LECTURES ON SUPERCONDUCTIVITY

Anthony J. Leggett
Department of Physics
University of Illinois at Urbana-Champaign, USA

Hong Kong University
Spring 2024

LECTURE 4
4/17/2024
GINZBURG-LANDAU THEORY

Ginzburg – Landau (GL) theory of superconductivity

Formulated 1950 (pre – BCS): even in 2023 mostly adequate for "engineering" applications of superconductivity (SC).

historically, takes its origins in general Landau-Lifshitz theory (1936) of 2nd order phase transitions, hence quantitative validity confined to T close to T_c. Here, I will start by arguing "with hindsight" for its qualitative validity at $T = 0$, and only later generalize to $T \neq 0$ and in particular $T \to T_c$.

Recall: could explain 3 major characteristics of SC state (persistent currents, Meissner effect, vanishing Peltier coefficient) by scenario in which fermionic pairs form effective bosons, and these undergo BEC. Suppose that N fermions form $N/2$ bosons, which are then condensed into the same 2 – particle state, Neglect for now relative wave function and df.

COM wave function of condensed pairs $\equiv \chi(\boldsymbol{R})$

COM coordinate

What is energy E_o (at $T = 0$) of pairs as function(al) of $\chi(\boldsymbol{R})$?

At first sight, for a single pair,

$$E_o\{\chi(\boldsymbol{R})\} = -\underset{\uparrow}{E_b}\int |\chi(\boldsymbol{R})|^2 d\boldsymbol{R} + \frac{\hbar^2}{2M}\int \left|\left(\boldsymbol{\nabla} - \frac{2ie}{\hbar}\boldsymbol{A}(\boldsymbol{R})\right)\chi(\boldsymbol{R})\right|^2$$

binding energy of pair

2 electrons involved!

$2m$ $\equiv \partial/\partial\boldsymbol{R}$

so , convenient to define

"order parameter" (OP)

$$\Psi(\boldsymbol{r}) \equiv \sqrt{\frac{N}{2}}\chi(\boldsymbol{R})$$

and to normalize so that $\int |\Psi(\boldsymbol{R})|^2 d\boldsymbol{R} = N/2$.

Also set $\alpha_{T=0} \equiv E_b$, $M \equiv 2m$

then energy of N/2 pairs is

$$E_o\{\Psi(\boldsymbol{R})\} = \text{const.} + \int d\boldsymbol{R}\left\{-\alpha_{T=0}|\Psi(\boldsymbol{R})|^2 + \frac{\hbar^2}{2m}\left|\left(\boldsymbol{\nabla} - \frac{2ie}{\hbar}\boldsymbol{A}(\boldsymbol{R})\right)\Psi(\boldsymbol{R})\right|^2\right\}$$

However, by itself this will not generate stability of supercurrents (lecture 3). To do so, must add (*e.g.*) term in $|\Psi|^4$… Adding also EM field term:

$$\mathrm{E}_0\{\Psi(\boldsymbol{R})\} = \int d\boldsymbol{R}\left\{\begin{array}{l} -\alpha_{\mathrm{T}=0}|\Psi(\boldsymbol{R})|^2 + \dfrac{1}{2}\beta_{\mathrm{T}=0}|\Psi(\boldsymbol{R})|^4 + \dfrac{\hbar^2}{4m}\left|\left(\boldsymbol{\nabla} - \dfrac{2ie}{\hbar}\boldsymbol{A}(\boldsymbol{R})\right)\Psi(\boldsymbol{R})\right|^2 \\ \qquad\qquad\qquad\qquad + \dfrac{1}{2}\mu_0^{-1}(\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{R}))^2 \end{array}\right\}$$

"Standard" form of GL (free) energy (at T=0)

Notes:

1. Normalization of $\Psi(R)$ is conventional and arbitrary. (see Appendix) leaves E_0 unchanged.

2. If $A(\boldsymbol{R}) = 0$ and $\Psi(\boldsymbol{R})$ = constant $\equiv \Psi^{eq}_{\mathrm{T}=0,}$ then value is

$$\Psi^{eq}_{\mathrm{T}=0} = (\alpha_{\mathrm{T}=0}/\beta_{\mathrm{T}=0})^{1/2}$$

and energy is $\mathrm{E}^{eq}_{\mathrm{T}=0} = -\alpha^2_{\mathrm{T}=0}/2\beta_{\mathrm{T}=0}$

3. Minimization with respect to $\boldsymbol{A}(\boldsymbol{R})(\delta E_o/\delta \boldsymbol{A}(\boldsymbol{R}) = 0)$ yields Maxwell's equation $\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{j}(\boldsymbol{R})$

provided that we identify

$$\boldsymbol{j}(\boldsymbol{R}) = \frac{e}{m}(\Psi^*(\boldsymbol{R})(-i\hbar\boldsymbol{\nabla} - 2e\mathbf{A})\Psi(\boldsymbol{R}) + c.c.)$$

4. Minimization with respect to $\Psi(\boldsymbol{R})(\delta E_0/\delta\Psi(\boldsymbol{R}) = 0)$ yields

$$-\alpha_{\mathrm{T}=0}\Psi(\boldsymbol{R}) + \beta_{\mathrm{T}=0}|\Psi(\boldsymbol{R})|^2\Psi(\boldsymbol{R}) - \frac{\hbar^2}{2m}\left(\boldsymbol{\nabla} - 2\frac{ie}{\hbar}\boldsymbol{A}(\boldsymbol{R})\right)^2\Psi(\boldsymbol{R}) = 0$$

Note that for $\boldsymbol{A}(\mathrm{R}) = 0$ this defines a characteristic length

$$\xi_{\mathrm{T}=0} \equiv \left(\frac{\hbar^2}{2m\alpha_{\mathrm{T}=0}}\right)^{1/2} \left(\equiv \left(\frac{\hbar^2}{2mE_b}\right)^{1/2} \sim pair\ radius\right)$$

A second characteristic length follows if we put $\Psi(R)$ = constant $\equiv \Psi_0$ and compare terms in A^2 and $(\nabla \times \mathrm{A})^2$: they are equal when $|\boldsymbol{\nabla} \times \boldsymbol{A}| = \lambda^{-1}_{T=0}$ with

$$\lambda_{\mathrm{T}=0} \equiv (2m/e^2\mu_o|\Psi_0|^2)^{1/2}$$

or since with our normalization $\Psi_0 = \sqrt{\mathrm{N}/2V}$,

$$\lambda_{\mathrm{T}=0} \equiv (m/\mu_0 e^2 n)^{1/2} \quad (\equiv \text{London penetration depth at } \mathrm{T} = 0)$$

electron density

(note that factors of 2 multiplying *e* and *m* cancel!)

Generalization to $\mathrm{T} \neq 0$:

Free energy

All we need do is to let $\mathrm{E}_0 \rightarrow \mathrm{F}(T)$ and let the coefficients α_0, β_0 be T – dependent:

$$\mathrm{F}(\Psi(\boldsymbol{R})\!:\mathrm{T}) = \mathrm{F}_0(\mathrm{T}) + \int d\boldsymbol{R}\mathcal{F}\{\Psi(\boldsymbol{R})T\}$$

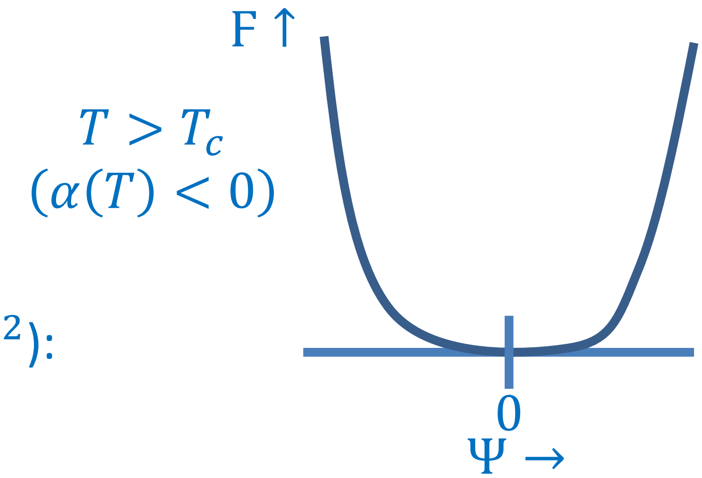
free energy density

$$\underbrace{\begin{aligned}&\mathcal{F}\{\Psi(\boldsymbol{R})T\}\\ &\equiv -\alpha(\mathrm{T})|\Psi(\boldsymbol{R})|^2 + \frac{1}{2}\beta(\mathrm{T})|\Psi(\boldsymbol{R})|^4 + \frac{\hbar^2}{2m}\left|\left(\boldsymbol{\nabla} - \frac{2ie}{\hbar}\boldsymbol{A}(\boldsymbol{R})\right)\Psi(\boldsymbol{R})\right|^2\\ &+ \mu_0^{-1}\left(\boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{R})\right)^2\end{aligned}}_{\text{standard form of GL free energy density}}$$

Note that there is now a contribution to $\alpha(\mathrm{T})$ (and possibly also $\beta(\mathrm{T})$) from the entropy term $-TS\{\Psi(\boldsymbol{R})\}$. (condensate itself carries no entropy, but electrons “liberated” from it do!)

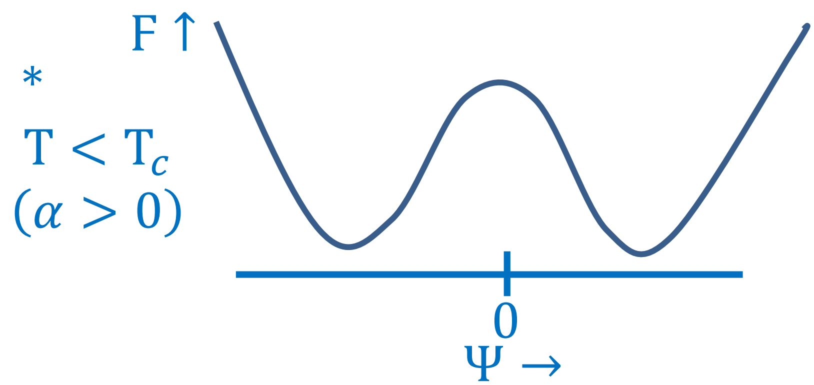
The limit $T \to T_c$

Since $\Psi(\boldsymbol{R})$ describes S state, should $\to 0$ at some T_c, then $\alpha(T)$ should change sign there. Most natural choice in limit $T \to T_c$ (corresponding to $E_b \sim$ constant, $S\{\Psi(R)\} \sim$ constant – constant $|\Psi|^2$):



$$\alpha(T) = \alpha_0(T_c - T)$$
$$\beta(T) = \beta_0 = \text{ind. of } T$$

For $T \neq 0$, all the $T = 0$ results go through with $\alpha_{T=0} \to \alpha(T)$, $\beta_{T=0} \to \beta(T)$. In particular in limit $T \to T_c$ from below:



"Mexican-hat" potential

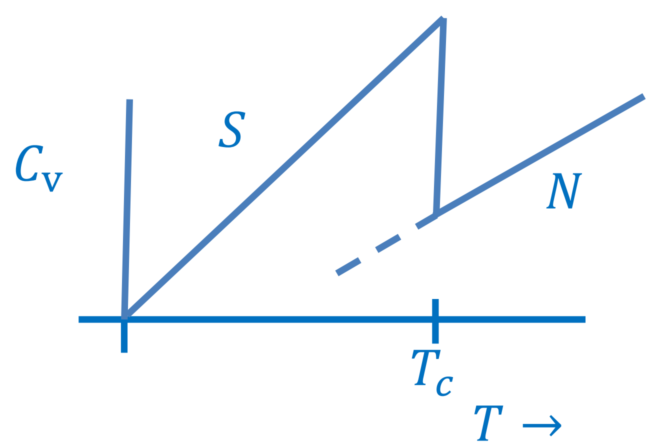
$$\Psi_{eq}(T) = \big(\alpha(T)/\beta(T)\big)^{1/2} = (\alpha_0/\beta_0)^{1/2}(T_c - T)^{1/2}$$
$$F_{eq}(T) - F_0(T) = -\alpha^2(T)/2\beta(T) = -(\alpha_0^2/2\beta_0)(T_c - T)^2$$

From $F_{eq}(T) - F_0(T) \propto (T_c - T)^2$, entropy S has no discontinuity at T_c,

$$\uparrow$$
$$\equiv F\{\Psi = 0\} \equiv F_N$$

but sp. ht. $C_v \equiv T dS/dT$ has discontinuity $T_c\alpha_0^2/\beta_0 \Rightarrow$ second order phase transition at T_c.

What about characteristic lengths $\xi \lambda$?

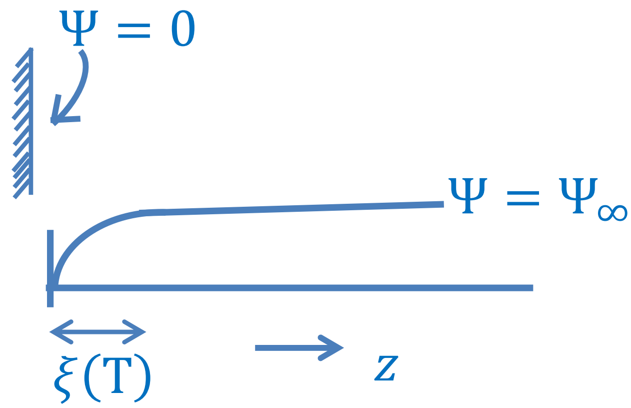


$$\xi(T) \propto [\alpha(T)]^{-1/2} \propto (T_c - T)^{-1/2}$$
$$\lambda(T) \propto \left[\Psi_{eq} \cdot (T)\right]^{-1} \propto (T_c - T)^{-1/2}$$

so ratio $\kappa \equiv \lambda/\xi$ is independent of T in limit $T \to T_c$.

Some simple applications of the GL theory

A. Zero magnetic field ($\boldsymbol{A} = 0$)



1. Recovery of OP near wall (etc.)

Suppose boundary condition at $z = 0$ is that Ψ must $\rightarrow 0$ (*e.g.* wall ferromagnetic).
Then must solve GL eq

$$\alpha(T)\Psi(z) + \beta_0|\Psi(z)|^2\Psi(z) - \frac{\hbar^2}{2m}\frac{\partial^2\Psi(z)}{\partial z^2} = 0$$

subject to boundary conditions

$$\Psi(z = 0) = 0$$

$$\Psi(z \rightarrow \infty) = \Psi_\infty \equiv (\alpha\,(T)/\beta_0)^{1/2}$$

Solution:

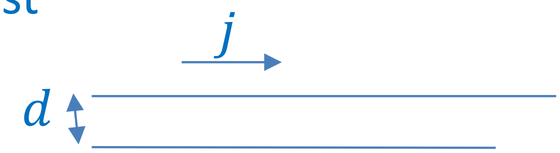
$$\Psi(z) = \Psi_\infty \tanh\,(z/2\xi(T))$$

$$\xi(T) \equiv (\hbar^2/2m\alpha(T))^{-1/2} \propto (T_c - T)^{-1/2}$$

Note: Ψ does not have to vanish at boundary with vacuum or insulator! [except on scale $\sim k_F^{-1}$ where N state wave functions do too] Thus, no objection to SC occurring in grains of dimension $\ll \xi(T)$. However, contact with N metal tends to suppress SC.

2. Current – carrying state in thin wire

If $d \ll \lambda(\mathrm{T})$, can neglect **A** to first approximation



By symmetry,

$$\Psi = |\Psi| \text{ exp } i\varphi, \; |\Psi| = \text{constant}, \; \boldsymbol{j} = \frac{2e\hbar}{m}|\Psi|^2 \boldsymbol{\nabla}\varphi$$

$$F = -\alpha(\mathrm{T})|\Psi|^2 + \tfrac{\beta_0}{2}|\Psi|^4 + \tfrac{\hbar^2}{2m}|\Psi|^2(\nabla\varphi)^2$$

"Superfluid velocity"

$$\left(\mathbf{v}_s = \frac{\hbar}{2m}\boldsymbol{\nabla}\varphi\right)$$

and we need to minimize this with respect to $|\Psi|$ for fixed $\boldsymbol{\nabla}\varphi$. Result:

$$|\Psi| = \left(\frac{\alpha(T) - \frac{\hbar^2}{2m}|\nabla\varphi|^2}{\beta_0}\right)^{1/2} \equiv \Psi_\infty(1 - \xi^2(T)(\nabla\varphi)^2)^{1/2}$$

$\Rightarrow |\Psi| \to 0$ for $\boldsymbol{\nabla}\varphi = \xi^{-1}(T)$ (condensation energy changes sign). However, j is nonmonotonic function of $\boldsymbol{\nabla}\varphi$, with maximum at point when $\boldsymbol{\nabla}\varphi = \frac{1}{\sqrt{3}}\xi^{-1}(\mathrm{T})$. (at which point $|\Psi| = \sqrt{2/3}\,\Psi_\infty$). Thus, critical current j_c given by

$$j_c = \frac{2e\hbar}{m} \cdot \frac{2}{3}\frac{\alpha(T)}{\beta_0}\frac{1}{\sqrt{3}}\xi^{-1}(T) \propto \alpha(T)\xi^{-1}(T) \propto (1 - T/T_c)^{3/2}$$

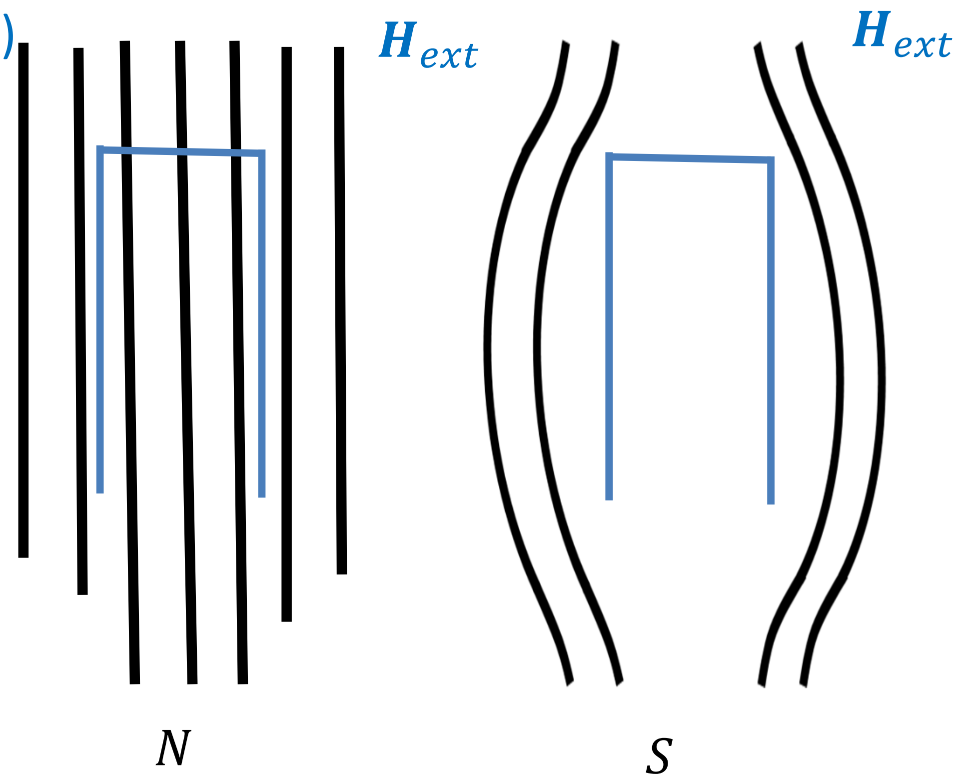
B. Behavior in magnetic field

Because of the Meissner effect, any superconductor, whether type-I or type-II will completely expel a weak magnetic field. If we consider just the competition between the resulting state and the Nstate, we have in a suitable geometry (long cylinder || to field) (per unit volume)

$$\Delta E_{magn} = +\frac{1}{2}\mu_0 H_{ext}^2 V$$

$$\Delta E_{cond} = -\alpha^2(T)/2\beta_0$$

V



and so "thermodynamic" critical field H_c is given by the value of H_{ext} at which $\Delta E_{magn} = \Delta E_{cond}$, i.e. by

$$\mathrm{H}_c(T) = (\alpha^2(T)/\mu_0\beta_0)^{1/2} \propto (1 - T/T_c)$$

For $H_{ext} > H_c(T)$ the sample simply reverts to the N phase.

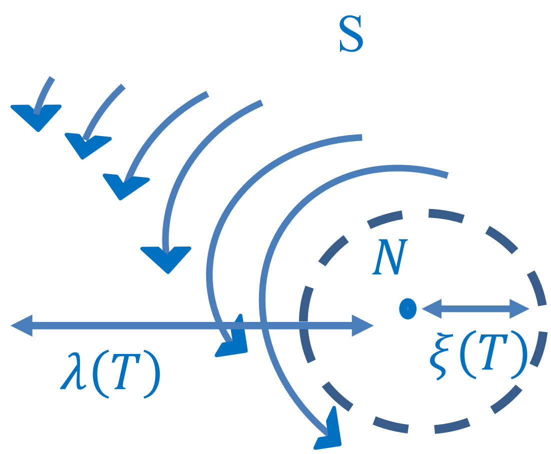
However, in general this works only for "type-I" superconductors in form of long cylinder parallel to field. More generally:

(a) in type-I superconductors, "intermediate" state forms with interleaved macroscopic regions of N and S.

(b) in type-II superconductors, magnetic field "punches through" sample in the form of vortex lines. "mixed state"

Isolated vortex line (Abrikosov)

Consider $\lambda(\mathrm{T}) \gg \xi(T)$
("extreme type-II")



Magnetic field turns region $\sim\xi^2(T)$ normal, and "punches through" there. Field is screened out of bulk by Meissner effect on scale $\lambda(T) \gg \xi(T)$

At distances $\gg \lambda(T)$, no current flows: however since

$$\boldsymbol{j} \propto \boldsymbol{\nabla}\varphi - \frac{2e}{\hbar}\boldsymbol{A}$$

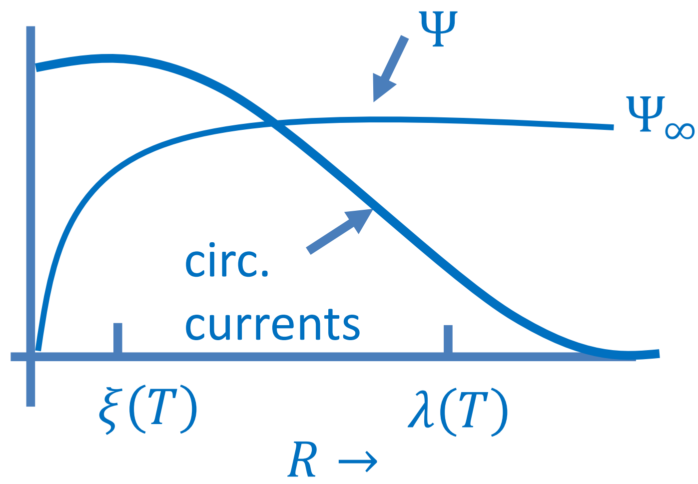
$\boldsymbol{\nabla}\varphi$ and $\boldsymbol{A}$ may be individually non zero, with $\boldsymbol{\nabla}\varphi = \frac{2e}{\hbar}\boldsymbol{A}$.
But φ must be single-valued mod. 2π, hence

$$\oint \boldsymbol{\nabla}\varphi \cdot \boldsymbol{d\ell} = 2n\pi \Longrightarrow \oint \boldsymbol{A} \cdot \boldsymbol{d\ell} \equiv \Phi = n\Phi_0$$

trapped flux

$h/2e$
"superconducting" flux quantum

$n = 0$ is trivial (no vortex!) and $|n| \geqslant 2$ is unstable, so restrict consideration to $n = (\pm)1$.
Since field extends over $\sim \lambda(T)$ into bulk metal, field at core $(\equiv \mathrm{H}_0) \sim \Phi_0/\lambda^2(T)$.



Energetics of single vortex line (per unit length)

(1) "intrinsic" energies:

(a) field energy $\sim \frac{1}{2}\mu_0 \mathrm{H}_0^2 \lambda^2 \sim \varphi_0^2/(\mu_o \lambda^2)$

(b) (minus) condensation energy $\sim -\frac{1}{2}\mu_o \mathrm{H}_c^2 \xi^2$

(c) flow energy: $\mathrm{v}_s \sim \frac{\hbar}{2m}\frac{1}{R}$, for $R \leq \lambda(T)$, so

$$(c) \sim |\Psi_\infty|^2 \left(\frac{\hbar}{2m}\right)^2 \int_\xi^\lambda \frac{RdR}{R^2} \sim |\Psi_\infty|^2 \frac{\hbar^2}{(2m)^2} \ell n(\lambda/\xi)$$

but $\lambda(T) = (\mu_0 |\Psi_\infty|^2 e^2/m)^{-1/2}$, so

$$(c) \sim \left(\Phi_0^2/\mu_0 \lambda^2(\mathrm{T})\right) \ell n(\lambda/\xi)$$

dominant over
(a) and (b) for $\lambda(T) \gg \xi(T)$

Thus "intrinsic" energy per unit length of vortex lines for $\lambda \gg \xi$ is

$$\mathrm{E}_0(T) \sim \left(\Phi_0^2/\left(\mu_o \lambda^2(T)\right) \cdot \ell n(\lambda/\xi)\right).$$

(2) On the other hand, the “extrinsic” energy saving due to admission of the external field is $\sim\mu_0 \mathrm{H}_{ext}\mathrm{H}_0 \sim \mathrm{H}_{ext}(\Phi_0/\lambda^2)$. Hence the condition for it to be energetically advantageous to admit a single vortex line is roughly

$$\mathrm{H}_{ext} \sim \left(\Phi_0/\lambda^2(T)\right)\cdot \ell n\kappa \qquad \left(\kappa \equiv \lambda(T)/\xi(T) \neq f(T)\right)$$

This defines the lower critical field H_{c1}.

What is the maximum field the superconductor can tolerate before switching to the N phase?

Roughly, defined by the point at which the vortex cores (area $\sim\xi^2(T)$) start to overlap. Since for near-complete penetration we must have $n\Phi_0 \sim \mathrm{H}_{ext}$, this gives the condition

number of vortices/unit area

$$\mathrm{H}_{ext} \sim \left(\Phi_0/\xi^2(\mathrm{T})\right)$$

This defines the upper critical field H_{c2}

Note that for $\lambda(T) \lesssim \xi(T)$, we have $\mathrm{H}_{c1} \gtrsim \mathrm{H}_{c2}$ and would expect type-I behavior.

Results of more quantitative treatment:

(a) condition for type-II behavior is $\kappa \equiv \left(\lambda(\mathrm{T})/\xi(\mathrm{T})\right) > 2^{-1/2}$

(b) in extreme type-II limit $\kappa \gg 1$,

$$\mathrm{H}_{c1}(T) = \left(\Phi_0/4\pi\lambda^2(\mathrm{T})\right)\ell n\kappa$$

(c) in same limit,

$$\mathrm{H}_{c2}(T) = \Phi_0/2\pi\xi^2(\mathrm{T}).$$

Note that quite generally we have up to logarithmic factors

$$\mathrm{H}_{c1}(T)\cdot\mathrm{H}_{c2}(T) \sim \mathrm{H}_c^2(T)$$

thermodynamic critical field

hence if $\mathrm{H}_c(T)$ held fixed, H_{c1} varies inversely to H_{C2}

Application: dirty superconductors

Alloying does not change $\mathrm{E}_{cond}(T)$, hence $\mathrm{H}_c^2(T)$, much. However, it drastically increases $\lambda(T)$ (and decreases $\xi(T)$). Hence κ is increased, and many elements which on type-I when pure become type-II when alloyed. As alloying increases, H_{c2} increases while H_{c1} decreases.

Summary of lecture 4

Ginzburg-Landau theory is special case of Landau-Lifshitz theory of 2nd order phase transitions: quantitatively valid only near T_c, but qualitatively over a much wider regime. Introduces order parameter ("macroscopic wave function")$\Psi(R)$ which couples to vector potential $\boldsymbol{A}(R)$with charge $2e$:

$$F\{\Psi(r)\} = \text{const.} - \alpha|\Psi(\boldsymbol{R})|^2 + \frac{1}{2}\beta|\Psi(\boldsymbol{R})|^4 +$$

$$\text{const.}\left|\left(\boldsymbol{\nabla} - \frac{2ie}{\hbar}\boldsymbol{A}(\boldsymbol{R})\right)\Psi(\boldsymbol{R})\right|^2 + \frac{1}{2}\mu_o^{-1}\big(\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{R})\big)^2$$

2 characteristic lengths:

healing length $\xi(T)$

penetration depth $\lambda(T)$

both $\propto (T_c - T)^{-1/2}$ for $T \to T_c$.

Magnetic behavior depends on ratio $\kappa = \lambda(T)/\xi(T)|_{T\to T_c}$

In a long cylindrical sample:

For $\kappa < 2^{-1/2}$, ("type-I") field completely expelled up to a thermodynamic critical field $H_c(T)$, at which point turns fully normal.

For $\kappa > 2^{-1/2}$, ("type-II") field starts to penetrate at H_c in form of vortices, continues to do so up to H_{c2} where turns completely normal.