

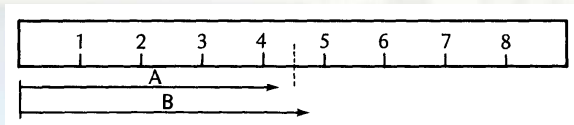
Fewer people use it

Errors of observation

- **Error**: the difference between the observed value of any physical quantity and the 'actual' value.
- **Human error**: due to observational reasons.
- **Systematic error**: due to the instruments and it determines the accuracy of the reading.
- **Random error**: due to environmental reasons or nature of the measurement
- Note difference between **accuracy** and **precision**.

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Estimation of Errors



- The maximum possible error is 0.5 cm
- Therefore $A = 4.0 \pm 0.5$ and $B = 5.0 \pm 0.5$
- Absolute error is 0.5 cm, but relative (percentage) error changes.
- Relative error of A is $0.5/4.0 \times 100\% = 12.5\%$
- If C is 50, then relative error of C is $0.5/50 \times 100\% = 1\%$.

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Propagation of Errors

If $x = a + b$, then $\Delta x = \Delta a + \Delta b$

If $x = a - b$, then $\Delta x = \Delta a + \Delta b$

If $x = a \times b$, then $\frac{\Delta x}{x} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$

If $x = \frac{a}{b}$, then $\frac{\Delta x}{x} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$

If $x = ka$, where k is a constant, then $\Delta x = k\Delta a$

In general, if $u = u(x, y, z, \dots)$,

$$\text{then } \delta u = \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y + \frac{\partial u}{\partial z} \delta z + \dots$$

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Examples

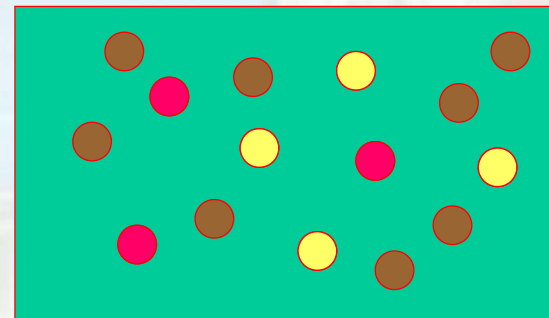
Given $a = 10 \pm 2$, $b = 7 \pm 3$.

- $d = a + b = 17 \pm (2+3)$.
- $e = a - b = 3 \pm (2+3)$.
- $f = a \times b = 70 \pm \Delta f$
where $\Delta f = (2/10 + 3/7) \times 70 = 44$.
- $g = a/b = 1.43 \pm \Delta g$
where $\Delta g = (2/10 + 3/7) \times 1.43 = 0.90$.

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Statistical Nature of Random Error

- No “accurate” value
- Mean value with uncertainty



e.g. radioactive decay

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Terminologies

- Expectation value
- Frequency distribution function
- Mean value

– For discrete x values, the mean

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$$

– For continuous distribution,

$$\bar{x} = \frac{\int_{x_1}^{x_2} x f(x) dx}{\int_{x_1}^{x_2} f(x) dx}$$

– The mean value is only a **sample mean**, it may be quite different from the **true mean** value.

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Terminologies

- Deviation = $x_i - \bar{x}$
- Sample variance s^2

$$s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1} \quad \text{or} \quad s^2 = \frac{\int_{x_1}^{x_2} (x - \bar{x})^2 f(x) dx}{\int_{x_1}^{x_2} f(x) dx}$$

- Standard deviation = (variance)^{1/2} = s
- Normalization – a distribution function is said to be normalized if the total area under the distribution curve is equal to 1. i.e.

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

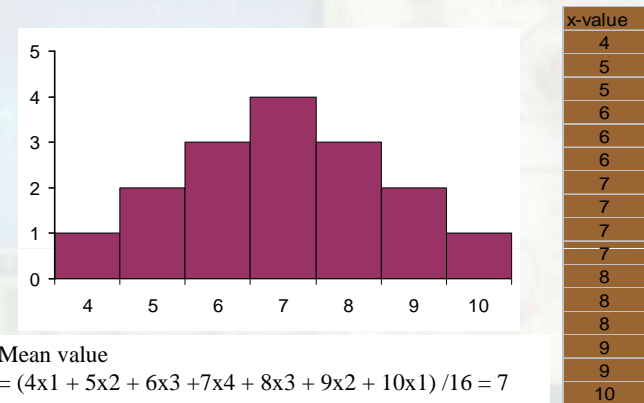
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Statistical Models

- Binomial distribution
- Poisson distribution
- Gaussian or Normal distribution
 - standard deviation $\sigma = (\text{mean})^{1/2}$
 - distribution is symmetric about mean

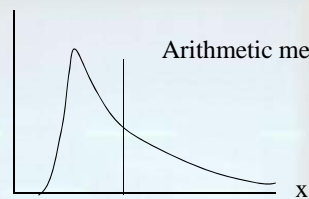
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Example - mean



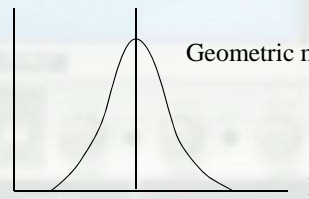
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Arithmetic & Geometric Mean



Arithmetic mean

e.g. the size distribution of atmospheric aerosols is Gaussian when the size is plotted in a log scale.



Geometric mean

log x

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Mean and Root-mean-squared Value

- R.M.S. equals the square root of the mean of the squared value.
- E.g. for five measurement results: 9.81, 9.80, 10.0, 9.60, 9.71

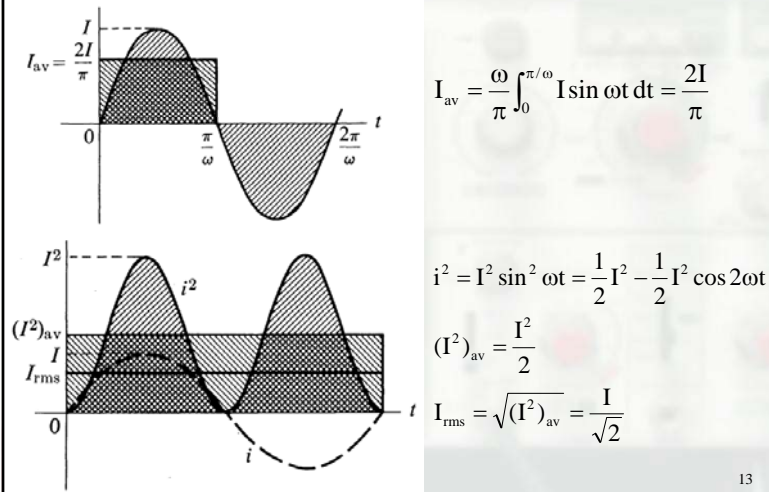
– Mean = $(9.81 + 9.80 + 10.0 + 9.60 + 9.71) / 5 = 9.784$

– R.M.S. =

$$\sqrt{\frac{9.81^2 + 9.80^2 + 10.0^2 + 9.60^2 + 9.71^2}{5}} = 9.785$$

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R.M.S. of AC



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Example - standard deviation

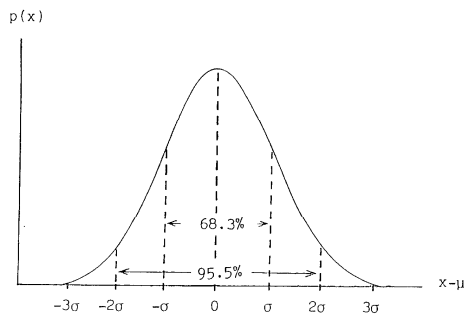
What is the standard deviation of the previous distribution?

- $$\sigma = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}} = 1.6$$
- $$\sigma = \sqrt{\bar{x}} = \sqrt{7} = 2.6$$

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Confidence Levels

The confidence of observing a value within the range $\mu-r$ to $\mu+r$ is equal to the probability of its occurrence given by $\int_{\mu-r}^{\mu+r} P(x) dx$.



Range r	0.675σ	σ	2σ	3σ
Confidence	50 %	68.3 %	95.4 %	99.7 %

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Example - relative error

$X = 100$

$\sigma = 10$

$\sigma/x = 0.1 = 10 \%$

$X = 100 \pm 10$

at 68.3 % C.L.

$X = 10000$

$\sigma = 100$

$\sigma/x = 0.01 = 1 \%$

$X = 10000 \pm 100$

at 68.3 % C.L.

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Useful statistical equations

$$\sigma_x = \sqrt{\bar{x}} \qquad \frac{\sigma_x}{\bar{x}} = \frac{1}{\sqrt{\bar{x}}}$$

$$\sigma_R = \frac{\sigma_x}{t} = \sqrt{\frac{R}{t}} \qquad \frac{\sigma_R}{R} = \frac{1}{\sqrt{Rt}}$$

$$\sigma_{\bar{x}} = \sqrt{\frac{\bar{x}}{n}} \qquad \frac{\sigma_{\bar{x}}}{\bar{x}} = \frac{1}{\sqrt{n\bar{x}}}$$

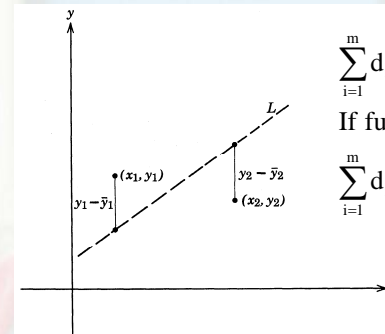
If $u = u(x, y, z, \dots)$ and if the errors are individually small and symmetric about zero, then

$$\sigma_u^2 = \left(\frac{\partial u}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial u}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial u}{\partial z}\right)^2 \sigma_z^2 + \Lambda$$

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Least Squares Approximations

- A curve fitting technique
- Minimize the sum of the squares of the deviations



$$\sum_{i=1}^m d_i^2 = \sum_{i=1}^m (y_i - \bar{y}_i)^2$$

If function is linear, then $\bar{y} = a_1 x + a_0$

$$\sum_{i=1}^m d_i^2 = \sum_{i=1}^m (y_i - a_1 x - a_0)^2$$

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Least Square Fit

To minimize S with respect to a & b

$$\frac{\partial S}{\partial a_0} = \sum_{i=1}^m 2(y_i - a_1 x_i - a_0)(-1) = 0$$

$$\frac{\partial S}{\partial a_1} = \sum_{i=1}^m 2(y_i - a_1 x_i - a_0)(-x_i) = 0$$

Rearranging terms we get

$$m a_0 + (\sum x_i) a_1 = \sum y_i \quad \text{and} \quad (\sum x_i) a_0 + (\sum x_i^2) a_1 = \sum x_i y_i$$

Solving the two linear equations, we have

$$a_0 = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{m \sum x_i^2 - (\sum x_i)^2}$$

$$a_1 = \frac{m \sum x_i y_i - \sum x_i \sum y_i}{m \sum x_i^2 - (\sum x_i)^2}$$

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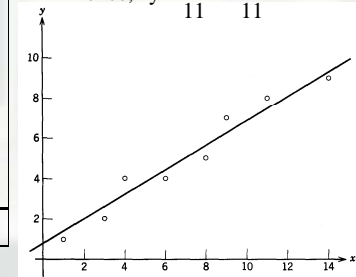
Example

x	y	x ²	xy	
1	1	1	1	
3	2	9	6	
4	3	16	16	
6	3	36	24	
8	4	64	40	
9	7	81	63	
11	8	121	88	
14	9	196	126	
sums:	56	40	524	364

$$a_0 = \frac{40 \times 524 - 56 \times 364}{8 \times 524 - (56)^2} = \frac{6}{11}$$

$$a_1 = \frac{8 \times 364 - 56 \times 40}{8 \times 524 - (56)^2} = \frac{7}{11}$$

$$\text{Hence, } y = \frac{7}{11}x + \frac{6}{11}$$



Other curves

Polynomial: $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Hyperbola: $y = \frac{1}{a_0 + a_1 x}$

Let $z = \frac{1}{y}$, then $z = a_0 + a_1 x$

Exponential: $y = a(b^x) \Rightarrow \log y = \log a + x \log b$

Geometric: $y = ax^b \Rightarrow \log y = \log a + b \log x$

Trigonometric: $y = a_0 + a_1 \cos \omega x$

or more generally $y = a_0 + \sum_{k=1}^n a_k \cos(k\omega x) + \sum_{k=1}^n b_k \sin(k\omega x)$