

Essential Maths for Physics Students Class B

Chapter 1 Vectors and ODEs

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⟨I⟩ Summary

1. Vectors and the Geometry of Space

(a) Three-dimensional Coordinate Systems

(b) Basic Properties of Vectors

If $\mathbf{u} = \langle u_x, u_y, u_z \rangle$, then $\|\mathbf{u}\| = \sqrt{u_x^2 + u_y^2 + u_z^2}$.

(c) Dot Products

$$\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z$$

i. Angle between vectors $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$

ii. Projections $\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \frac{\mathbf{v}}{\|\mathbf{v}\|} = (\mathbf{u} \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}}$

iii. Direction cosines $\cos \alpha = \frac{u_x}{\|\mathbf{u}\|}$, $\cos \beta = \frac{u_y}{\|\mathbf{u}\|}$, $\cos \gamma = \frac{u_z}{\|\mathbf{u}\|}$

(d) Cross Products

$$\mathbf{u} \times \mathbf{v} = (u_y v_z - u_z v_y) \hat{\mathbf{i}} + (u_z v_x - u_x v_z) \hat{\mathbf{j}} + (u_x v_y - u_y v_x) \hat{\mathbf{k}}$$

i. Geometric interpretation $\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta \hat{\mathbf{n}}$

ii. Area and volume

Area of parallelogram formed by \mathbf{u} and $\mathbf{v} = \|\mathbf{u} \times \mathbf{v}\|$

Volume of parallelepiped formed by \mathbf{u} , \mathbf{v} , and $\mathbf{w} = \|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})\|$

(e) Lines and Planes

i. Straight lines in space

$$\begin{aligned} (x - x_1) \hat{\mathbf{i}} + (y - y_1) \hat{\mathbf{j}} + (z - z_1) \hat{\mathbf{k}} \\ = t [(x_2 - x_1) \hat{\mathbf{i}} + (y_2 - y_1) \hat{\mathbf{j}} + (z_2 - z_1) \hat{\mathbf{k}}] \end{aligned}$$

$$x = x_1 + t(x_2 - x_1), \quad y = y_1 + t(y_2 - y_1), \quad z = z_1 + t(z_2 - z_1)$$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

ii. Flat planes in space

$$Ax + By + Cz = Ax_0 + By_0 + Cz_0 = D$$

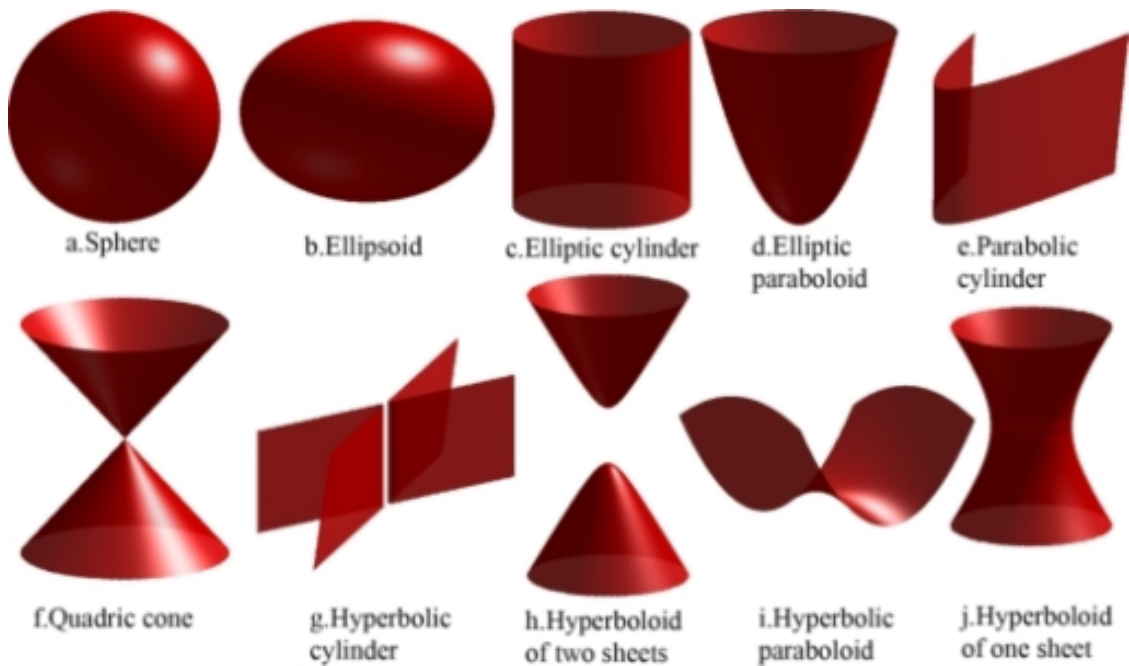
iii. Vector methods for measuring distance

Let P and S be two points. Then

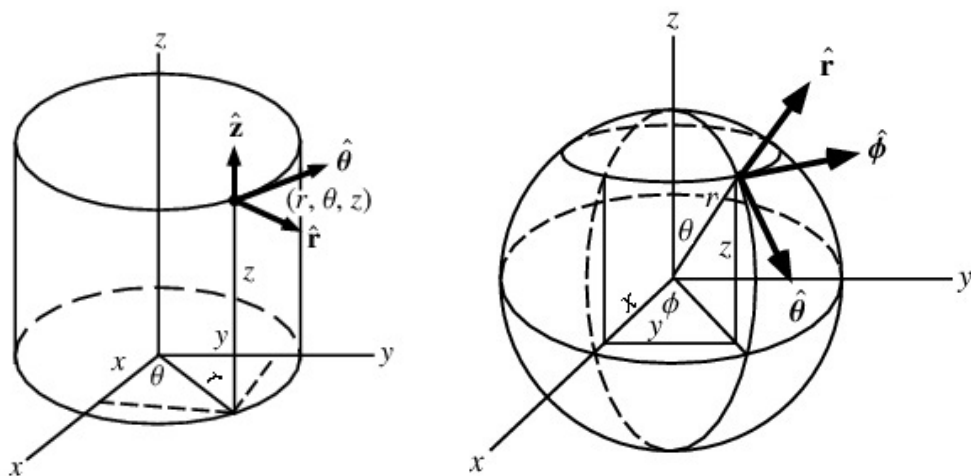
$$\text{distance from } S \text{ to a line through } P \text{ and } // \text{ to } \mathbf{u} = \frac{\|\vec{PS} \times \mathbf{u}\|}{\|\mathbf{u}\|}$$

$$\text{distance from } S \text{ to the plane with } P \text{ and normal } \mathbf{n} = \frac{\|\vec{PS} \cdot \mathbf{n}\|}{\|\mathbf{n}\|}$$

(f) Quadric Surfaces



(g) Cylindrical and Spherical Coordinates



2. Vector Functions and Their Properties

(a) Vector Functions and Space Curve

A space curve can be represented by $\mathbf{r}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$.

(b) Calculus of Vector Functions

i. Vector derivatives $\mathbf{r}'(t) = (df/dt)\hat{\mathbf{i}} + (dg/dt)\hat{\mathbf{j}} + (dh/dt)\hat{\mathbf{k}}$

Note that $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ for any t if $\|\mathbf{r}(t)\| = \text{constant}$.

ii. Vector integrals $\int \mathbf{r}(t) dt = (\int f(t) dt)\hat{\mathbf{i}} + (\int g(t) dt)\hat{\mathbf{j}} + (\int h(t) dt)\hat{\mathbf{k}}$

Similar equation applies for definite integrals.

(c) Unit Tangent and Normal Vectors

Unit tangent vector $\mathbf{T} = \mathbf{v}(t)/\|\mathbf{v}(t)\|$

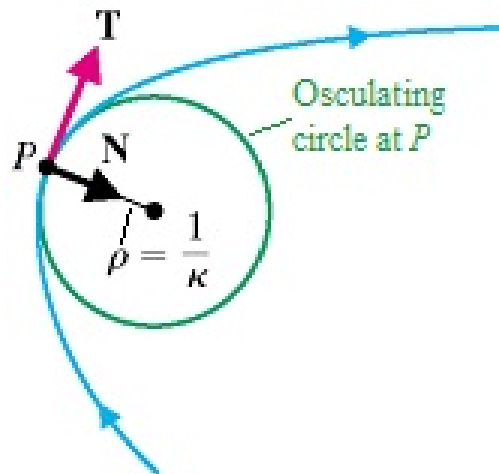
Principal unit normal vector $\mathbf{N} = \mathbf{T}'(t)/\|\mathbf{T}'(t)\|$

(d) Curvature and Acceleration

i. Arc length parameter $s(t) = \int_{t_0}^t \|\mathbf{v}(\tau)\| d\tau$ where $\mathbf{T}(t) = \frac{d\mathbf{r}}{ds}$

ii. Curvature and radius of curvature

Curvature $\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$, Radius of curvature $\rho = \frac{1}{\kappa} = \frac{1}{\|d\mathbf{T}/ds\|}$



iii. Centripetal and normal acceleration

The acceleration of a moving body can be written as

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N} \quad \text{where} \quad a_T = \frac{d\|\mathbf{v}\|}{dt}, \quad a_N = \kappa \|\mathbf{v}\|^2.$$

Alternatively,

$$a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}, \quad a_N = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}, \quad \kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$$

3. Ordinary Differential Equations (ODE)

(a) Basic Terminology and Properties

(b) Solving First Order ODEs

Type	Form of DE	Method of Solution
Linear	$\frac{dy}{dx} + a(x)y = f(x)$	Multiplying the DE by the integrating factor $e^{\int a(x)dx}$
Separable	$\frac{dy}{dx} = M(x)N(y)$	Separation of variables
Homogeneous	$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$	Using the substitution $v = y/x$
Exact	$M(x, y)dx + N(x, y)dy = 0$ where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$	Solve $\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$
Modified exact	$M(x, y)dx + N(x, y)dy = 0$ where $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$	Multiplying with integrating factor* to make the DE exact

*Case 1: If $\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$ is a function of x only, say $g(x)$, then $\mu(x) = \exp(\int g(x) dx)$ is an integrating factor for the DE.

Case 2: If $\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$ is a function of y only, say $h(y)$, then $\mu(y) = \exp(\int h(y) dy)$ is an integrating factor for the DE.

(c) Theory for Linear ODEs

i. Existence and Uniqueness Theorem

Suppose we have a n th order linear ODE

$$\frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x) \frac{dy}{dx} + a_0(x)y = f(x)$$

where $a_1(x), \dots, a_{n-1}(x)$ and $f(x)$ are continuous functions on the interval $[x_1, x_2]$. Then we will have an unique solution over this interval if and only if there exists n initial conditions:

$$y(x_0) = c_0, y'(x_0) = c_1, y''(x_0) = c_2, \dots, y^{(n-1)}(x_0) = c_n$$

where c_0, c_1, \dots are some real numbers and x_0 is a constant whose value is in the interval $[x_1, x_2]$.

ii. Linear dependence of solutions (Test by Wronskian)

$$\text{Wronskian } W(y_1, y_2, \dots, y_n; x) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

If $y_1(x)$ and $y_2(x)$ are the solutions of the ODE

$$y'' + a_1(x)y' + a_0(x)y = 0$$

where $a_0(x)$ and $a_1(x)$ are continuous functions of x on an interval I , then

$$W(y_1, y_2; x) \begin{cases} = 0 \text{ for all } x \in I \Rightarrow y_1, y_2 \text{ are linearly dependent} \\ \neq 0 \text{ for all } x \in I \Rightarrow y_1, y_2 \text{ are linearly independent} \end{cases}$$

iii. General solutions of second order ODEs

If $a_0(x)$ and $a_1(x)$ are continuous functions of x on an interval I , then the second-order homogeneous linear ODE

$$y'' + a_1(x)y' + a_0(x)y = 0$$

has exactly two linearly independent solutions $y_1(x)$ and $y_2(x)$. In addition, the general solution of this ODE can be written as

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

for arbitrary constants c_1 and c_2 .

(d) Solving Second Order Homogeneous Linear ODEs

i. Method of Reduction of Order

If $y_1(x)$ is a known solution, find the other linearly independent solution $y_2(x)$ by substituting $y_2(x) = v(x)y_1(x)$ (which reduces the ODE to a first order separable ODE).

ii. Solutions of the constant coefficient ODE $y'' + ay' + by = 0$

Solve the characteristic equation $\lambda^2 + a\lambda + b = 0$ to find the linearly independent solutions.

(e) Solving Second Order Nonhomogeneous Linear ODE

The general solution to the nonhomogeneous ODE

$$y'' + a_1(x)y' + a_0(x)y = f(x)$$

is of the form

$$y(x) = y_c(x) + y_p(x)$$

where y_p is its particular solution and y_c is its complementary solution, i.e. the general solution of the associated homogeneous equation $y'' + a_1(x)y' + a_0(x)y = 0$.

To find the particular solution y_p , we can use the following methods:

i. Method of Variation of Parameters

Assume $y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$ where y_1 and y_2 are the two linearly independent solutions of the associated homogeneous equation.

ii. Method of Undetermined Coefficients

Guess $y_p(x)$ using the basic rule if $f(x)$ is a product of one or more of these functions: polynomials in x , exponential functions of x , sines and cosines of x . Beware when to use the modification rule.

⟨II⟩ Examples

1. The compass in an airplane indicates that it is headed due east and its air speed indicator reads 215 km/hr. A steady wind of 65 km/hr is blowing due north.

(a) What is the velocity of the plane with respect to the ground?

(b) If the pilot wishes to fly due east, what must be the heading? That is to say, what must the compass read?

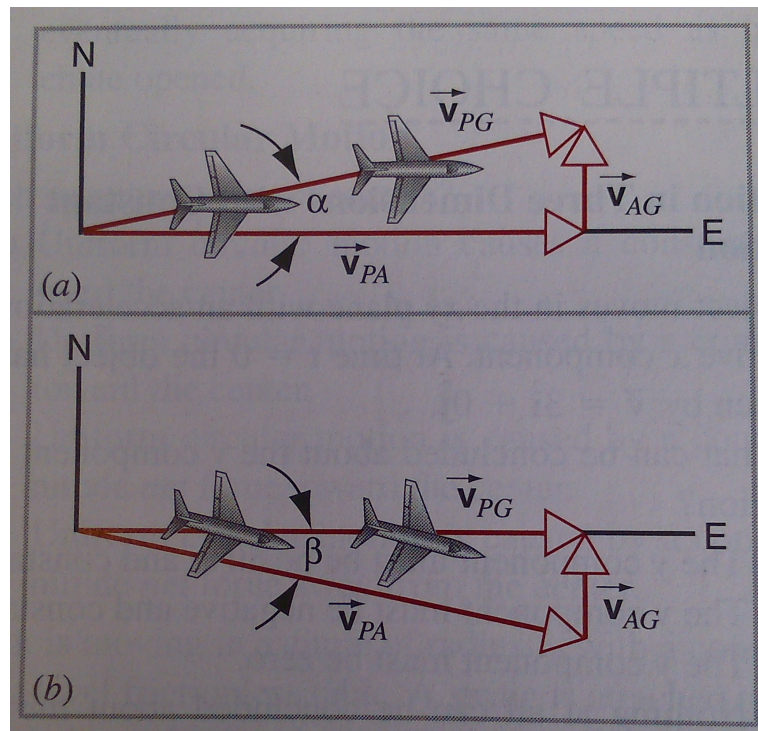
Solution:

(a) The moving “particle” in this problem is the plane P . There are two reference frames, the ground (G) and the air (A). The velocity of the

plane P with respect to the ground can be written as:

$$\mathbf{v}_{PG} = \mathbf{v}_{PA} + \mathbf{v}_{AG}$$

where \mathbf{v}_{PA} is the velocity of the plane with respect to the air and \mathbf{v}_{AG} is the velocity of the air with respect to the ground (i.e. the wind velocity). Figure (a) on the next page shows these vectors which form a right-angled triangle. Note that the orientation of the plane is consistent with a due east reading on its compass.



The reading of the ground velocity (the ground speed) is found from

$$\begin{aligned} \|\mathbf{v}_{PG}\| &= \sqrt{\|\mathbf{v}_{PA}\|^2 + \|\mathbf{v}_{AG}\|^2} \\ &= \sqrt{(215 \text{ km/hr})^2 + (65 \text{ km/hr})^2} = 224.6 \text{ km/hr} . \end{aligned}$$

The angle α in Figure (a) is given by

$$\alpha = \tan^{-1} \left(\frac{\|\mathbf{v}_{AG}\|}{\|\mathbf{v}_{PA}\|} \right) = \tan^{-1} \left(\frac{65 \text{ km/hr}}{215 \text{ km/hr}} \right) = 16.8^\circ .$$

Therefore, the plane is flying at a speed of 224.6 km/hr in a direction 16.8° north of east with respect to the ground.

- (b) In this case, the pilot must head into the wind so that the velocity of the plane with respect to the ground points east. The wind velocity

remains the same and the vector diagram for the relative velocities is as shown in Figure (b) above. Note that the three vectors still form a right-angled triangle, as they did in Figure (a), but in this case the hypotenuse is $\|\mathbf{v}_{PA}\|$ rather than $\|\mathbf{v}_{PG}\|$.

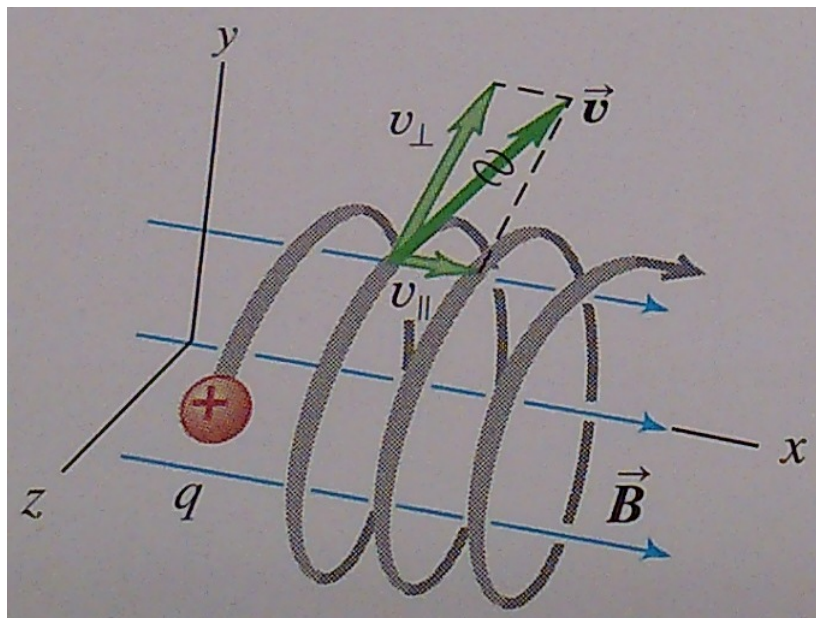
So the pilot's ground speed is now

$$\begin{aligned}\|\mathbf{v}_{PG}\| &= \sqrt{\|\mathbf{v}_{PA}\|^2 - \|\mathbf{v}_{AG}\|^2} \\ &= \sqrt{(215 \text{ km/hr})^2 - (65 \text{ km/hr})^2} = 204.9 \text{ km/hr} .\end{aligned}$$

As the orientation of the plane in Figure (b) indicates, the pilot must head into the wind by an angle β given by

$$\beta = \sin^{-1} \left(\frac{\|\mathbf{v}_{AG}\|}{\|\mathbf{v}_{PA}\|} \right) = \sin^{-1} \left(\frac{65 \text{ km/hr}}{204.9 \text{ km/hr}} \right) = 17.6^\circ .$$

2. A proton (charge $q = 1.60 \times 10^{-19}$ C, mass $m = 1.67 \times 10^{-27}$ kg) is moving along a helical path in a uniform magnetic field \mathbf{B} of magnitude 0.500 T directed along the x -axis as shown in the figure below. Assume only the magnetic force acts on the proton. At time $t = 0$, the proton has velocity components $v_x = 1.50 \times 10^5$ m/s, $v_y = 0$, and $v_z = 2.00 \times 10^5$ m/s.



- (a) Determine the force acting on the proton and its acceleration at $t = 0$.
- (b) Find the radius of the helical path, the angular speed of the proton, and the distance travelled along the helix axis per revolution.

Solution:

- (a) Since $v_y = 0$, the initial velocity vector $\mathbf{v} = v_x \hat{\mathbf{i}} + v_z \hat{\mathbf{k}}$. Recalling that $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \mathbf{0}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$, the magnetic force acting on the proton is equal to

$$\begin{aligned}\mathbf{F} &= q\mathbf{v} \times \mathbf{B} \\ &= q(v_x \hat{\mathbf{i}} + v_z \hat{\mathbf{k}}) \times \|\mathbf{B}\| \hat{\mathbf{i}} \\ &= qv_z \|\mathbf{B}\| \hat{\mathbf{j}} \\ &= (1.60 \times 10^{-19} \text{ C})(2.00 \times 10^5 \text{ m/s})(0.500 \text{ T}) \hat{\mathbf{j}} \\ &= (1.60 \times 10^{-14} \text{ N}) \hat{\mathbf{j}}\end{aligned}$$

This may seem like a weak force. But the resulting acceleration \mathbf{a} is tremendous because the proton mass is so small:

$$\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{(1.60 \times 10^{-14} \text{ N}) \hat{\mathbf{j}}}{(1.67 \times 10^{-27} \text{ kg})} = (9.58 \times 10^{12} \text{ m/s}^2) \hat{\mathbf{j}}$$

- (b) At $t = 0$, the component of velocity perpendicular to \mathbf{B} is v_z . Thus,

$$\begin{aligned}\frac{mv_z^2}{R} &= qv_z \|\mathbf{B}\| \\ \Rightarrow R &= \frac{mv_z}{q\|\mathbf{B}\|} = \frac{(1.67 \times 10^{-27} \text{ kg})(2.00 \times 10^5 \text{ m/s})}{(1.60 \times 10^{-19} \text{ C})(0.500 \text{ T})} = 4.18 \text{ mm}\end{aligned}$$

The angular speed is then given by

$$\omega = \frac{v_z}{R} = \frac{q\|\mathbf{B}\|}{m} = \frac{(1.60 \times 10^{-19} \text{ C})(0.500 \text{ T})}{(1.67 \times 10^{-27} \text{ kg})} = 4.79 \times 10^7 \text{ rad/s}$$

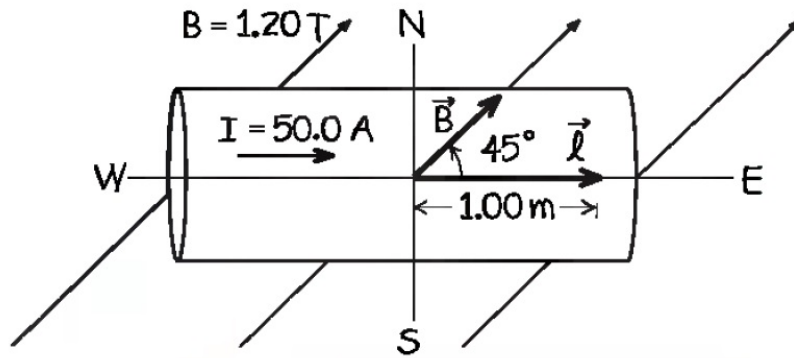
The time required for one revolution is the period

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{(4.79 \times 10^7 \text{ s}^{-1})} = 1.31 \times 10^{-7} \text{ s}.$$

So the distance travelled along the helix axis per revolution is

$$v_x T = (1.50 \times 10^5 \text{ m/s})(1.31 \times 10^{-7} \text{ s}) = 19.7 \text{ mm}$$

3. A straight horizontal copper rod carries a current of 50.0 A from west to east in a region between the poles of a large electromagnet. Inside this region, there is a horizontal magnetic field of magnitude 1.20 T pointing toward the north-east, i.e. 45° north of east (see below figure).



- (a) Find the magnitude and direction of the force on a 1.00-m cross section of the rod. What keeps the rod horizontal?
- (b) How should it be oriented to maximize the magnitude of the force? What is the magnitude of the force in such case?

Solution:

- (a) The angle ϕ between the directions of current and field is 45° . So the magnitude of the force acting on a 1.00-m cross section of the rod is

$$\begin{aligned}
 \|\mathbf{F}\| &= I\|\mathbf{L} \times \mathbf{B}\| \\
 &= I\|\mathbf{L}\|\|\mathbf{B}\| \sin \phi \\
 &= (50.0 \text{ A})(1.00 \text{ m})(1.20 \text{ T}) \sin 45^\circ \\
 &= 42.4 \text{ N}
 \end{aligned}$$

The force is perpendicular to the plane of the current and the field, both of which lie in the horizontal plane. Thus the force must be vertical and the right-hand rule shows that it is vertically upward.

Alternatively, we can use a coordinate system with the x -axis pointing east, the y -axis north, and the z -axis up. Then we have

$$\begin{aligned}
 \mathbf{L} &= (1.00 \text{ m})\hat{\mathbf{i}}, & \mathbf{B} &= (1.20 \text{ T})(\cos 45^\circ \hat{\mathbf{i}} + \sin 45^\circ \hat{\mathbf{j}}), \\
 \mathbf{F} &= I(\mathbf{L} \times \mathbf{B}) \\
 &= (50.0 \text{ A})[(1.00 \text{ m})\hat{\mathbf{i}} \times (1.20 \text{ T})(\cos 45^\circ \hat{\mathbf{i}} + \sin 45^\circ \hat{\mathbf{j}})] \\
 &= (42.4 \text{ N})\hat{\mathbf{k}}
 \end{aligned}$$

If the rod is in mechanical equilibrium under the action of its weight and the upward magnetic force, then its weight $W = 42.4 \text{ N}$.

- (b) The magnitude of the force is maximum if $\phi = 90^\circ$ so that $\mathbf{L} \perp \mathbf{B}$. To have the force still be upward, we rotate the rod clockwise by 45°

from its orientation in the figure so that the current runs toward the southeast. Then the magnetic force has magnitude

$$\|\mathbf{F}\| = I\|\mathbf{L}\|\|\mathbf{B}\| \sin \phi = (50.0 \text{ A})(1.00 \text{ m})(1.20 \text{ T}) = 60.0 \text{ N}.$$

4. A circular coil of radius $r = 0.0500 \text{ m}$ with $N = 30$ turns of wire lies in a horizontal plane. It carries a current of $I = 5.00 \text{ A}$ in a counterclockwise sense when viewed from above. The coil is in a uniform magnetic field of magnitude $B = 1.20 \text{ T}$ directed towards the right. Find the magnitude of the magnetic moment and the torque on the coil.

Solution:

The area of the coil is

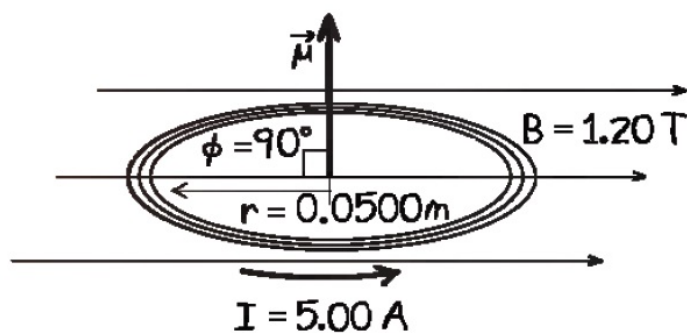
$$A = \pi r^2 = \pi(0.0500 \text{ m})^2 = 7.85 \times 10^{-3} \text{ m}^2.$$

The magnitude of magnetic moment of each turn of the coil is

$$\|\boldsymbol{\mu}\| = IA = (5.00 \text{ A})(7.85 \times 10^{-3} \text{ m}^2) = 3.93 \times 10^{-2} \text{ A} \cdot \text{m}^2$$

So the magnitude of the total magnetic moment of all 30 turns is

$$\|\boldsymbol{\mu}_{\text{total}}\| = N\|\boldsymbol{\mu}\| = (30)(3.93 \times 10^{-2} \text{ A} \cdot \text{m}^2) = 1.18 \text{ A} \cdot \text{m}^2.$$



As shown in the above figure, the angle ϕ between the direction of \mathbf{B} and the direction of $\boldsymbol{\mu}$ (which is along the normal to the plane of the coil) is 90° . Therefore, the magnitude of the total torque on the coil is

$$\begin{aligned} \|\boldsymbol{\tau}_{\text{total}}\| &= \|\boldsymbol{\mu}_{\text{total}} \times \mathbf{B}\| \\ &= \|\boldsymbol{\mu}_{\text{total}}\|\|\mathbf{B}\| \sin \phi \\ &= (1.18 \text{ A} \cdot \text{m}^2)(1.20 \text{ T}) \sin 90^\circ \\ &= 1.42 \text{ N} \cdot \text{m} \end{aligned}$$

Alternatively, the magnitude of the torque on each turn of the coil is

$$\|\boldsymbol{\tau}\| = \|\boldsymbol{\mu}\|\|\mathbf{B}\| \sin \phi = (3.93 \times 10^{-2} \text{ A}\cdot\text{m}^2)(1.20 \text{ T}) \sin 90^\circ = 0.0472 \text{ N}\cdot\text{m}$$

and the magnitude of the total torque on the coil is

$$\|\boldsymbol{\tau}_{\text{total}}\| = N\|\boldsymbol{\tau}\| = (30)(0.0472 \text{ N}\cdot\text{m}) = 1.42 \text{ N}\cdot\text{m}.$$

5. If the coil in the previous example rotates from its initial position to a position where its magnetic moment is parallel to \mathbf{B} , what is the change in its potential energy?

Solution:

The initial potential energy U_i of the coil is

$$\begin{aligned} U_i &= -\boldsymbol{\mu}_{\text{total}} \cdot \mathbf{B} \\ &= -\|\boldsymbol{\mu}_{\text{total}}\|\|\mathbf{B}\| \cos \phi_i \\ &= -(1.18 \text{ A}\cdot\text{m}^2)(1.20 \text{ T}) \cos 90^\circ \\ &= 0 \end{aligned}$$

Similarly, the final potential energy U_f of the coil is

$$\begin{aligned} U_f &= -\boldsymbol{\mu}_{\text{total}} \cdot \mathbf{B} \\ &= -\|\boldsymbol{\mu}_{\text{total}}\|\|\mathbf{B}\| \cos \phi_f \\ &= -(1.18 \text{ A}\cdot\text{m}^2)(1.20 \text{ T}) \cos 0^\circ \\ &= -1.42 \text{ J} \end{aligned}$$

So the change in the potential energy of the coil is equal to

$$\Delta U = U_f - U_i = -1.42 \text{ J}$$

6. An air gun is pointed at a target which is released in free fall as the bullet leaves the muzzle. Use vector method to show that the bullet always hits the falling target no matter what is its initial speed if the air resistance is negligible.

Solution:

As shown in the figure on the next page, the position and velocity of the projectile P (the bullet) and the target T at any time t are described by

the formulae:

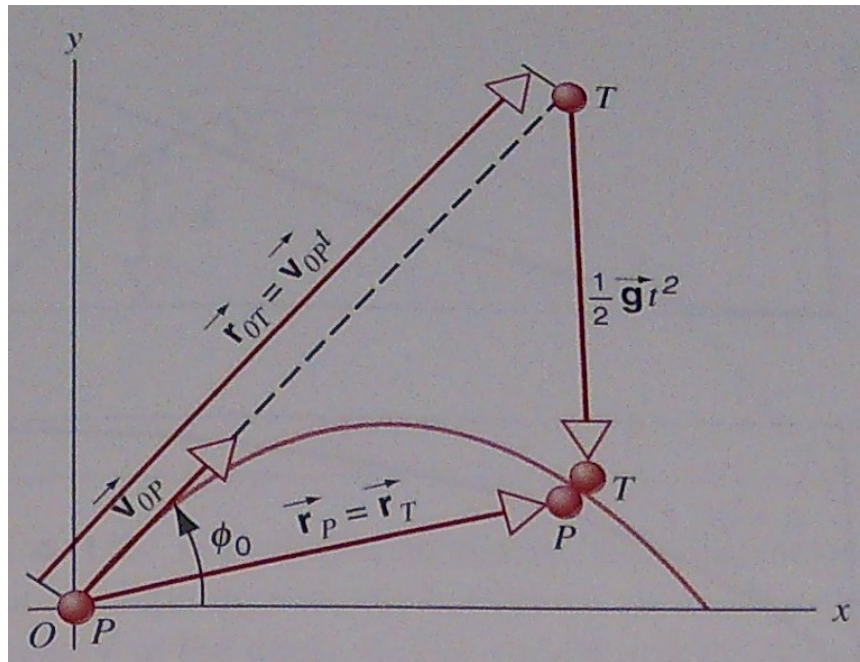
$$\mathbf{r} = \int_0^t \mathbf{v} dt + \mathbf{r}_0 = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{a} t^2,$$

$$\mathbf{v} = \int_0^t \mathbf{a} dt + \mathbf{v}_0 = \mathbf{v}_0 + \mathbf{a} t,$$

since they both move with constant acceleration $\mathbf{a} = \mathbf{g}$.

Note that the initial position of the projectile P is $\mathbf{r}_{0P} = \mathbf{0}$ while the initial position and initial velocity of the target T are \mathbf{r}_{0T} and $\mathbf{v}_{0T} = \mathbf{0}$. So their positions at time t are

$$\mathbf{r}_P = \mathbf{v}_{0P} t + \frac{1}{2} \mathbf{g} t^2, \quad \mathbf{r}_T = \mathbf{r}_{0T} + \frac{1}{2} \mathbf{g} t^2.$$



If there is a collision, we must have $\mathbf{r}_P = \mathbf{r}_T$. This will always occur at a time t given by $\mathbf{r}_{0T} = \mathbf{v}_{0P} t$, i.e. in the time $t = \|\mathbf{r}_{0T}\|/\|\mathbf{v}_{0P}\|$ that it would take for the projectile to travel to the target position along the line of sight. Because multiplying a vector by a positive scalar gives another vector in the same direction, the equation $\mathbf{r}_{0T} = \mathbf{v}_{0P} t$ tells us that \mathbf{r}_{0T} and \mathbf{v}_{0P} must point along the same direction. In other words, the gun must be aimed at the initial position of the target.

7. An object of mass m that moves in an elliptical path with constant angular speed ω has position vector

$$\mathbf{r}(t) = a \cos(\omega t) \hat{\mathbf{i}} + b \sin(\omega t) \hat{\mathbf{j}}.$$

Find the force acting on the object and show that its directed towards the origin.

Solution:

The velocity and acceleration of the object are

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -a\omega \sin(\omega t) \hat{\mathbf{i}} + b\omega \cos(\omega t) \hat{\mathbf{j}}$$

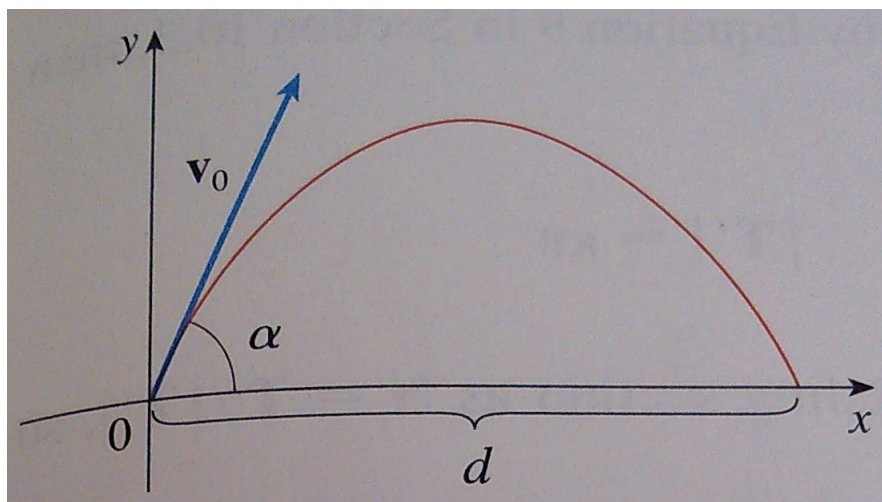
$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = -a\omega^2 \cos(\omega t) \hat{\mathbf{i}} - b\omega^2 \sin(\omega t) \hat{\mathbf{j}}$$

Then applying Newton's second law gives the force as

$$\mathbf{F}(t) = m\mathbf{a}(t) = -m\omega^2 [a \cos(\omega t) \hat{\mathbf{i}} + b \sin(\omega t) \hat{\mathbf{j}}] = -m\omega^2 \mathbf{r}(t)$$

It shows that the force acts in the direction opposite to the radial vector $\mathbf{r}(t)$ and thus points towards the origin.

8. A projectile is fired with angle of elevation α and initial velocity \mathbf{v}_0 as shown in the figure below. Assuming that air resistance is negligible and the only external force acting on the projectile is due to gravity, find the position vector $\mathbf{r}(t)$ of the projectile. What value of α maximizes the range (i.e. the horizontal distance travelled) of the projectile?



Solution:

We set up the axes so that the projectile starts at the origin. Since the force due to gravity is pointing downward, we have

$$\begin{aligned}\mathbf{F} &= m\mathbf{a} = -mg\hat{\mathbf{j}} \\ \Rightarrow \mathbf{v}'(t) &= \mathbf{a} = -g\hat{\mathbf{j}}\end{aligned}$$

where $g = 9.81 \text{ m/s}^2$.

Integrating the above equation, we find that

$$\mathbf{r}'(t) = \mathbf{v}(t) = -gt\hat{\mathbf{j}} + \mathbf{v}_0$$

as the integrating constant $\mathbf{C} = \mathbf{v}(0) = \mathbf{v}_0$.

Integrating again, we obtain

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\hat{\mathbf{j}} + t\mathbf{v}_0 + \mathbf{D}.$$

But $\mathbf{D} = \mathbf{r}(0) = \mathbf{0}$. So the position vector of the projectile is given by

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\hat{\mathbf{j}} + t\mathbf{v}_0$$

If we write $\|\mathbf{v}_0\| = v_0$, then the initial velocity of the projectile

$$\mathbf{v}_0 = v_0 \cos \alpha \hat{\mathbf{i}} + v_0 \sin \alpha \hat{\mathbf{j}}$$

$$\therefore \mathbf{r}(t) = (v_0 \cos \alpha)t\hat{\mathbf{i}} + \left[(v_0 \sin \alpha)t - \frac{1}{2}gt^2 \right] \hat{\mathbf{j}}$$

The parametric equations of the projectile's trajectory are therefore

$$x = (v_0 \cos \alpha)t, \quad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2.$$

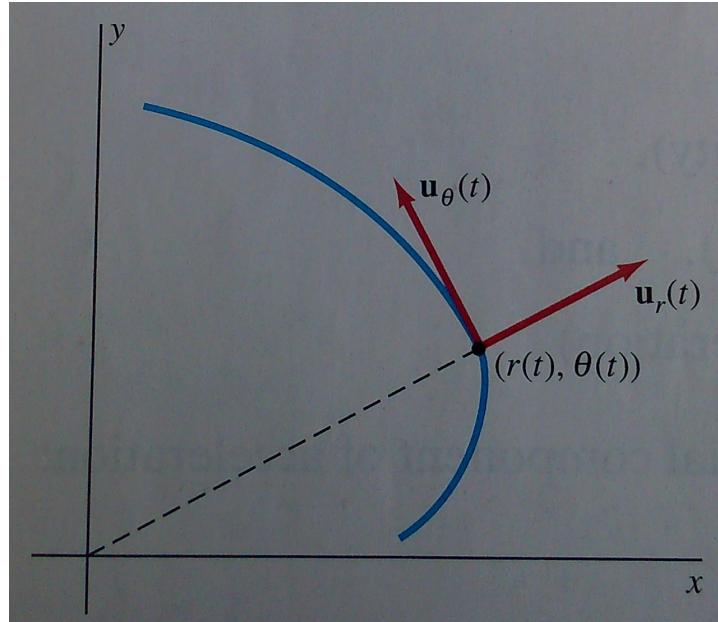
The range d is the value of x when $y = 0$. Setting $y = 0$, we obtain $t = 0$ or $t = 2v_0 \sin \alpha / g$. The latter value of t then yields

$$d = (v_0 \cos \alpha) \left(\frac{2v_0 \sin \alpha}{g} \right) = \frac{v_0^2 (2 \sin \alpha \cos \alpha)}{g} = \frac{v_0^2 \sin 2\alpha}{g}$$

Obviously, d reaches the maximum value when $\sin 2\alpha = 1$, i.e. $\alpha = \pi/4$.

9. A particle is moving in a plane with position specified by polar coordinates r and θ . As shown in the figure below, the radial and transverse unit vectors at each point of the plane are

$$\hat{\mathbf{u}}_r = (\cos \theta) \hat{\mathbf{i}} + (\sin \theta) \hat{\mathbf{j}}, \quad \hat{\mathbf{u}}_\theta = -(\sin \theta) \hat{\mathbf{i}} + (\cos \theta) \hat{\mathbf{j}}.$$



Show that the velocity and acceleration vectors of the particle are

$$\mathbf{v} = \left(\frac{dr}{dt} \right) \hat{\mathbf{u}}_r + \left(r \frac{d\theta}{dt} \right) \hat{\mathbf{u}}_\theta,$$

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right] \hat{\mathbf{u}}_r + \left[\frac{1}{r} \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right) \right] \hat{\mathbf{u}}_\theta.$$

Solution:

By differentiation of $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\theta$ with respect to time t , we obtain

$$\frac{d\hat{\mathbf{u}}_r}{dt} = \hat{\mathbf{u}}_\theta \frac{d\theta}{dt} \quad \text{and} \quad \frac{d\hat{\mathbf{u}}_\theta}{dt} = -\hat{\mathbf{u}}_r \frac{d\theta}{dt}. \quad (1)$$

The position vector \mathbf{r} points away from the origin and has length $\|\mathbf{r}\| = r$. Therefore,

$$\mathbf{r} = r \hat{\mathbf{u}}_r. \quad (2)$$

Differentiating both sides of Eq. (2) with respect to t gives

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \hat{\mathbf{u}}_r \frac{dr}{dt} + r \frac{d\hat{\mathbf{u}}_r}{dt}. \quad (3)$$

Combining Equations (1) and (3) yields

$$\mathbf{v} = \left(\frac{dr}{dt} \right) \hat{\mathbf{u}}_r + \left(r \frac{d\theta}{dt} \right) \hat{\mathbf{u}}_\theta \quad (4)$$

Next, we differentiate both sides of Eq. (4) and find that

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \left(\hat{\mathbf{u}}_r \frac{d^2r}{dt^2} + \frac{dr}{dt} \frac{d\hat{\mathbf{u}}_r}{dt} \right) + \left(\frac{dr}{dt} \frac{d\theta}{dt} \hat{\mathbf{u}}_\theta + r \frac{d^2\theta}{dt^2} \hat{\mathbf{u}}_\theta + r \frac{d\theta}{dt} \frac{d\hat{\mathbf{u}}_\theta}{dt} \right)$$

Using Eq. (1) and collecting the coefficients of $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\theta$, we obtain the expression

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right] \hat{\mathbf{u}}_r + \left[\frac{1}{r} \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right) \right] \hat{\mathbf{u}}_\theta$$

10. Kepler's first law states that a planet revolves around the Sun in an elliptical orbit with the Sun at one focus. Use the following steps to prove this law.

(a) Let \mathbf{r} and \mathbf{v} be the position and velocity vectors of the object, respectively. Show that $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ is a constant vector, which implies that the planet moves in one plane.

(b) Using the result of (a), deduce that

$$\mathbf{v} \times \mathbf{h} = GM\hat{\mathbf{u}} + \mathbf{c}, \quad \mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = GM\|\mathbf{r}\| + \|\mathbf{r}\|\|\mathbf{c}\| \cos \theta$$

where the unit vector $\hat{\mathbf{u}} = \mathbf{r}/\|\mathbf{r}\|$, \mathbf{c} is a constant vector, and θ is the angle between the vectors \mathbf{c} and \mathbf{r} .

(c) Hence, show that

$$r = \frac{ed}{1 + e \cos \theta}$$

where $r = \|\mathbf{r}\|$, $e = \|\mathbf{c}\|/(GM)$ and $d = \|\mathbf{h}\|^2/\|\mathbf{c}\|$. This equation is the polar equation of a conic section with focus at the origin and eccentricity e ; and thus proves the Kepler's first law.

Solution:

(a) Since the gravitational force of the Sun on a planet is so much larger than the forces exerted by other celestial bodies, we can safely ignore

all bodies in the universe except the Sun and the planet revolving about it. We use a coordinate system with the Sun at the origin. Let $\mathbf{r} = \mathbf{r}(t)$ be the position vector of the planet. Then the velocity vector is $\mathbf{v} = \mathbf{r}'$ and the acceleration vector is $\mathbf{a} = \mathbf{r}''$.

Next, we apply the following Newton's laws:

$$\text{Second Law of Motion: } \mathbf{F} = m\mathbf{a}$$

$$\text{Law of Gravitation: } \mathbf{F} = -\frac{GMm}{r^3}\mathbf{r} = -\frac{GMm}{r^2}\hat{\mathbf{u}}$$

where \mathbf{F} is the gravitational force acting on the planet, m and M are the masses of the planet and the Sun, G is the gravitational constant, $r = \|\mathbf{r}\|$, and $\hat{\mathbf{u}} = (1/r)\mathbf{r}$ is the unit vector in the direction of \mathbf{r} .

Equating the expressions for \mathbf{F} in Newton's two laws yields

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

which implies \mathbf{a} is parallel to \mathbf{r} . It follows that

$$\mathbf{r} \times \mathbf{a} = \mathbf{0}.$$

Differentiating the vector $\mathbf{r} \times \mathbf{v}$ gives

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{r}' \times \mathbf{v} + \mathbf{r} \times \mathbf{v}' = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

$$\Rightarrow \mathbf{r} \times \mathbf{v} = \mathbf{h}$$

where \mathbf{h} is a constant vector. It means that the vector $\mathbf{r} = \mathbf{r}(t)$ is perpendicular to \mathbf{h} for all values of t and so the planet always lies in the plane through the origin perpendicular to \mathbf{h} . Therefore, the orbit of the planet is a plane curve.

(b) We rewrite the vector \mathbf{h} as follows:

$$\begin{aligned} \mathbf{h} &= \mathbf{r} \times \mathbf{r}' \\ &= r\hat{\mathbf{u}} \times (r\hat{\mathbf{u}})' \\ &= r\hat{\mathbf{u}} \times (r'\hat{\mathbf{u}} + r\hat{\mathbf{u}}') \\ &= rr'(\hat{\mathbf{u}} \times \hat{\mathbf{u}}) + r^2(\hat{\mathbf{u}} \times \hat{\mathbf{u}}') \\ &= r^2(\hat{\mathbf{u}} \times \hat{\mathbf{u}}') \end{aligned}$$

Then the cross product $\mathbf{a} \times \mathbf{h}$ is given by

$$\begin{aligned}\mathbf{a} \times \mathbf{h} &= -\frac{GM}{r^2} \hat{\mathbf{u}} \times [r^2(\hat{\mathbf{u}} \times \hat{\mathbf{u}}')] \\ &= -GM \hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \hat{\mathbf{u}}') \\ &= -GM[(\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}')\hat{\mathbf{u}} - (\hat{\mathbf{u}} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}']\end{aligned}$$

However, since $\|\hat{\mathbf{u}}\| = 1$, $\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = \|\hat{\mathbf{u}}\|^2 = 1$ and $\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}' = 0$. Therefore,

$$\mathbf{a} \times \mathbf{h} = GM\hat{\mathbf{u}}'$$

As \mathbf{h} is a constant vector,

$$(\mathbf{v} \times \mathbf{h})' = \mathbf{v}' \times \mathbf{h} = \mathbf{a} \times \mathbf{h} = GM\hat{\mathbf{u}}'$$

Integrating both sides of this equation, we obtain

$$\mathbf{v} \times \mathbf{h} = GM\hat{\mathbf{u}} + \mathbf{c} \tag{5}$$

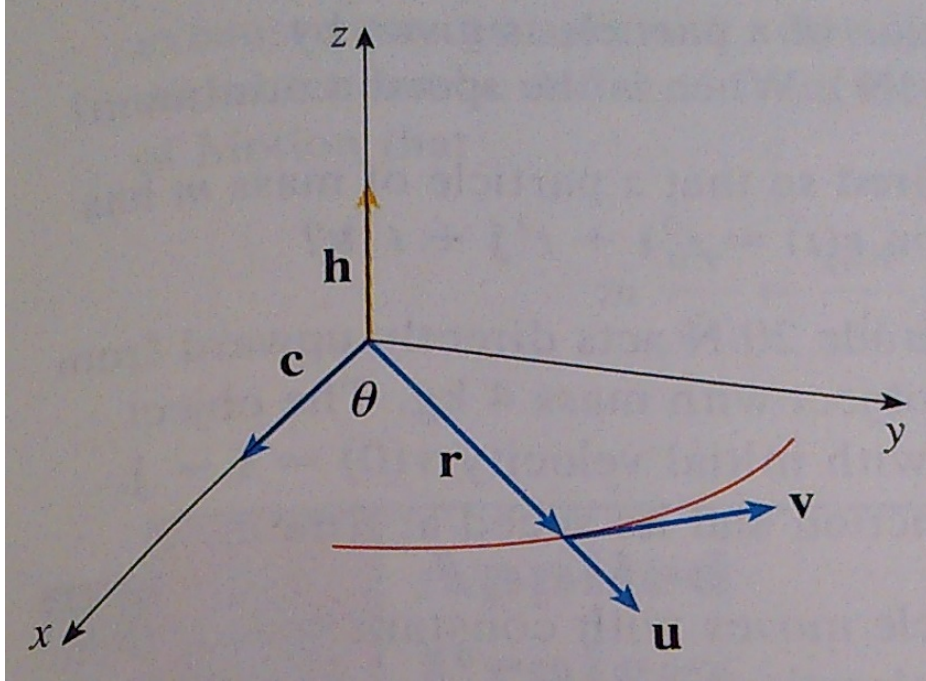
where \mathbf{c} is a constant vector.

For convenience, we choose the z axis so that the unit vector $\hat{\mathbf{k}}$ points in the direction of the vector \mathbf{h} . Then the planet moves in the xy -plane. Since both $\mathbf{v} \times \mathbf{h}$ and $\hat{\mathbf{u}}$ are perpendicular to \mathbf{h} , Equation (5) shows that \mathbf{c} lies in the xy -plane.

Next, we choose the x - and y -axes so that the vector $\hat{\mathbf{i}}$ lies in the direction of \mathbf{c} , as shown in the figure below. If θ is the angle between the vectors \mathbf{c} and \mathbf{r} , then (r, θ) are polar coordinates of the planet. From Equation (5), we then have

$$\begin{aligned}\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= \mathbf{r} \cdot (GM\hat{\mathbf{u}} + \mathbf{c}) \\ &= GM\mathbf{r} \cdot \hat{\mathbf{u}} + \mathbf{r} \cdot \mathbf{c} \\ &= GMr\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} + \|\mathbf{r}\|\|\mathbf{c}\| \cos \theta \\ &= GMr + rc \cos \theta\end{aligned}$$

where $c = \|\mathbf{c}\|$.



(c) From the result of part (b),

$$r = \|\mathbf{r}\| = \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{GM + \|\mathbf{c}\| \cos \theta} = \frac{1}{GM} \left[\frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{1 + e \cos \theta} \right]$$

where $e = \|\mathbf{c}\|/(GM)$. However,

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = \|\mathbf{h}\|^2 = h^2$$

where $h = \|\mathbf{h}\|$. Therefore,

$$r = \frac{h^2/(GM)}{1 + e \cos \theta} = \frac{e\|\mathbf{h}\|^2/\|\mathbf{c}\|}{1 + e \cos \theta}$$

Writing $d = \|\mathbf{h}\|^2/\|\mathbf{c}\|$, we obtain the expression

$$r = \frac{ed}{1 + e \cos \theta}.$$

11. A particle moving along a straight line is attracted to the origin by a force F . If the force of attraction F is proportional to the distance x of the particle from the origin, show that the particle will execute simple harmonic motion, i.e. its equation of motion satisfies a differential equation of the form

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

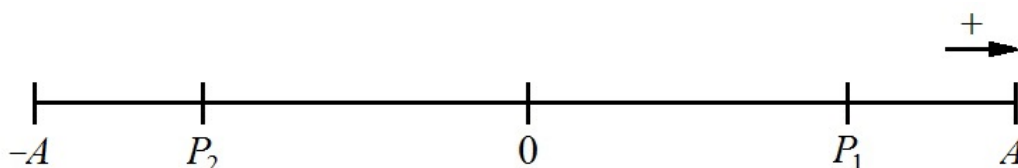
where ω_0 is a positive constant. Find the solution of this equation of motion. Hence, describe the motion of the particle.

Solution:

From the given information, the particle experiences the attractive force

$$F = -kx$$

where $k > 0$ is a proportionality constant. The negative sign is necessary because when the particle is at P_1 , x is positive and F acts in the negative direction; when it is at P_2 , x is negative and F acts in the positive direction (see below figure). So F and x always have opposite signs.



Thus the equation of motion for the particle is

$$F = m \frac{d^2x}{dt^2} = -kx$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Since both k and m are positive constants, we may replace k/m by a new constant ω_0^2 . Then the equation of motion becomes

$$\frac{d^2x}{dt^2} + \omega_0^2x = 0.$$

which implies the particle executes simple harmonic motion.

Obviously, the general solution of the above differential equation is

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

We can rewrite the solution as

$$x(t) = A \cos(\omega_0 t + \delta),$$

where $\cos \delta = c_1/A$, $\sin \delta = -c_2/A$, $A = \sqrt{c_1^2 + c_2^2}$.

Since $\cos \theta \in [-1, 1]$ for any θ , we can see that $|x| \leq A$ at any time t . So the particle never goes beyond the points A and $-A$. These points are thus the maximum displacement of the particle from the origin.

Besides, differentiating the solution gives the velocity of the particle

$$\frac{dx}{dt} = v(t) = -A\omega_0 \sin(\omega_0 t + \delta),$$

When $x = \pm A$,

$$|\cos(\omega_0 t + \delta)| = 1 \quad \Rightarrow \quad \sin(\omega_0 t + \delta) = 0.$$

Thus the speed of the particle v is zero at the end points $x = \pm A$.

Similarly, when $x = 0$,

$$\cos(\omega_0 t + \delta) = 0 \quad \Rightarrow \quad \sin(\omega_0 t + \delta) = \pm 1.$$

Thus the particle reaches its maximum speed $A\omega_0$ at the origin $x = 0$. For other values of x , the speed of the particle would be between 0 and $A\omega_0$. Its speed increases as the particle goes from $x = A$ to the origin. After crossing the origin, its speed decreases until it drops to zero again at $x = -A$. But the particle now moves in the other direction (recall that the force F is always opposite to the displacement x). Then its speed increases again until it reaches its maximum speed at the origin, and then decreases until it is zero again at $x = A$. The particle thus oscillates back and forth, moving in an endless cycle over the interval $x \in [-A, A]$.

12. A particle executes simple harmonic motion with natural (undamped) frequency $\omega_0 = 6$ rad/s. If the particle starts from the equilibrium position with an initial speed of 3 m/s, find
- (a) the solution of its equation of motion,
 - (b) the amplitude A of the motion,
 - (c) the phase angle δ , and
 - (d) the period T of the motion.

Solution:

- (a) Since the particle executes simple harmonic motion, its equation of motion is

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0.$$

where the natural frequency $\omega_0 = 6$ rad/s. Solving this DE yields

$$x(t) = A \cos(\omega_0 t + \delta),$$
$$v(t) = \frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \delta).$$

Putting in the initial conditions $x(0) = 0$ and $v(0) = 3$ m/s, we obtain

$$0 = A \cos \delta, \quad 3 = -6A \sin \delta.$$

Since the amplitude $A > 0$, we find from the first equation that $\delta = 3\pi/2$. With this value of δ , the second equation gives $A = 1/2$. Hence, the solution of its equation of motion is

$$x(t) = \frac{1}{2} \cos \left(6t + \frac{3\pi}{2} \right).$$

(b) As shown in part (a), the amplitude of the motion is $A = 0.5$ m.

(c) As shown in part (a), the phase angle is $\delta = 3\pi/2$ rad.

(d) The period of the motion is

$$T = \frac{1}{f} = \frac{2\pi}{\omega_0} = \frac{2\pi}{6} = \frac{\pi}{3} \text{ (s)}$$

13. By Hooke's law, the force F in an elastic spring is proportional to the distance ℓ by which the spring has been stretched, i.e.

$$F = -k\ell,$$

where $k > 0$ is the spring constant. Suppose an object of mass $m = 2$ kg stretches an elastic helical spring by a distance $\ell = 0.175$ m. After it comes to rest, it is stretched an additional distance $y = 0.200$ m and then released. Find the solution of its equation of motion. Hence, determine the period T and amplitude A of the motion.

Solution:

When the spring is in equilibrium, the upward force due to the spring must be equal to the downward force due to the weight attached to it. Thus, according to the given information,

$$k\ell = mg$$
$$\Rightarrow k(0.175 \text{ m}) = (2 \text{ kg})(9.81 \text{ m/s}^2)$$
$$\Rightarrow k = 112 \text{ N/m}$$

Let downward be the positive y direction and the equilibrium position of the spring be $y = 0$. Then the equation of motion for the object is

$$\begin{aligned} m \frac{d^2 y}{dt^2} &= mg - k(\ell + y) = -ky \\ \Rightarrow \frac{d^2 y}{dt^2} + \frac{k}{m} y &= 0 \end{aligned}$$

Since it has the same form as the equation of motion for the simple harmonic motion, we know that the elastic spring with weight attached will execute such motion about the equilibrium position $y = 0$ with natural (undamped) frequency $\omega_0 = \sqrt{k/m}$.

It can be shown that the solution of the equation of motion is

$$y(t) = c \cos \left(\sqrt{k/m} t + \delta \right).$$

Plugging in the values of m and k (in SI units), we have

$$\begin{aligned} y(t) &= c \cos \left(\sqrt{56} t + \delta \right), \\ v(t) = \frac{dy}{dt} &= -\sqrt{56} c \sin \left(\sqrt{56} t + \delta \right). \end{aligned}$$

Inserting the initial conditions $y(0) = 0.200$ m and $v(0) = 0$ into these equations yields

$$0.200 = c \cos \delta, \quad 0 = -\sqrt{56} c \sin \delta.$$

Since $c > 0$, the second equation implies $\delta = 0$. With this value of δ , the first equation gives $c = 0.200$ m. Substituting these values back into the solution, we find the solution of the equation of motion:

$$y(t) = 0.200 \cos(\sqrt{56} t).$$

The period of the motion is thus

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{k/m}} = \frac{2\pi \text{ rad}}{\sqrt{56} \text{ rad/s}} = \frac{\pi}{\sqrt{14}} \text{ (s)}$$

Moreover, the amplitude of the motion is $c = 0.200$ m.

14. A motion in which the departure or displacement increases beyond all bounds as time passes is called an unstable motion. In such case, a mechanical breakdown of this system is bound to occur. This condition,

where the frequency ω of the driving force equals the frequency ω_0 of the natural (undamped) frequency of the system, is known as undamped resonance, and ω_0 is called the undamped resonant frequency. Suppose the equation of motion of a system is

$$\frac{d^2y}{dt^2} + 4y = \cos 2t.$$

Find the general solution of this equation. Is the motion stable or unstable? What is the undamped resonant frequency ω_0 ?

Solution:

Setting the left side of the equation of motion equal to zero and solving it, we obtain the complementary solution

$$y_c(t) = c \cos(2t + \delta).$$

We are going to find the particular solution y_p of the equation of motion by the method of undetermined coefficients. Since the right side of this equation agrees with the complementary solution, the guess for y_p must be of the form

$$y_p(t) = At \sin 2t + Bt \cos 2t.$$

Plugging it back into the equation of motion, it can be shown that

$$y_p(t) = \frac{1}{4} t \sin 2t.$$

Hence the general solution of the equation of motion is

$$y(t) = c \cos(2t + \delta) + \frac{1}{4} t \sin 2t.$$

The presence of the factor $t/4$ in the amplitude of the second term implies that the displacement y increases with time t . The motion is thus unstable. A comparison of the equation of motion and the complementary solution shows that the condition of undamped resonance is present since the frequency of the driving force and the natural (undamped) frequency of the system are both 2 rad/s. Hence, the undamped resonant frequency is $\omega_0 = 2$ rad/s.

15. A capacitor of capacitance $C = 1/505$ F, an inductor of inductance $L = 1/20$ H, and a resistor of resistance $R = 1 \Omega$ are connected in series. If the

current in the circuit $I = 0$ and the charge on the capacitor $q = 1$ C at time $t = 0$, find the charge and current in the circuit (due to the discharge of the capacitor) when $t = 0.01$ s.

Solution:

Kirchhoff's second law states that the sum of the voltage drops in a closed circuit is equal to the electromotive force of the source of energy $E(t)$. So the current in the circuit is governed by the equation

$$RI + L\frac{dI}{dt} + \frac{1}{C}q = E(t)$$

It can be rewritten as

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t),$$

which is the differential equation for the motion of the charge q in the circuit as a function of time t .

Plugging in the given values, the above equation becomes

$$\begin{aligned} \frac{1}{20}\frac{d^2q}{dt^2} + \frac{dq}{dt} + 505q &= 0, \\ \Rightarrow \frac{d^2q}{dt^2} + 20\frac{dq}{dt} + 10100q &= 0 \end{aligned}$$

Its general solution is

$$q(t) = e^{-10t}(c_1 \sin 100t + c_2 \cos 100t).$$

Thus the current in the circuit is

$$I(t) = \frac{dq}{dt} = 10e^{-10t}[(10c_1 - c_2) \cos 100t - (c_1 + 10c_2) \sin 100t]$$

Substituting the initial conditions $q(0) = 1$ C and $I(0) = 0$ into the above equations, we obtain

$$\begin{aligned} 1 &= c_2, \quad 0 = 10(10c_1 - c_2) \\ \Rightarrow c_1 &= 0.1, \quad c_2 = 1 \end{aligned}$$

When $t = 0.01$ s, the charge and current in the circuit are

$$\begin{aligned} q(0.01) &= e^{-0.1}(0.1 \sin 1 + \cos 1) = 0.565 \text{ C} \\ I(0.01) &= 10e^{-0.1}[-(0.1 + 10) \sin 1] = -76.9 \text{ A} \end{aligned}$$

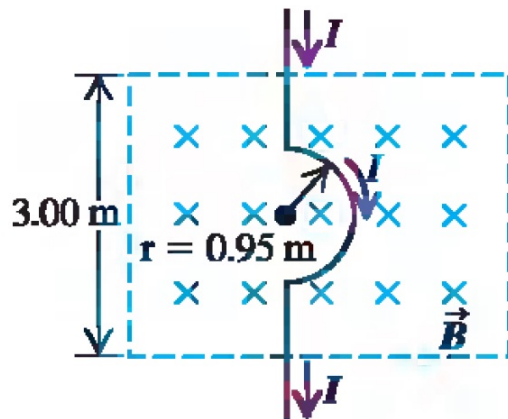
The negative current indicates that the capacitor is discharging, i.e. the charged particles are moving in a direction opposite to the one in which they moved when the capacitor was being charged.

⟨III⟩ Problems

- A proton (charge $q = 1.60 \times 10^{-19} \text{ C}$, $m = 1.67 \times 10^{-27} \text{ kg}$) moves in a uniform magnetic field $\mathbf{B} = (0.500 \text{ T})\hat{\mathbf{i}}$ in the $+x$ -direction. In addition to the magnetic field, there is a uniform electric field $\mathbf{E} = (+2.00 \times 10^4 \text{ V/m})\hat{\mathbf{i}}$ in the same direction. Assume the proton starts its motion at time $t = 0$ with velocity components $v_x = 1.50 \times 10^5 \text{ m/s}$, $v_y = 0$, and $v_z = 2.00 \times 10^5 \text{ m/s}$.

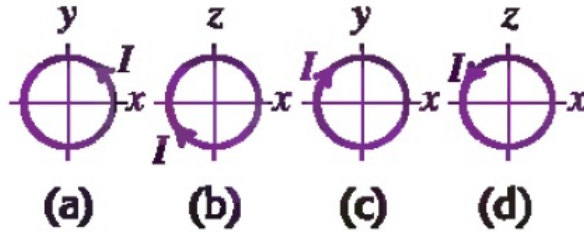
 - What are the magnitude and direction of the magnetic force acting on the proton at time $t = 0$?
 - Will the proton have a component of acceleration in the direction of the electric field?
 - Describe the path of the proton. How does the electric field affect the path? Explain your answer briefly.
 - What is the x -component of the displacement of the proton at $t = T/2$, where T is the period of its circular motion, from its position at $t = 0$?

- A long, straight wire containing a semi-circular region of radius $r = 0.95 \text{ m}$ is placed in a uniform magnetic field of magnitude $B = 2.20 \text{ T}$ pointing into the page as shown in the figure at the top of next page. What would be the magnetic force acting on the wire if it carries a current $I = 3.40 \text{ A}$?

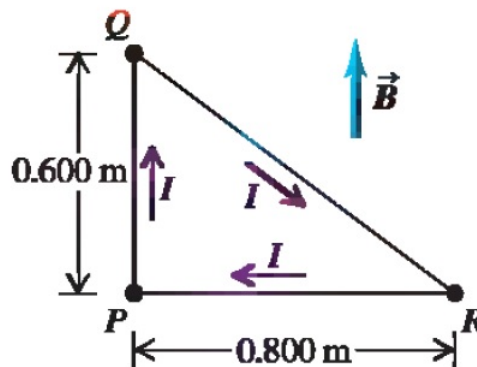


- A circular coil with area A and N turns is free to rotate about a diameter that coincides with the x -axis. Suppose a current I is circulating in the coil and there is a uniform magnetic field \mathbf{B} in the positive y -direction.

Calculate the magnitude and direction of the torque τ acting on the coil and the value of its potential energy U when the coil is oriented as shown in parts (a) to (d) of the below figure.

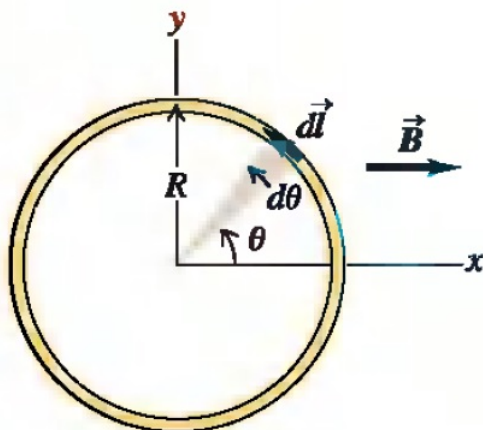


4. A triangular loop of uniform wire carries a current $I = 5.00 \text{ A}$ in the direction as shown in the figure below. The loop is inside a uniform magnetic field which has magnitude $B = 3.00 \text{ T}$ in the same direction as the current in the side PQ of the loop.



- Find the force exerted by the magnetic field on each side of the loop. If the force is not zero, specify its direction.
- What is the net force acting on the loop?
- The loop is pivoted about an axis that lies along the side PR . Using the forces calculated in part (a), compute the torque acting on each side of the loop.
- What is the magnitude of the net torque acting on the loop? Calculate the net torque from the torques calculated in part (c) and also from the equation $\tau = NIAB \sin \phi$. Do these two results agree?
- Is the net torque directed to rotate point Q into the plane or out of the plane of the figure?

5. A wire ring lies in the xy -plane with its center at the origin carries a counterclockwise current I as viewed from the top (see below figure). Assume there is a uniform magnetic field $\mathbf{B} = B_x \hat{\mathbf{i}}$ in the $+x$ -direction.



- (a) Show that the element in the figure is described mathematically by

$$d\mathbf{l} = R d\theta (-\sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}}),$$

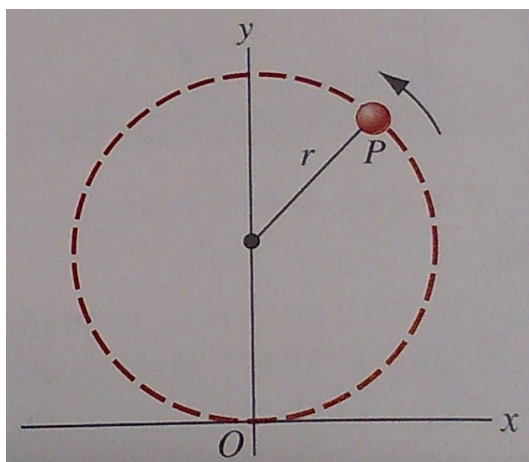
and then find the force acting on it $d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}$.

- (b) Integrate $d\mathbf{F}$ around the loop to show that the net force \mathbf{F} acting on the loop is zero.
- (c) Using the result of part (a), find the torque acting on the element $d\boldsymbol{\tau} = \mathbf{r} \times d\mathbf{F}$ where $\mathbf{r} = R(\cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}})$ is the vector pointing from the center of the loop to the element $d\mathbf{l}$. (Note that $d\mathbf{l} \perp \mathbf{r}$.)
- (d) Integrate $d\boldsymbol{\tau}$ over the loop to find the total torque $\boldsymbol{\tau}$ on the loop. Show that the result can be rewritten as $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$ where $\boldsymbol{\mu} = IA$.

6. A particle P travels with constant speed on a circle of radius $r = 3.0$ m and completes one revolution in a time interval of $T = 20$ s (see the figure below). Suppose the particle passes through the origin O at time $t = 0$. With respect to the origin O , find

- (a) the magnitude and direction of the vectors describing its position 5.0 s, 7.5 s, and 10 s later,
- (b) the magnitude and direction of the displacement in the 5.0 s interval from the fifth to the tenth second,
- (c) the average velocity vector in this interval,
- (d) the instantaneous velocity vector at the beginning and at the end of this interval, and

- (e) the instantaneous acceleration vector at the beginning and at the end of this interval. Measure angles counterclockwise from the x -axis.



7. A captain is driving a spaceship with position vector given by

$$\mathbf{r}(t) = (3 + t)\hat{\mathbf{i}} + (2 + \ln t)\hat{\mathbf{j}} + \left[7 - \frac{4}{(t^2 + 1)}\right]\hat{\mathbf{k}}$$

for any time $t > 0$. The captain wants the spaceship to coast into the space station located at $(6, 4, 9)$. When should the engines be turned off?

8. A particle moves on a circle of fixed radius r . Prove that the velocity and acceleration vectors of the particle are given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \quad \text{and} \quad \mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}$$

where $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are the particle's angular velocity and acceleration vectors.

9. Kepler's second law states that the line joining the Sun to a planet sweeps out equal areas in equal times. Use the following steps to prove this law. Note that the notation is the same as in Example 10. In particular, use polar coordinates with radial vector $\mathbf{r} = (r \cos \theta)\hat{\mathbf{i}} + (r \sin \theta)\hat{\mathbf{j}}$.

- (a) Show that the vector $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ has magnitude

$$h = r^2 \frac{d\theta}{dt}.$$

- (b) If $A \equiv A(t)$ is the area swept out by the radial vector $\mathbf{r} = \mathbf{r}(t)$ in the time interval $[t_0, t]$, verify that

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}.$$

- (c) Hence, deduce that

$$\frac{dA}{dt} = \frac{1}{2}h = \text{constant}.$$

It means that the rate at which A is swept out is constant and thus proves Kepler's second law.

10. Kepler's third law states that the square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit. Use the following steps to prove this law. Let T be the period of a planet around the Sun, i.e. the time required for it to travel once around its elliptical orbit. Suppose that the lengths of the major and minor axes are $2a$ and $2b$, respectively.

- (a) Using the result of part (c) of previous problem, prove that

$$T = \frac{2\pi ab}{h}.$$

- (b) Deduce that h and a, b are related by

$$\frac{h^2}{GM} = \frac{b^2}{a}.$$

- (c) Hence, show that

$$T^2 = \frac{4\pi^2}{GM}a^3.$$

This proves Kepler's third law. (Notice that the proportionality constant $4\pi^2/(GM)$ is independent of the planet.)

11. A particle executes simple harmonic motion with frequency $f = 4$ Hz. If it is 1 m from the equilibrium position and moving with a speed of 4 m/s at time $t = 1$ s, find the general solution of its equation of motion.

12. A particle of mass m moving in a straight line is repelled from the origin O by a force F which is proportional to the distance of the particle from O . At time $t = 0$, the velocity of the particle is $-c\sqrt{k}$ m/s and it is c m from the origin. Find the position x of the particle as a function of time t if the magnitude of the repelling force is kx . Show that the particle will never reach the origin if $m < 1$; and the particle will approach the origin but never reach it if $m = 1$.
13. A heavy rubber band of natural length l_0 is stretched 1.2 m when a weight W is attached to it. Find the solution of its equation of motion and the maximum stretch of the band if the weight is given an initial downward velocity of 3 m/s at the position that the band is unstretched.
14. An object of mass $m = 5$ kg stretches a helical spring 0.9 m. After it is brought to rest, the system is set into vibration from the point of zero stretch by a driving force $F(t) = \sin 6t$. Find
- the distance x and velocity v as functions of time t ,
 - the natural angular frequency ω_0 of the system,
 - the forcing angular frequency ω , i.e. the angular frequency of the driving force,
 - the maximum possible displacement or departure of the object from its equilibrium position, and
 - whether the motion is stable or unstable.
15. If an inductor of inductance L and a capacitor of capacitance C are connected in series in a circuit with voltage source $E(t)$, the charge q in the circuit is governed by

$$L \frac{d^2q}{dt^2} + \frac{q}{C} = E(t).$$

It is the differential equation of the harmonic oscillator for the electric charge q which corresponds to the forced undamped motion of a mechanical system.

- Solve for q and I as functions of time t if $E(t) = 0$, $q(0) = q_0$, and $I(0) = 0$. What is the natural (undamped) frequency of the system?

- (b) Solve for q and I as functions of time t if $E(t) =$ a constant emf E , $q(0) = 0$, and $I(0) = 0$.
- (c) Solve for q and I as functions of time t if $E(t) = E \sin \omega t$, $q(0) = q_0$, and $I(0) = 0$. What value of ω will produce (undamped) resonance?