of velocities**. One can easily check that if $v = c$ or the speed of the bullet $\sqrt{{u'_x}^2 + {u'_y}^2 + {u'_z}^2} = c$, the resulting speed is still c. This is consistent with the fact that speed of light is constant in any inertial frame.

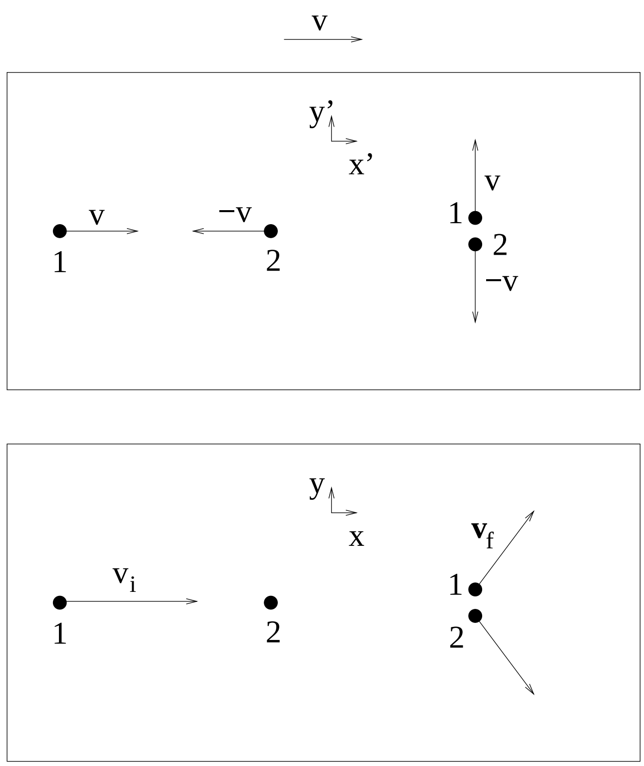

Figure 3.4: Collisions of two identical particles in two frames.

3.4 Energy and Momentum

We will examine the conservation of momentum and energy in this section. Consider the collision of two identical particles of rest mass m in two frames. The upper half of Fig. 3.4 is the center of mass frame. Suppose before the collision, the speeds of the two particles are v and $-v$ along the x'-axis, as indicated in the left of the figure. After the collision, suppose the velocities of the particles are along the y'-axis, as indicated in the right. The lower half of the figure is the laboratory frame. Before the collision, particle 2 is at rest, and let the speed of particle 1 be v_i. After the collision, let the velocity of particle 1 be $\mathbf{v}_f$. Then, by Eq. (3.33), we have

$$v_i = \frac{2v}{1+v^2/c^2} \tag{3.34}$$

$$v_{fx} = v \tag{3.35}$$

$$v_{fy} = \frac{v}{\gamma_v} \tag{3.36}$$

where v_{fx} and v_{fy} are the components of $\mathbf{v}_f$. (Substitute $(u'_x, u'_y, u'_z) = (v, 0, 0)$ to get the first equation, and $(u'_x, u'_y, u'_z) = (0, v, 0)$ to get the last two.) If we defined the momentum by the classical way, $\mathbf{p} \equiv m\mathbf{v}$, then the conservation of momentum does not hold in the laboratory frame.

$$\frac{2mv}{1+v^2/c^2} \neq 2mv_{fx} = 2mv \ . \tag{3.37}$$

To retain the conservation of momentum (and experiments verify that this is the correct way), we define the **momentum** as

$$\mathbf{p} \equiv \frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}} \ . \tag{3.38}$$

Note that in this definition, $\mathbf{v}$ is always the velocity of the particle relative to the frame, and v is its speed. Then, momentum is conserved in our case because the x-component of the initial total momentum is

$$\begin{aligned} & \frac{mv_i}{\sqrt{1-v_i^2/c^2}} \\ = \ & \frac{2mv}{1+v^2/c^2} \frac{1}{\sqrt{1-\frac{4v^2}{(1+v^2/c^2)^2c^2}}} \\ = \ & \frac{2mv}{\sqrt{(1+v^2/c^2)^2 - 4v^2/c^2}} \\ = \ & \frac{2mv}{\sqrt{(1-v^2/c^2)^2}} \\ = \ & \frac{2mv}{1-v^2/c^2} \ , \end{aligned} \tag{3.39}$$

and the x-component of the final total momentum is

$$\begin{aligned} & 2\frac{mv_{fx}}{\sqrt{1-(v_{fx}^2+v_{fy}^2)/c^2}} \\ = \ & \frac{2mv}{\sqrt{1-(v^2+v^2(1-v^2/c^2))/c^2}} \\ = \ & \frac{2mv}{\sqrt{1-2v^2/c^2+v^4/c^4}} \\ = \ & \frac{2mv}{1-v^2/c^2} \ . \end{aligned} \tag{3.40}$$

(The factor of 2 is due to the fact that there are two particles.)

Let us *assume* that the correct generalization of the Newton's second law is

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{mv}{\sqrt{1-v^2/c^2}} \ . \tag{3.41}$$

where t is the time of rest frame of the observer and we just consider the magnitude. (This is verified by experiments for charged particles in electric field, for example.) Then, denote the displacement by r, the kinetic energy of the particle is

$$\begin{aligned}
K &= \int F\,\mathrm{d}r = \int F\frac{\mathrm{d}r}{\mathrm{d}t}\,\mathrm{d}t = \int Fv\,\mathrm{d}t \\
&= \int v\,\mathrm{d}\frac{mv}{\sqrt{1-v^2/c^2}} \\
&= \int mv\left(\frac{1}{\sqrt{1-v^2/c^2}} - \frac{v}{2}\frac{-2v/c^2}{(1-v^2/c^2)^{3/2}}\right)\mathrm{d}v \\
&= \int \frac{mv}{(1-v^2/c^2)^{3/2}}\,\mathrm{d}v \\
&= m\int \frac{\mathrm{d}v^2}{2(1-v^2/c^2)^{3/2}} \\
&= \frac{mc^2}{\sqrt{1-v^2/c^2}} + \text{constant}\ . \qquad (3.42)
\end{aligned}$$

Usually, we define that kinetic energy is zero when the speed is zero, we have

$$K = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2\ . \qquad (3.43)$$

We define the total **relativistic energy** or just energy E of a particle as

$$E = \frac{mc^2}{\sqrt{1-v^2/c^2}}\ , \qquad (3.44)$$

and the **rest energy** E_0 as

$$E_0 = mc^2\ . \qquad (3.45)$$

As a simple calculation, we try to find the speed at which the relativistic energy is twice the rest energy. Let the required speed be v, then

$$\frac{mc^2}{\sqrt{1-v^2/c^2}} = 2mc^2\ , \qquad (3.46)$$

and it gives $v = \frac{\sqrt{3}}{2}c$.

The energy, momentum and mass of a particle are related by

$$E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

$$\begin{aligned}
E^2 &= \frac{m^2c^4}{1-v^2/c^2} \\
E^2 &= \frac{m^2c^4}{1-v^2/c^2}\,(1-v^2/c^2+v^2/c^2) \\
E^2 &= m^2c^4+\frac{m^2v^2c^2}{1-v^2/c^2} \\
E^2 &= m^2c^4+p^2c^2\ . \qquad (3.47)
\end{aligned}$$

We find by experiments that the total relativistic energy of an isolated system remains constant. This is the relativistic form of conservation of energy. (If you do not like to assume the generalized form of the Newton's second law, you could consult, for example, Section 11.4 and 11.5 of John David Jackson, Classical Electrodynamics, 3rd ed., Wiley, 1999 for a proof of the famous equation $E = mc^2$.)

We will, in fact, prove in the next chapter that energy and momentum are conserved in any frame if they are conserved in any one frame.

One conclusion of the special relativity is that the speed of light is the fastest speed that could be attained. When the speed of an object approaches the speed of light, it gets more and more difficult to accelerate because of Eq. (3.41).

The conservation of the relativistic energy implies that rest mass is not conserved in general, because energy can escape in the form of photons, for example. One important example is the thermonuclear fusion in the core of a star, including the Sun,

$$4\mathrm{p} \rightarrow {}^4\mathrm{He} + 2\mathrm{e}^+ + 2\nu_e + 2\gamma\ , \qquad (3.48)$$

where four protons fuse to one helium nucleus, two positrons, two neutrinos and two photons. The sum of the rest masses of particles in the right hand side is less than the sum in the left. The difference is the energy released, about 4.2×10^{-12}J per helium nucleus created. This is also the **binding energy** of the helium nucleus, because we need this amount of energy to break apart the nucleus into its components. Hence, the Sun is losing mass by radiating. Given that the solar luminosity is 4×10^{26}W, the Sun is losing mass with rate 4×10^{9}kg/s.

Chapter 4

Relativistic Kinematics

We will further develop the theory in this chapter.

4.1 Relativistic Doppler Effect

Let recall the formula for the classical Doppler effect. Let the speed of the wave be C, and the emitted wavelength of some source be λ_0. We have to distinguish whether the observer is moving or the source is moving relative to the medium of the wave. If the source and the observer are moving away from each other with speeds v_S and v_O respectively, then the observed wavelength is

$$\lambda = \frac{1 + v_S/C}{1 - v_O/C}\,\lambda_0 \ . \tag{4.1}$$

Now we consider the relativistic Doppler effect for light. Since light does not need a medium, the correct formula should only depend on the relative speed between the source and the observer. Suppose the source is at rest, let the time taken for it to emit N periods of wave be t_0 and the wavelength be λ_0, Fig. 4.1, then we have

$$N\lambda_0 = ct_0 \ . \tag{4.2}$$

If it is moving away at a speed v, and it takes time t to emit the same N periods of wave, we have

$$N\lambda = (c + v)t \ , \tag{4.3}$$

where λ is the wavelength we, the stationary observers, observed. The times taken t and t_0 are related by Eq. (2.14)

$$t_0 = t\,\sqrt{1 - v^2/c^2} \ . \tag{4.4}$$

(For non-relativistic Doppler effect, we take $t = t_0$.) Thus,

$$\frac{\lambda}{\lambda_0} = (1 + v/c)\frac{t}{t_0} = \left(\frac{1 + v/c}{1 - v/c}\right)^{1/2} \ . \tag{4.5}$$

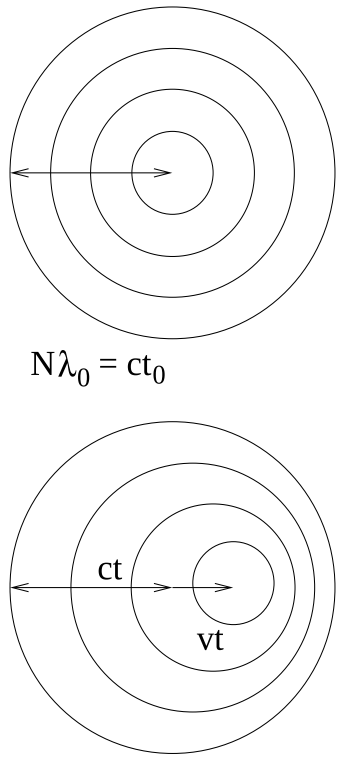

Figure 4.1: Relativistic Doppler effect.

This formula is valid for any object moving directly away or toward the observer (v positive if it is receding and negative if it is approaching). The formula for general direction of velocity is more complicated and not shown here.

If v is much less than c, we can approximate the right hand side by $1+v/c$ and the formula reduces to Eq. (4.1), because for small v_S and v_O,

$$\frac{\lambda}{\lambda_0} = \frac{1+v_S/C}{1-v_O/C} \approx (1+v_S/C)(1+v_O/C) \approx 1+(v_S+v_O)/C \tag{4.6}$$

and $v_S + v_O$ is the relative speed between them. If v is close to c, the shift of wavelength will be infinite.

As an example, suppose two spaceships are moving toward each other, with speeds v_1 and v_2, relative to us. If the spaceship with speed v_1 shines a laser with frequency f_1 in its frame to the other spaceship, what will be the observed frequency in our frame, and the observed frequency in frame of the other spaceship? Well, the observed frequency in our frame is simply

$$f_1 \left(\frac{1+v_1/c}{1-v_1/c} \right)^{1/2} . \tag{4.7}$$

Their relative speed in their frame is given by

$$\frac{v_1 + v_2}{1 + v_1 v_2/c^2} \; . \tag{4.8}$$

Hence, the observed frequency is

$$\begin{aligned} & f_1 \left(\frac{1 + \frac{(v_1+v_2)/c}{1+v_1v_2/c^2}}{1 - \frac{(v_1+v_2)/c}{1+v_1v_2/c^2}} \right)^{1/2} \\ = \; & f_1 \left(\frac{1 + v_1/c + v_2/c + v_1 v_2/c^2}{1 - v_1/c - v_2/c + v_1 v_2/c^2} \right)^{1/2} \\ = \; & f_1 \left(\frac{(1 + v_1/c)(1 + v_2/c)}{(1 - v_1/c)(1 - v_2/c)} \right)^{1/2} \; . \end{aligned} \tag{4.9}$$

Another way to explain this formula is that, in the frame of the other spaceship, the observed frequency is just the Doppler shifted frequency of the one in Eq. (4.7) of speed v_2, and this is exactly given by the above formula.

There are many applications of the Doppler effect. For example, police uses radar to monitor speed of vehicles by emitting radio wave of known frequency and detect the frequency of the reflected wave. The two are related by the Doppler effect. Also, astronomers detect the speeds of galaxies by measuring the shift of the spectrum of light emitted.

4.2 Conservation of Energy and Momentum

We are going to prove that energy and momentum are conserved in any frame if they are in one frame. Let the two frames, one moving one stationary, be related as in Eq. (3.15) to Eq. (3.18), as usual, and let the velocity of a particle be $\mathbf{u}$ and $\mathbf{u}'$ relative to the stationary and moving frames as in Eq. (3.29) to Eq. (3.33). We first prove an useful relation

$$\begin{aligned} & 1 - \frac{\mathbf{u}^2}{c^2} \\ = \; & \frac{1}{c^2} \left(c^2 - \frac{(u'_x + v)^2}{(1 + v u'_x/c^2)^2} \right. \\ & \left. - \frac{{u'_y}^2}{(1 + v u'_x/c^2)^2} (1 - v^2/c^2) - \frac{{u'_z}^2}{(1 + v u'_x/c^2)^2} (1 - v^2/c^2) \right) \\ = \; & \frac{1}{c^2 (1 + v u'_x/c^2)^2} \Big(c^2 (1 + v u'_x/c^2)^2 - (u'_x + v)^2 \\ & - {u'_y}^2 (1 - v^2/c^2) - {u'_z}^2 (1 - v^2/c^2) \Big) \\ = \; & \frac{1}{c^2 (1 + v u'_x/c^2)^2} \Big(c^2 + 2 v u'_x + v^2 {u'_x}^2/c^2 - {u'_x}^2 - 2 u'_x v - v^2 \end{aligned}$$

$$
\begin{aligned}
& -{u'_y}^2(1-v^2/c^2) - {u'_z}^2(1-v^2/c^2)\Big) \\
= \; & \frac{1}{c^2(1+vu'_x/c^2)^2}\left(c^2 - v^2 - {u'_x}^2(1-v^2/c^2)\right. \\
& \left. -{u'_y}^2(1-v^2/c^2) - {u'_z}^2(1-v^2/c^2)\right) \\
= \; & \frac{1-v^2/c^2}{(1+vu'_x/c^2)^2}\left(1 - ({u'_x}^2 + {u'_y}^2 + {u'_z}^2)/c^2\right) \; . \qquad (4.10)
\end{aligned}
$$

Another way to write this is

$$
\gamma_{\mathbf{u}} = (1 + \frac{vu'_x}{c^2})\,\gamma_{\mathbf{u}'}\gamma_v \qquad (4.11)
$$

where, for example, $\gamma_{\mathbf{u}} = 1/\sqrt{1-\mathbf{u}^2/c^2}$. Hence, the energy of a particles of mass m in the two frames, E and E' are

$$
E = \gamma_{\mathbf{u}} mc^2 = (1 + \frac{vu'_x}{c^2})\,\gamma_{\mathbf{u}'}\gamma_v mc^2 = (1 + \frac{vu'_x}{c^2})\,\gamma_v E' \; . \qquad (4.12)
$$

Its momentum in the two frames, $\mathbf{p} \equiv (p_x, p_y, p_z)$ and $\mathbf{p}' \equiv (p'_x, p'_y, p'_z)$ are related by

$$
\begin{aligned}
& \mathbf{p} \\
= \; & \gamma_{\mathbf{u}} m \mathbf{u} \\
= \; & (1 + \frac{vu'_x}{c^2})\,\gamma_{\mathbf{u}'}\gamma_v m \left(\frac{u'_x + v}{1 + vu'_x/c^2}, \frac{u'_y}{\gamma_v(1+vu'_x/c^2)}, \frac{u'_z}{\gamma_v(1+vu'_x/c^2)}\right) \\
= \; & \gamma_{\mathbf{u}'} m \left(\gamma_v(u'_x + v), u'_y, u'_z\right) \\
= \; & (\gamma_v p'_x + \gamma_v \gamma_{\mathbf{u}'} m v, p'_y, p'_z) \; . \qquad (4.13)
\end{aligned}
$$

Let $p_t \equiv E/c = \gamma_{\mathbf{u}} mc$ and $p'_t \equiv E'/c = \gamma_{\mathbf{u}'} mc$. (We will justify the notations below.) Then, we can rewrite Eq. (4.12) and Eq. (4.13) as

$$
\left\{
\begin{array}{lcl}
p_x & = & \gamma_v(p'_x + \gamma_{\mathbf{u}'} m v) \\
p_y & = & p'_y \\
p_z & = & p'_z \\
p_t & = & (1 + \frac{vu'_x}{c^2})\,\gamma_v p'_t
\end{array}
\right.
$$

$$
\left\{
\begin{array}{lcl}
p_x & = & \gamma_v(p'_x + (v/c)\gamma_{\mathbf{u}'} mc) \\
p_y & = & p'_y \\
p_z & = & p'_z \\
p_t & = & \gamma_v(1 + vu'_x/c^2) p'_t
\end{array}
\right.
$$

$$
\left\{
\begin{array}{lcl}
p_x & = & \gamma_v(p'_x + (v/c)p'_t) \\
p_y & = & p'_y \\
p_z & = & p'_z \\
p_t & = & \gamma_v(p'_t + (v/c)p'_x)
\end{array}
\right. \; . \qquad (4.14)
$$

These are the transformations of energy and momentum between frames. Suppose initially there are N particles, labeled by (i), and after they interact, there are M particles, labeled by (j). Conservation of energy and momentum in one frame means

$$\left\{ \begin{array}{lcl} \sum_i E'_{(i)} & = & \sum_j E'_{(j)} \\ \sum_i \mathbf{p}'_{(i)} & = & \sum_j \mathbf{p}'_{(j)} \end{array} \right. . \tag{4.15}$$

The energy and momentum of each particle will transform according to Eq. (4.14). For conservation of energy, consider

$$\begin{aligned} & \sum_i E_{(i)} - \sum_j E_{(j)} \\ = \;& c\left(\sum_i p_{t(i)} - \sum_j p_{t(j)}\right) \\ = \;& c\left(\sum_i \gamma_v(p'_{t(i)} + (v/c)p'_{x(i)}) - \sum_j \gamma_v(p'_{t(j)} + (v/c)p'_{x(j)})\right) \\ = \;& c\left(\gamma_v(\sum_i p'_{t(i)} - \sum_j p'_{t(j)}) + \frac{\gamma_v v}{c}(\sum_i p'_{x(i)} - \sum_j p'_{x(j)})\right) \\ = \;& 0 \end{aligned} \tag{4.16}$$

by Eq. (4.15). The crucial point is that the coefficients in the transformation, γ_v and $\gamma_v\, v/c$, only depend on the velocity between the two frames, not on the properties of the particles. Similarly, we can prove the conservation of momentum.

4.3 Four Vectors

Let us rewrite Eq. (3.15) to Eq. (3.18) as

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} \gamma_v & 0 & 0 & \gamma_v v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma_v v/c & 0 & 0 & \gamma_v \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} . \tag{4.17}$$

The transformation matrix is exactly equal to the one for the momentum in Eq. (4.14)

$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ p_t \end{pmatrix} = \begin{pmatrix} \gamma_v & 0 & 0 & \gamma_v v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma_v v/c & 0 & 0 & \gamma_v \end{pmatrix} \begin{pmatrix} p'_x \\ p'_y \\ p'_z \\ p'_t \end{pmatrix} . \tag{4.18}$$

Any physical quantity whose components relative to different frames transform in this way is called a **four vector**. Hence, the space-time point of an event is a four vector, the momentum-energy is also a four vector. The four-velocity is defined to be $(\gamma_{\mathbf{u}}\mathbf{u}, \gamma_{\mathbf{u}}c)$ if the usual velocity of the particle is $\mathbf{u}$. This concept of four vectors may not appear to be very useful in this stage, but you will find it extremely important in further development of relativity. Just to let the readers to have an idea, the electric field and magnetic field are not four vectors. They are something more general than vectors, called tensors.

Chapter 5

Paradoxes

In this last chapter, we will use two paradoxes to clarify some of the difficult points of special relativity. In the last section, we will discuss an illusion of faster-than-light speed, which confused astronomers for a while.

5.1 Twin Paradox

The paradox is that a pair of twin brothers were born on the Earth. One of them, say A, stayed on the Earth. The other, B, immediately took a space ship. He went away from the Earth in high speed for a long time, then turned back to the Earth in high speed. From the point of view of A, B was traveling in high speed and his time would be dilated. As a result, when B was back to the Earth, B should be younger than A. However, from the point of view of B, A and the Earth were also traveling in high speed away from B. So, B concluded that A should be younger when they met. Who is correct?

The simple resolution is that the space ship had to turn back. When it did, it was not an inertial frame anymore. Thus, B would actually experience the effect and B would be younger. The longer resolution is as follows.

Suppose A and B agree that they would send light signals to each other every ten years, according to their own clocks. How many signals would they received? Since trajectory of light on space-time diagram is diagonal. For A on the Earth, the signals he sent out would be like what is shown as dash lines in Fig. 5.1. The trajectory of B is also shown in the same space-time diagram as a bold line. We could count from the diagram that B would receive 8 light signals from A. Thus, both A and B knew that it was 80 years for A.

For B in the space ship, his clock ran slower relative to A's. Hence, the separation of his light signals in the space-time diagram along his trajectory is larger than that of A's, as shown in Fig. 5.2. But he would keep the separation of his signals even on *all* of his journey. One can easily see that A would receive less signals from B. In the diagram, A only received 4 signals,

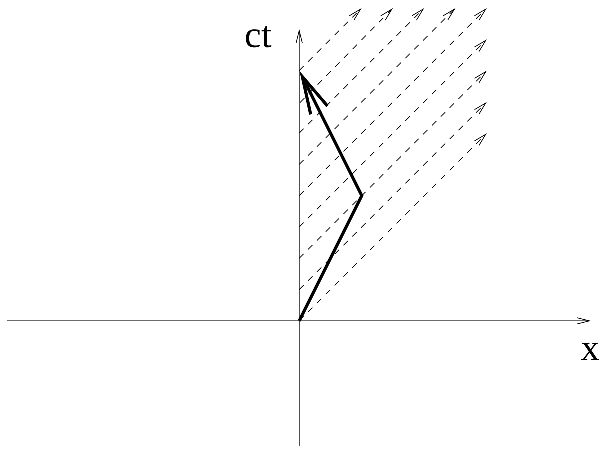

Figure 5.1: Observer on Earth sent out light signals every ten years. The bold line represents the trajectory of a space ship going away and then coming back.

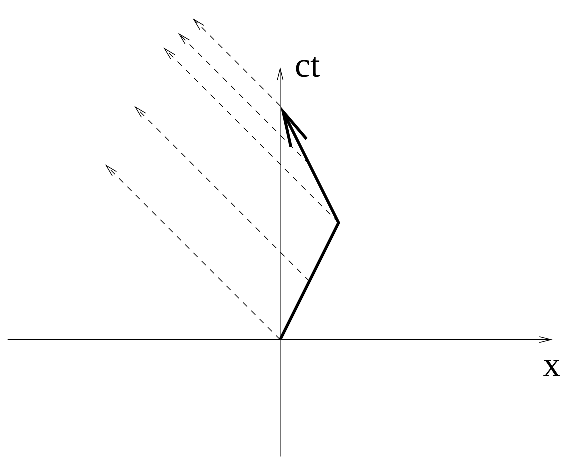

Figure 5.2: Observer in a space ship sent out light signals every ten years.

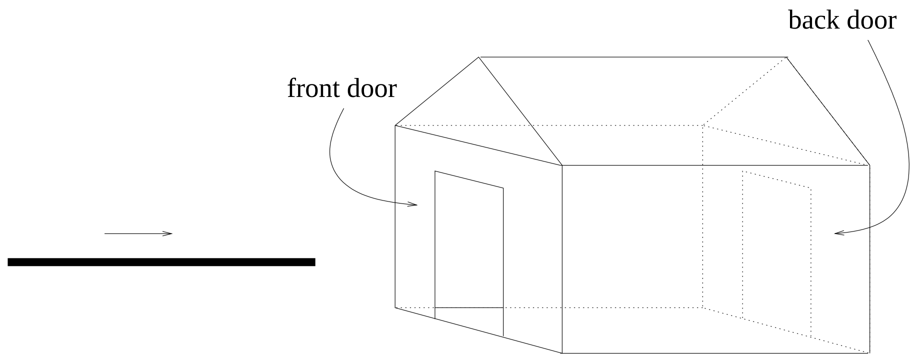

Figure 5.3: A pole at the left is moving very fast to pass through the barn at the right.

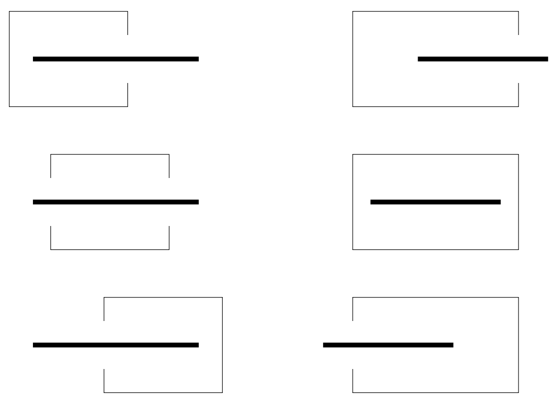

Figure 5.4: This is the top view of the pole and barn paradox. The left column is the view of a moving observer, while the right is the view of a stationary observer. The time is running upward.

or it was only 40 years for B. In conclusion, B would be younger than A. (The frequency of the light signals is just the Doppler effect. Could you translate our argument to the language of Doppler effect?)

5.2 Pole and Barn Paradox

The paradox is that a pole has the same proper length as a barn, Fig. 5.3. The pole is moving very fast to try to pass through the barn. The barn, however, has two doors, one at the front and one at the back. From the point of view of a stationary observer, due to Lorentz contraction, the length of the pole is less than that of the barn. Hence, for some brief moment, the pole is totally inside the barn and he can close the two doors. From the point of view of the pole, the length of the barn is contracted. Hence, there is no time that it is totally inside the barn. Who is correct?

The resolution is that the paradox involves the concept of closing the

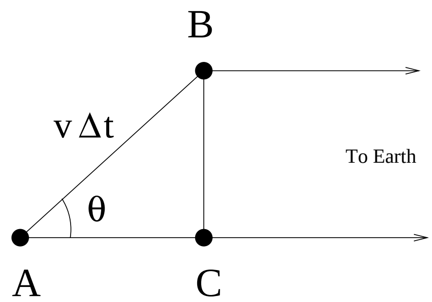

Figure 5.5: An illusion of superluminal motion.

two doors simultaneously. We know that this depends on observers. In fact, both views are correct, Fig. 5.4. The stationary observer will see that the two doors are closed for a brief moment, but an observer moving with the pole will not see that the two doors are closed at the same time. The moving observer will see that the back door is closed before the front end of the pole reaches it. Then, the back door opens for the pole to pass through. And then, the front door closes after the back end of the pole completely is inside the barn. The readers could work out the space-time diagram for the whole process.

5.3 An Illusion of Superluminal Motion

We very often see jets associated with black holes and quasars. The jets will produce gas clouds, which move in a very high speed. We did observe that the gas clouds appear to be moving with speed greater than the speed of light, called superluminal motion. This is an illusion. How can we explain it?

Usually, the velocity of the gas cloud will make an angle θ with our line of sight, Fig. 5.5. Let the distance between point B or C from the Earth be L, and the time at which the gas cloud was at point A be t_1. Also, assume that the gas cloud will move from point A to point B in time Δt with speed v. Then, the time that we see the gas cloud at point B is

$$t_1 + \Delta t + L/c \; . \tag{5.1}$$

However, the light emitted when the cloud was at point A will take a bit more time to reach Earth. Because the distance between point A and C is $v\Delta t \cos\theta$, the time that we see the gas cloud at point A is

$$t_1 + (v\Delta t \cos\theta + L)/c \; . \tag{5.2}$$

Because point A and C are on the same line of sight, we cannot distinguish between the two points. Thus, from our point of view, the time taken for the gas cloud to move from point A or C to point B is

$$(t_1 + \Delta t + L/c) - (t_1 + (v\Delta t \cos\theta + L)/c) = \Delta t - v\Delta t \cos\theta/c \ . \tag{5.3}$$

The distance between B and C is $v\Delta t \sin\theta$. Hence, the apparent speed for the gas cloud to move from point C to B is

$$\frac{v\Delta t \sin\theta}{\Delta t - v\Delta t \cos\theta/c} = \frac{v \sin\theta}{1 - v\cos\theta/c} \ . \tag{5.4}$$

If $v = 0.8c$ and $\theta = 30°$, the apparent speed is $1.3c$.

Index