Lectures on superconductivity

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Lecture 3
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Bose-Einstein condensation: the problem of supercurrent metastability

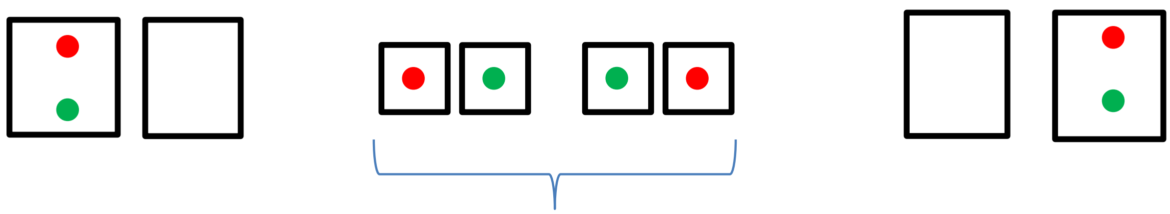
WHY BEC? ← Bose-Einstein condensation

1. Qualitative argument:

   ultra-toy example: how many ways of distributing 2 objects between 2 boxes?

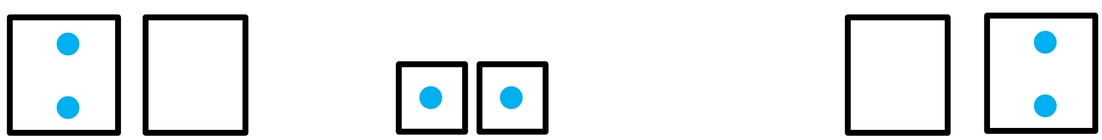
A. Objects distinguishable (classical): (● , ●)

same box | different boxes | same box



1 state | 2 states | 1 state

B. Objects indistinguishable (bosons): (● , ●)



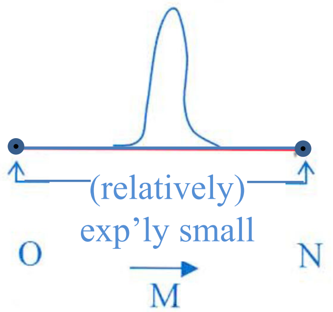
1 state | 1 state | 1 state

So: indistinguishability doesn't favor same state; it disfavors different states! (but effect on relative count is the same)

Generalization: Distribute N objects between 2 boxes: what is probability P(M) of finding M in one box? (add possible energy bias, e.g. e^{-E/k_BT})

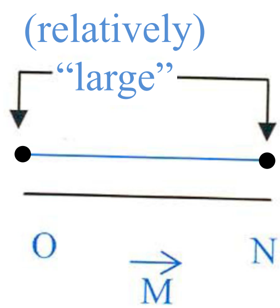
A. Objects distinguishable (≡ coin toss):

$$P(M) = \frac{N!}{M!\,(N-M)!}$$



B. Objects indistinguishable (bosons):

P(M) = 1/N



Note that statistics alone does not guarantee BEC: some kind of energy bias (e.g. due to temperature) is necessary.

II. Quantitative argt. (Einstein, 1925):

$$n_i(T) = [\exp(\varepsilon_i - \mu(T)/k_BT - 1)]^{-1}$$

chemical potential, ≤ 0

$$\mu(T) \text{ fixed by: } \sum_i n_i\,(T{:}\,\mu(T))) = N$$ ⟵ total no. of particles

$T \to \infty \Rightarrow \mu \to -\infty$: $T\downarrow \Rightarrow \mu\uparrow$. But what if

$$\sum_i [\exp(\varepsilon_i/k_BT) - 1]^{-1} < N?$$

Einstein: Macroscopic no. of particles occupy lowest (ε=0) state!

Condition for this to happen: roughly, $T \lesssim h^2/2mk_Ba^2 \equiv T_0$

distance between particles ↗ (a)

Hence for any given total number and volume, there exists a temperature T_0 such that $N_{max}(T_0) = N$. Below this temperature in thermal equilibrium, lowest-energy single-particle state (usually $\boldsymbol{k} = 0$) is macroscopically occupied:

$n_0 \sim N$ ← definition of BEC

Now consider a gas of charged (but noninteracting) bosons on a ring in flux Φ: single-particle energies are given by

$$\left.\begin{aligned} E_\ell &= \text{const.}\left(\ell - \Phi/\Phi_0^{sp}\right)^2, \\ j_\ell &= \text{const.}\left(\ell - \Phi/\Phi_0^{sp}\right) \end{aligned}\right\} \quad \ell = 0, \pm 1, \pm 2 \ldots$$

So when BEC takes place in thermal equilibrium, it does so in the state which minimizes $E(\ell)$, *i.e.* that with ℓ the closest integer to Φ/Φ_0^{sp}; in particular, for $\Phi < \frac{1}{2}\Phi_0^{sp}$, condensation is into $\ell = 0$ state, and contributes an amount $\propto N$ to the circulating current.

Thus, prima facie, if one could invoke BEC one could explain Meissner effect (and vanishing of Peltier coefficient, since a single state carries no entropy). But…

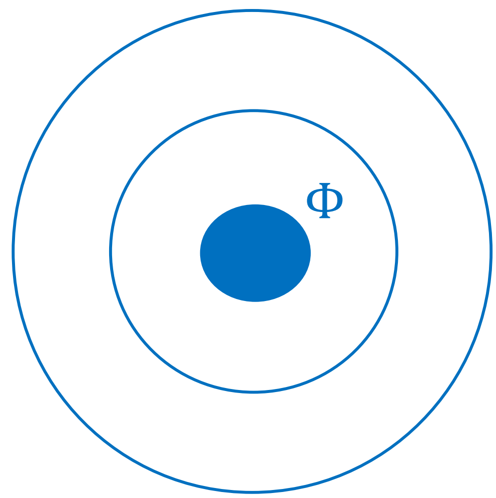
Two problems with BEC as an explanation of superconductivity:

1. Does not (by itself) explain metastability of supercurrents.
2. Electrons are not bosons but fermions!

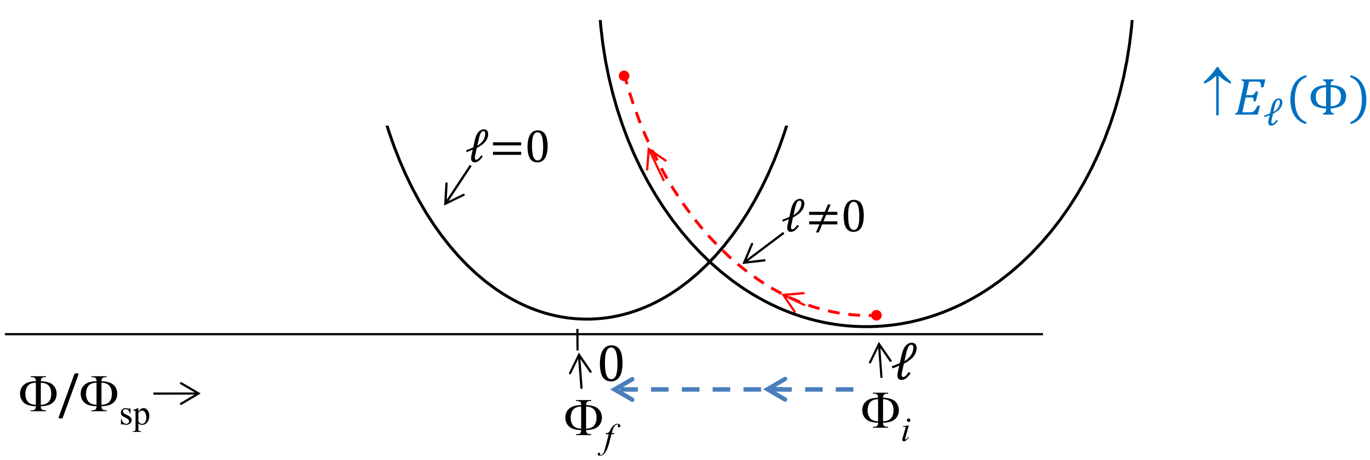
1. The problem of supercurrent metastability

A. Formulation of problem:

Imagine we start with the system in equilibrium at $T = 0$ with some trapped flux $\Phi > \frac{1}{2}\Phi_o^{sp}$. According to the analysis of lecture 2, the single-particle energy levels and currents are given by



$$\left.\begin{aligned} \epsilon(\ell:\Phi) &= \frac{\hbar^2}{2mR^2}\left(\ell - \Phi/\Phi_o^{sp}\right)^2 \\ j(\ell;\Phi) &= \frac{e\hbar}{mR}\left(\ell - \Phi/\Phi_o^{sp}\right) \end{aligned}\right\} \quad (\ell = 0, \pm 1, \pm 2 \ldots\ )$$



so BEC takes place in the state which has ℓ closest to Φ/Φ_o^{sp}; by construction this $\ell \neq 0$. Now we adiabatically turn off the flux: in this process we assume ℓ does not change, thus the final value of ℓ is still $\neq 0$. But since Φ is now zero, we now have

$$\epsilon'(\ell) = \frac{\hbar^2}{2mR^2}\ell^2 \qquad \left(\text{and } j'(\ell) = \frac{e\hbar}{mR}\,\ell\right)$$

and it is clear that the GS has $\ell = 0$, so that our resultant state $(\ell \neq 0)$ cannot be stable.

B. Can the system relax to $\ell = 0$? Prima facie, situation is exactly similar to relaxation of excited state of single electron in atom.* ($\tau \sim 10^{-9}$ secs!) In that case, looking at behavior of wave function in xy-plane, for $\ell = 1$ ($p-$state).

$$\left.\begin{aligned} \psi_{in}(\theta) &\equiv \psi_p(\theta) \sim \exp i\theta \\ \psi_f(\theta) &\equiv \psi_s(\theta) \sim \text{const.} \end{aligned}\right\}$$ "topologically" distinct

Let's form a function which interpolates between these forms:

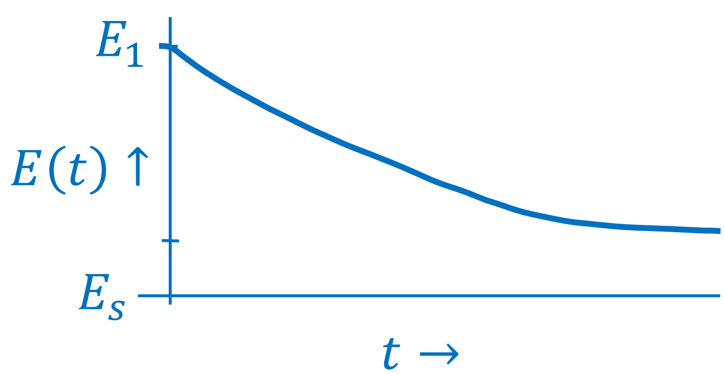
$$\psi\,(\theta\colon t) = a(t)\psi_p(\theta) + b(t)\psi_s(\theta)$$

$$\begin{cases} |a(t)|^2 + |b(t)|^2 = 1 \\ a(-\infty) = 1, b(-\infty) = 0 \\ a(+\infty) = 0, b(+\infty) = 1 \end{cases}$$

Because of linearity of Schrödinger equation

$$E(t) = |a(t)|^2 E_p + |b(t)|^2 E_s$$

– downhill all the way! Note always ∃ a value of θ and t for which we get a node. For the free Bose gas, can do exactly the same for the single-particle state $\psi(\theta\colon t)$ in which BEC is realized



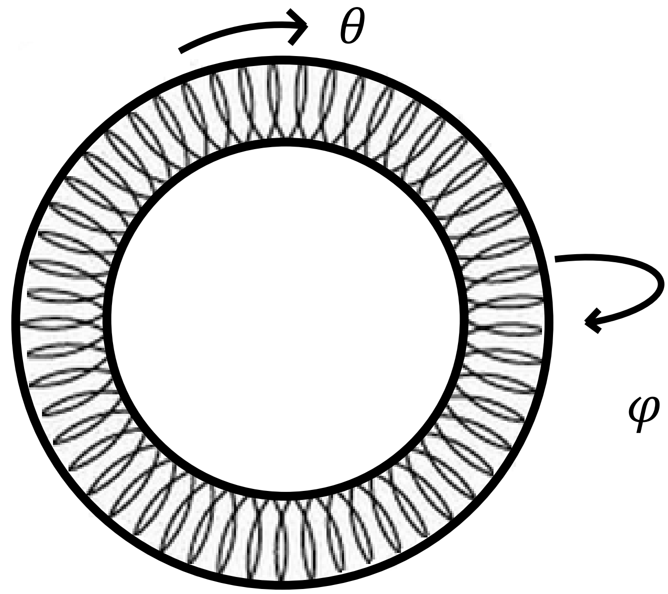
$$\Rightarrow \text{no metastability.}$$

Yet in experiment, $\tau \gtrsim 10^{15}$ secs!!

*with EM field treated as classical.

Stability of supercurrents:

C. Topological argument



Let's consider a more general annular geometry, so that angular momentum is not necessarily conserved. Nevertheless, for any single-particle wave function $\psi(\theta)$ (with $\psi(\theta + 2\pi) = \psi(\theta)$) we can define

(for any point at which $|\psi(\theta)| \neq 0$)

$$\varphi(\theta) \equiv \arg \psi(\theta)$$

and thus the winding number

$$\mathcal{N} \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \varphi(\theta)}{\partial \theta} d\theta \equiv \frac{\varphi(2\pi) - \varphi(0)}{2\pi} = 0, \pm 1, \pm 2 \ldots$$

(because $\varphi\,(2\pi) - \varphi(o)$ mod. 2π)

(Analogy: string wound around hula-hoop)

Crucial point: $\mathcal{N}$ is topologically conserved, *i.e.* the only way to change it is to "cut the string" (*i.e.* let $|\psi(\theta)| \to 0$ for some value of θ). This is exactly what the electron in the atom did… Why can the condensate not do the same?

In Schrödinger mechanics energy is bilinear in ψ, ψ^*. What if we add a quartic term, so that $E\{\psi\} = E_{schr} + \int_o^{2\pi} \kappa\, |\psi(\theta)|^4 d\theta$ with $\kappa > 0$? Then for $p \to s$ transitions repeat interpolation

$$\psi(\theta\!: t) = a(t)\psi_p(\theta) + b(t)\psi_s(\theta), \qquad |a(t)|^2 + |b(t)|^2 = 1.$$

Now we have:

$$E(b) = E_{Schr}(t) + \int_o^{2\pi} d\theta \kappa\, |\psi(\theta\!: t)|^4$$

$$= E_{Schr}(t) + \kappa \int_o^{2\pi} \left|a(t)e^{i\theta} + b(t)\right|^4 d\theta/2\pi$$

The term in κ *is*

$$\kappa \int_0^{2\pi} d\theta/2\pi \left\{|a|^2 + |b|^2 + 2Re\left(ab^* e^{i\theta}\right)\right\}^2$$

$$\equiv \kappa \int \frac{d\theta}{2\pi}\left(1 + 2|a| \cdot |b| cos(\theta - \theta_0)\right)^2 \qquad \theta_0 \equiv -\arg ab^*$$

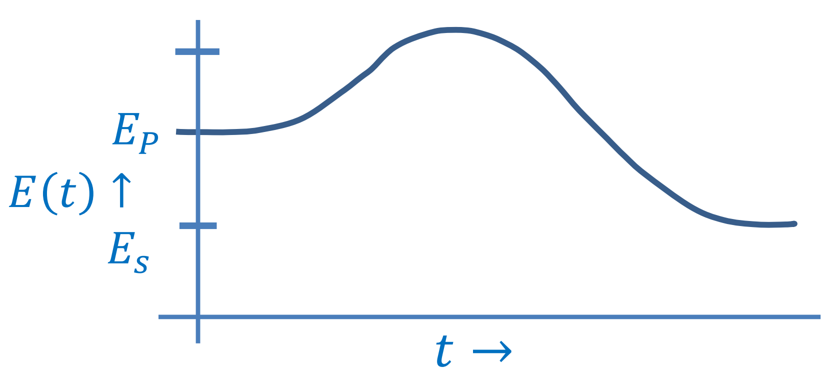
$$= \kappa(1 + 2|a|^2 \cdot |b|^2)$$

Hence

$$\mathrm{E}(t) = \kappa + E_{Schr}(t) + 2\kappa|a(t)|^2 \cdot |b(t)|^2$$

$$\equiv const. - (|b(t)|^2 - |a(t)|^2)\left(E_p - E_s\right) + 2\kappa|a(t)|^2 \cdot |b(t)|^2$$

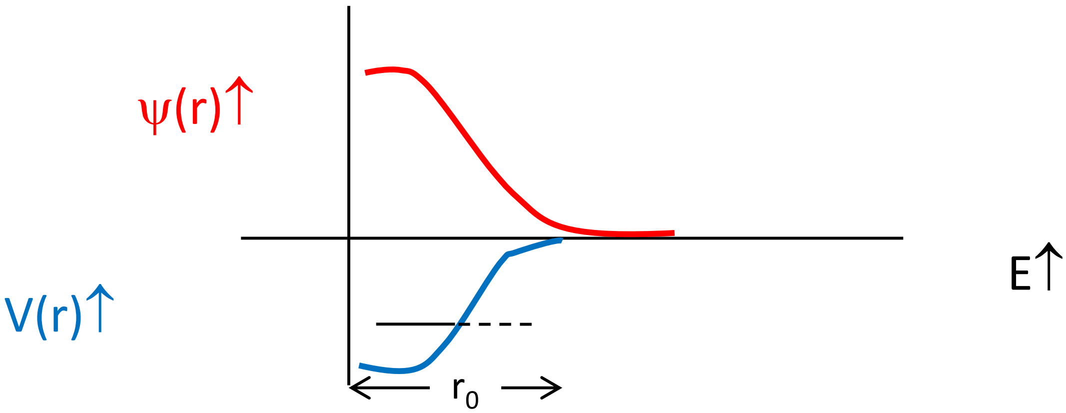
which for $\kappa > \left(E_p - E_s\right)$ is nonmonotonic $\Rightarrow$ metastability!



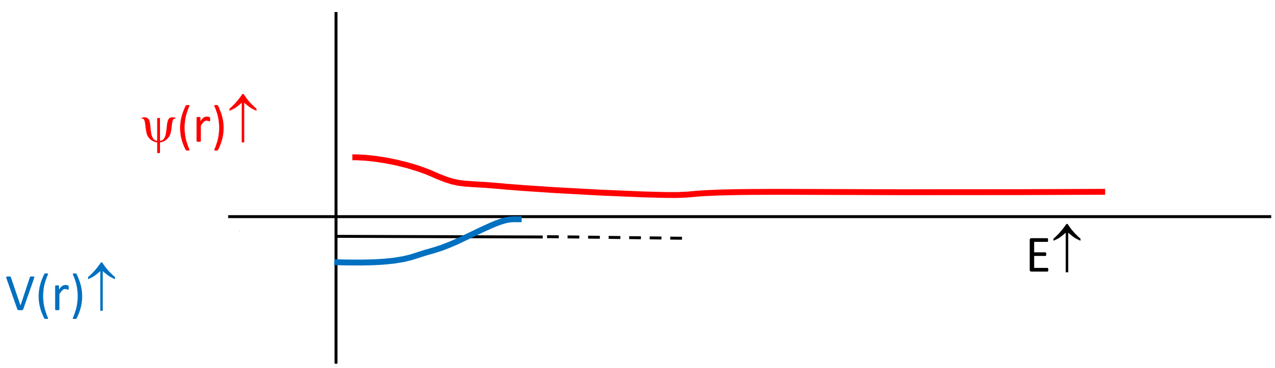
2. The problem of statistics: the "BEC-BCS crossover"

Recap: 2 fermions in 3D free space, interacting via short-range attractive potential $V(r)$ of controllable strength with range $\sim r_o$. If fermions, can form spin singlet$\left(\frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)\right)$, then spatial wave function symmetric → s-state possible.

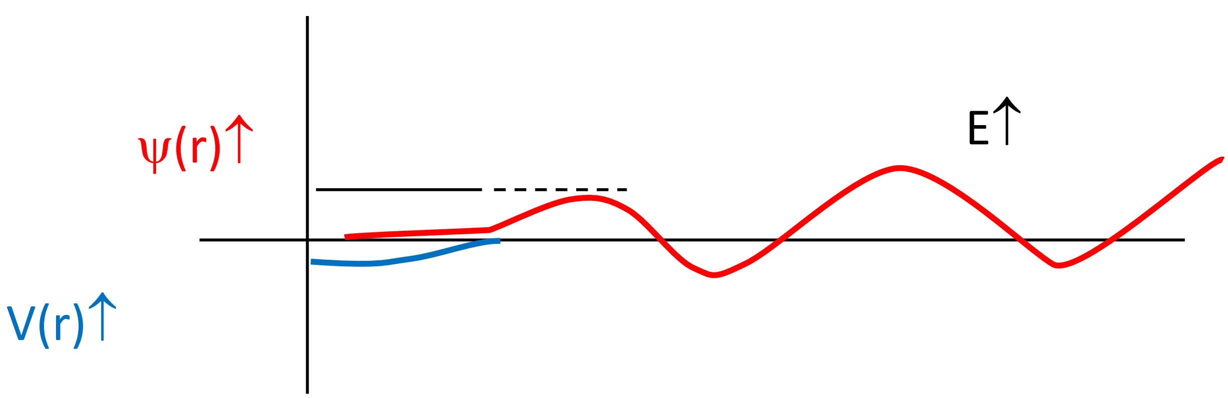
1. For sufficiently strong attraction, particles form bound state ("molecule") with radius $\sim r_o$, binding energy $\gtrsim \hbar^2/mr_o^2$. (2 fermions ⇒ 1 boson)



2. As attraction weakened, radius of bound state becomes $\gg r_o$ and eventually → ∞; binding energy → 0. ("unitary limit")



3. Beyond this point no bound state is formed.



Expectations in system of N such fermions (assume $nr_o^3 \ll 1$ *i.e.* "dilute") *e.g.* ultracold Fermi alkali gas

1. In region where attractive potential is strong, since $r_o \ll$ interparticle spacing, can treat fermion pairs as bosons and (at $T = 0$) expect simple BEC of molecules. – no effect of Fermi statistics.

2. As bound state radius → ∞, "molecules" start to overlap strongly, so expect nontrivial effects of (a) inter-molecular interactions and (b) underlying Fermi statistics. But plausible that (some kind of) BEC possible.

3. The $64K question: What happens in the unitary limit (the point where a single molecule becomes unbound)?

Apparent answer (from theory and experiment in ultracold Fermi gases)

nothing!

i.e. in many-particle system, onset of 2-particle bound state is just not seen.

in fact, now believed that "BEC of pairs" persists right up to "BCS limit" (a_s negative and small), *i.e.* ultraweak attraction)

Why?

Partial clue: statements for 2-particle system are valid only in 3D. In 2D or 1D a bound state is formed for arbitrarily weak attraction (but in 2D case, binding energy exponentially small in interaction strength).
(cf. problem 1.4)

So: can we regard superconductivity as a sort of BEC of pairs of electrons? (Blatt, Schafroth …)

Summary of lecture 3

Bose-Einstein condensation (BEC) can explain Meissner effect and vanishing Peltier coefficient, but

(a) cannot by itself explain metastability of supercurrents.

(b) Electrons are fermions not bosons. (“statistics problem”)

Metastability of supercurrents can be explained if there is a term in the energy proportional to $|\psi(r)|^4$ with a positive coefficient. “Statistics problem” might be explained if tightly-bound difermionic molecules evolve smoothly into much more weakly bound collective state.