# Essential Maths for Physics Students Class B
# Chapter 2 Multivariable Calculus

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## ⟨I⟩ Summary

1. Partial Differentiation

   (a) Limits and Continuity of Multivariable Functions

      i. If $(x_0, y_0) = (x(t_0), y(t_0))$, then

$$\lim_{\substack{(x,y)\ \to\ (x_0,y_0) \\ \text{along curve } C}} f(x,y) = \lim_{t\to t_0} f(x(t), y(t))$$

      ii. If the limit of $f(x,y)$ fails to exist as $(x,y) \to (x_0, y_0)$ along some smooth curves, or if $f(x,y)$ has different limits as $(x,y) \to (x_0, y_0)$ along two different smooth curves, then the limit of $f(x,y)$ doesn't exist as $(x,y)$ approaches $(x_0, y_0)$.

      iii. A function $f$ is continuous at the point $(x_0, y_0)$ if all the following conditions are satisfied: (1)$f$ is defined at $(x_0, y_0)$, (2) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists, (3) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$. A function is continuous everywhere if it's continuous at every point of its domain.

   (b) Partial Derivatives of Multivariable Functions

$$\begin{aligned}\frac{\partial f(x_1, x_2, \ldots, x_i, \ldots, x_n)}{\partial x_i} &= f_{x_i} \\ &= \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots, x_n)}{\Delta x_i}\end{aligned}$$

   If $f$ is a function of two variables $x$ and $y$, then

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right),$$
$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$$

   (c) Chain Rule for Multivariable Functions

   Suppose $f$ is a differentiable function of $n$ variables $x_1, x_2, \ldots, x_n$ and each $x_i$ is a differentiable functions of $m$ variables $t_1, t_2, \ldots, t_m$. Then $f$ is also a differentiable functions of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots m$.

(d) Implicit Differentiation

Suppose $y = f(x)$ is a function defined implicitly by the equation $F(x, y) = 0$. If $\partial F/\partial y \neq 0$, then

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}.$$

Suppose $z = f(x, y)$ is a function defined implicitly by the equation $F(x, y, z) = 0$. If $\partial F/\partial z \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}, \quad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}.$$

(e) The Gradient Vector

i. Rectangular coordinates $\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$

ii. Cylindrical coordinates $\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$

iii. Spherical coordinates $\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$

(f) Tangent Planes and Normal Vectors to Surfaces

i. The normal line to the surface $F(\mathbf{r}) = c$, where $c$ is any constant, at the point $\mathbf{r}_0 = (x_0, y_0, z_0)$ is the line through $\mathbf{r}_0$ parallel to $\nabla F(\mathbf{r}_0)$.

ii. The tangent plane to the surface $F(\mathbf{r}) = c$, where $c$ is any constant, at the point $\mathbf{r}_0 = (x_0, y_0, z_0)$ is the plane through $\mathbf{r}_0$ perpendicular to $\nabla F(\mathbf{r}_0)$.

(g) Directional Derivatives and Total Differentials

i. The directional derivative of a differentiable function $f(x, y)$ at $(x_0, y_0)$ in the direction of a unit vector $\hat{\mathbf{u}}$ is

$$D_{\hat{\mathbf{u}}}f(\mathbf{r}_0) = \lim_{h\to 0}\frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} = \nabla f(\mathbf{r}_0)\cdot\hat{\mathbf{u}}$$

where $\hat{\mathbf{u}} = a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}}$ and $\sqrt{a^2 + b^2} = 1$.

ii. The increment of a multivariable differentiable function $f(\mathbf{r})$ as $\mathbf{r}_0 \to \mathbf{r}_0 + \Delta\mathbf{r}$ can be approximated by the total differential $df$, i.e.

$$\Delta f = f(\mathbf{r}_0 + \Delta\mathbf{r}) - f(\mathbf{r}_0) \approx df = \nabla f(\mathbf{r}_0) \cdot d\mathbf{r}$$

where $d\mathbf{r} = \Delta\mathbf{r}$ if $\|d\mathbf{r}\|$ is sufficiently small.

(h) Critical Points of Two-variable Functions

Second Derivatives Test

Suppose $(x_0, y_0)$ is a critical point of a two-variable function $f(x, y)$ and $\nabla f(x_0, y_0) = \mathbf{0}$. Let

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2.$$

(1) If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f(x_0, y_0)$ is a local minimum.
(2) If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then $f(x_0, y_0)$ is a local maximum.
(3) If $D < 0$, then $f(x_0, y_0)$ is a saddle point of $f$.
(4) If $D = 0$, then all the previous cases are possible.

2. Multiple Integrals

(a) Double Integrals

i. Definition and physical interpretations

If $f$ is an integrable function defined on a closed, bounded region $R$ in the $xy$-plane, then the double integral of $f$ over $R$ is defined to be:

$$\iint_R f(x, y)\, dxdy = \iint_R f(x, y)\, dA = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\, \Delta A_{ij}$$

ii. Double integrals in rectangular coordinates

Fubini's Theorem (2D cases)

If a function $f$ is continuous over the rectangular region $R : x_1 \leq x \leq x_2, y_1 \leq y \leq y_2$, then

$$\begin{aligned} \iint_R f(x, y)\, dA &= \int_{x_1}^{x_2} \left[ \int_{y_1}^{y_2} f(x, y)\, dy \right] dx \\ &= \int_{y_1}^{y_2} \left[ \int_{x_1}^{x_2} f(x, y)\, dx \right] dy \end{aligned}$$

If a function $f$ is continuous over a non-rectangular region $R$, then

$$\iint_R f(x,y)\,dA = \int_{x_1}^{x_2}\left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right]dx$$

$$\text{if } R := \{x_1 \le x \le x_2, g_1(x) \le y \le g_2(x)\},$$

$$\text{or } \iint_R f(x,y)\,dA = \int_{y_1}^{y_2}\left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right]dy$$

$$\text{if } R := \{h_1(y) \le x \le h_2(y), y_1 \le y \le y_2\}.$$

iii. Double integrals in polar coordinates

If a function $f$ is continuous over the polar region $R$, then

$$\iint_R f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\,rdrd\theta$$

(b) Triple Integrals

i. Definition and physical interpretations

If $f$ is an integrable function defined on a closed, bounded solid region $R$, then the triple integral of $f$ over $R$ is defined to be:

$$\iiint_R f(x,y,z)\,dxdydz = \iiint_R f(x,y,z)\,dV$$

$$= \lim_{n,m,l\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} f(x_i, y_j, z_k)\,\Delta V_{ijk}$$

ii. Triple integrals in rectangular coordinates

Fubini's Theorem (3D cases)

If a function $f$ is continuous over the rectangular region $R: x_1 \le x \le x_2, y_1 \le y \le y_2, z_1 \le z \le z_2$, then

$$\iiint_R f(x,y,z)\,dV = \int_{x_1}^{x_2}\left\{\int_{y_1}^{y_2}\left[\int_{z_1}^{z_2} f(x,y,z)\,dz\right]dy\right\}dx$$

$$= \int_{x_1}^{x_2}\left\{\int_{z_1}^{z_2}\left[\int_{y_1}^{y_2} f(x,y,z)\,dy\right]dz\right\}dx = \dots$$

If a function $f$ is continuous over a non-rectangular region $R := \{x_1 \le x \le x_2, g_1(x) \le y \le g_2(x), h_1(x,y) \le z \le h_2(x,y)\}$, then

$$\iiint_R f(x,y,z)\,dV = \int_{x_1}^{x_2}\left\{\int_{g_1(x)}^{g_2(x)}\left[\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\right\}dy\right]dx$$

If the order of integration is switched, the integral would be given by a similar expression with proper change in the limits of integration.

iii. Triple integrals in cylindrical and spherical coordinates

If a function $f$ is continuous over the cylindrical region $R$, then

$$\iiint_R f(x, y, z)\, dV = \iiint_R f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

If a function $f$ is continuous over the spherical region $R$, then

$$\iiint_R f(x, y, z)\, dV$$
$$= \iiint_R f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\, r^2 \sin\theta\, dr\, d\phi\, d\theta$$

(c) Substitutions in Multiple Integrals

i. Change of variables in double integrals

Suppose a region $S$ in the $uv$-plane is mapped onto the region $R$ in the $xy$-plane by a one-to-one transformation. If $x = g(u, v)$ and $y = h(u, v)$, then

$$\iint_R f(x, y)\, dA = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

where the mapping factor is the Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

ii. Change of variables in triple integrals

Suppose a region $S$ in the $uvw$-space is mapped onto the region $R$ in the $xyz$-space by a one-to-one transformation. If $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$, then

$$\iiint_R f(x, y, z)\, dV$$
$$= \iiint_S f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

where the mapping factor is the Jacobian

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

## ⟨II⟩ Examples

1. The wind-chill index $I$ is the perceived temperature when the actual temperature is $T$ and the wind speed is $v$. So we can write $I = f(T, v)$. Below is the table excerpted from a table of values of $I$ compiled by the National Atmospheric and Oceanic Administration.

| | $v$ (km/hr) | | | | |
|---|---|---|---|---|---|
| $T(°C)$ | 10 | 20 | 30 | 40 | 50 |
| 20 | 18 | 16 | 14 | 13 | 13 |
| 16 | 14 | 11 | 9 | 7 | 7 |
| 12 | 9 | 5 | 3 | 1 | 0 |
| 8 | 5 | 0 | −3 | −5 | −6 |

(a) Estimate the values of $f_T(12, 20)$ and $f_v(12, 20)$.

(b) In general, what can you say about the signs of $\partial I/\partial T$ and $\partial I/\partial v$?

**Solution:**

(a) The derivatives $f_T(12, 20)$ and $f_v(12, 20)$ are the rates of change of $f$ with respect to $T$ and $v$ when $T = 12$ and $v = 20$, i.e.

$$f_T(12, 20) = \lim_{h\to 0}\left[\frac{f(12+h, 20) - f(12, 20)}{h}\right]$$

$$f_v(12, 20) = \lim_{k\to 0}\left[\frac{f(12, 20+k) - f(12, 20)}{k}\right]$$

We can approximate them by taking $h = 4$ and $-4$ as well as $k = 10$ and $-10$ respectively:

$$f_T(12, 20) \approx \frac{f(16, 20) - f(12, 20)}{4} = \frac{11-5}{4} = 1.5$$

$$f_T(12,20) \approx \frac{f(8,20)-f(12,20)}{-4} = \frac{0-5}{-4} = 1.25$$

$$f_v(12,20) \approx \frac{f(12,30)-f(12,20)}{10} = \frac{3-5}{10} = -0.2$$

$$f_v(12,20) \approx \frac{f(12,10)-f(12,20)}{-10} = \frac{9-5}{-10} = -0.4$$

By averaging these values, we get the estimates

$$f_T(12,20) \approx 1.38, \qquad f_v(12,20) \approx -0.3.$$

(b) From the table, we can observe that the value of $I$ decreases as $T$ decreases while $v$ kept constant. However, it decreases as $v$ increases while $T$ kept constant. Thus, in general, the sign of $\partial I/\partial T$ is positive while the sign of $\partial I/\partial v$ is negative.

2. Verify that the function $u = e^{-\alpha^2 k^2 t} \sin kx$ is a solution of the heat conduction equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}.$$

**Solution:**

We first compute the partial derivatives:

$$\frac{\partial u}{\partial t} = \sin kx \left[\frac{d}{dt}\left(e^{-\alpha^2 k^2 t}\right)\right] = -\alpha^2 k^2 e^{-\alpha^2 k^2 t} \sin kx,$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-\alpha^2 k^2 t} \left[\frac{d^2}{dx^2}(\sin kx)\right] = -k^2 e^{-\alpha^2 k^2 t} \sin kx.$$

Obviously, $u$ is a solution of the heat conduction equation.

3. The kinetic energy of a body with mass $m$ and speed $v$ is $K = \frac{1}{2}mv^2$. Show that

$$\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} = K.$$

**Solution:**

We first compute the partial derivatives:

$$\frac{\partial K}{\partial m} = \frac{1}{2}v^2\left[\frac{d}{dm}(m)\right] = \frac{1}{2}v^2, \qquad \frac{\partial^2 K}{\partial v^2} = \frac{1}{2}m\left[\frac{d^2}{dv^2}(v^2)\right] = m.$$

Obviously, $K$ satisfies the equation

$$\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} = K.$$

4. The pressure $P$ (in kilopascals), volume $V$ (in liters), and temperature $T$ (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

   **Solution:**

   Let $t$ be the time elapsed in seconds. Then at the given instant we have $T = 200, dT/dt = 0.1, V = 100, dV/dt = 0.2$ (with $T$ in K and $V$ in L). Since $P = 8.31T/V$, the Chain Rule gives

$$\begin{aligned}\frac{dP}{dt} &= \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt}\\ &= \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31T}{V^2}\frac{dV}{dt}\\ &= \frac{8.31}{100}(0.1) - \frac{8.31(300)}{(100)^2}(0.2)\\ &= -0.0416 \ \ (\text{kPa/s})\end{aligned}$$

   It implies that the pressure is decreasing at a rate of 0.0416 kPa/s.

5. Consider a given quantity of liquid whose pressure $P$, volume $V$, and temperature $T$ satisfy a given "state equation" of the form $F(P, V, T) = 0$. The thermal expansivity $\alpha$ and isothermal compressivity $\beta$ of the liquid are defined by

$$\alpha = \frac{1}{V}\frac{\partial V}{\partial T} \quad \text{and} \quad \beta = -\frac{1}{V}\frac{\partial V}{\partial P}.$$

   Apply implicit differentiation first to calculate $\partial V/\partial P$ and $\partial V/\partial T$, and then to calculate $\partial P/\partial V$ and $\partial P/\partial T$. Deduce from these results that

$$\frac{\partial P}{\partial T} = \frac{\alpha}{\beta}.$$

**Solution:**

Consider $V \equiv V(P,T)$ to be a function defined implicitly by the "state equation" $F(P,V,T)=0$. Applying implicit differentiation yields

$$\frac{\partial V}{\partial P} = -\frac{\partial F/\partial P}{\partial F/\partial V}, \qquad \frac{\partial V}{\partial T} = -\frac{\partial F/\partial T}{\partial F/\partial V}.$$

Next, consider $P \equiv P(V,T)$ to be a function defined implicitly by the "state equation" $F(P,V,T)=0$. Applying implicit differentiation yields

$$\frac{\partial P}{\partial V} = -\frac{\partial F/\partial V}{\partial F/\partial P}, \qquad \frac{\partial P}{\partial T} = -\frac{\partial F/\partial T}{\partial F/\partial P}.$$

Combining the above results, we obtain

$$\frac{\partial P}{\partial T} = -\frac{\partial F/\partial T}{\partial F/\partial P} = -\left(\frac{\partial F/\partial T}{\partial F/\partial V}\right)\Big/\left(\frac{\partial F/\partial P}{\partial F/\partial V}\right) = -\left(\frac{\partial V}{\partial T}\right)\Big/\left(\frac{\partial V}{\partial P}\right) = \frac{\alpha}{\beta}$$

6. Suppose the temperature at a point $(x,y,z)$ in space is given by

$$T(x,y,z) = \frac{80}{(1+x^2+2y^2+3z^2)},$$

where $T$ is measured in degree Celsius and $x$, $y$, $z$ are in meters. In which direction does the temperature increase the fastest at the point $(1,1,-2)$? What is the maximum rate of increase?

**Solution:**

The gradient of $T$ is

$$\nabla T = \frac{\partial T}{\partial x}\hat{\mathbf{i}} + \frac{\partial T}{\partial y}\hat{\mathbf{j}} + \frac{\partial T}{\partial z}\hat{\mathbf{k}} = \frac{160(-x\,\hat{\mathbf{i}} - 2y\,\hat{\mathbf{j}} - 3z\,\hat{\mathbf{k}})}{(1+x^2+2y^2+3z^2)^2}.$$

At the point $(1,1,-2)$, the gradient vector is

$$\nabla T(1,1,-2) = 160(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})/16^2 = 5(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})/8.$$

The directional derivative $D_{\hat{\mathbf{u}}}f(\mathbf{r})$ of a differentiable function $f$ attains the maximum value $\|\nabla f(\mathbf{r})\|$ when $\hat{\mathbf{u}}$ is in the same direction as the gradient vector $\nabla f(\mathbf{r})$. So the temperature increases the fastest in the direction of the gradient vector $\nabla T(1,1,-2) = 5(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})/8$ or, equivalently, in the direction of $-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ or the unit vector $(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})/\sqrt{41}$. The maximum rate of increase is the length of the gradient vector:

$$\|\nabla T(1,1,-2)\| = 5\|-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}\|/8 = 5\sqrt{41}/8 \;\; (^\circ\text{C/m}).$$

7. The volume $V$ (in cubic centimeters) of 1 mole of an ideal gas is given by

$$V = \frac{82.06T}{P}$$

where $P$ is the pressure (in atmospheres) and $T$ is the absolute temperature (in kelvins). Find the approximate change in $V$ when $P$ is increased from 5 atm to 5.2 atm and $T$ is increased from 300 K to 310 K.

**Solution:**

The total differential of $V = V(P, T)$ is

$$dV = \frac{\partial V}{\partial P}\,dP + \frac{\partial V}{\partial T}\,dT = -\frac{82.06T}{P^2}\,dP + \frac{82.06}{P}\,dT.$$

Putting in the given parameters $P = 5$, $T = 300$, $dP = 0.2$, and $dT = 10$ (with $P$ in atm and $T$ in $K$), we obtain

$$dV = -\frac{82.06(300)}{(5)^2}(0.2) + \frac{82.06}{5}(10) = -32.8 \ \ (\mathrm{cm}^3).$$

This indicates that the gas will decrease in volume by about $33\,\mathrm{cm}^3$. Indeed, the actual change is

$$\Delta V = \frac{(82.06)(310)}{5.2} - \frac{(82.06)(300)}{5} = 4892.0 - 4923.6 = -31.6 \ \ (\mathrm{cm}^3).$$

8. A rectangular box without a lid is to be made from $15\,\mathrm{m}^2$ of cardboard. Find the maximum volume of such a box.

**Solution:**

Let the length, width, and height of the box (in meters) be $x$, $y$, and $z$, respectively. Then the volume of the box is

$$V = xyz.$$

We can express $V$ as a function of just two variables $x$ and $y$ by using the fact that the area of four sides and the bottom of the box is

$$xy + 2yz + 2xz = 15.$$

Solving this equation for $z$ yields $z = (15-xy)/[2(x+y)]$. So the expression for $V$ becomes

$$V = xy\left[\frac{(15-xy)}{2(x+y)}\right] = \frac{(15xy - x^2y^2)}{2(x+y)}.$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(15 - 2xy - x^2)}{2(x+y)^2}, \qquad \frac{\partial V}{\partial y} = \frac{x^2(15 - 2xy - y^2)}{2(x+y)^2}.$$

If $V$ reaches the maximum, then $\partial V/\partial x = \partial V/\partial y = 0$. But $x = 0$ or $y = 0$ gives $V = 0$. So we must solve the equations

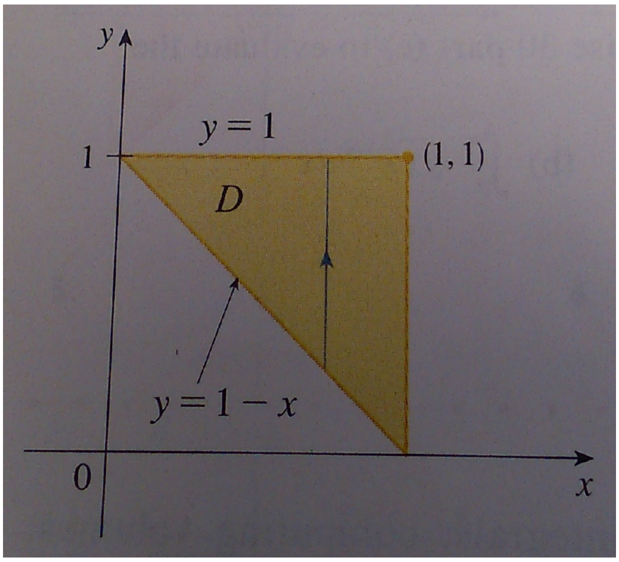
$$15 - 2xy - x^2 = 0 \quad \text{and} \quad 15 - 2xy - y^2 = 0.$$

The solution is $x^2 = y^2$ and so $x = y$. (Note that both $x$ and $y$ must be positive in this problem.) If we put $x = y$ in either equation, we get $15 - 3x^2 = 0$, which gives $x = y = \sqrt{5}$ and $z = \sqrt{5}/2$.

We could use the Second Derivatives Test to show that this gives a local maximum of $V$, which has to occur at a critical point of $V$. So it must occur when $x = y = \sqrt{5}$ and $z = \sqrt{5}/2$. Thus the maximum volume is

$$V = \sqrt{5} \cdot \sqrt{5} \cdot \frac{\sqrt{5}}{2} = \frac{5\sqrt{5}}{2}.$$

9. Charge is distributed over the triangular region $D$ as shown in the figure below. Find the total charge $Q$ in $D$ if the charge density at $(x, y)$ is $\sigma(x, y) = xy$ measured in coulombs per square meter ($\mathrm{C/m^2}$).



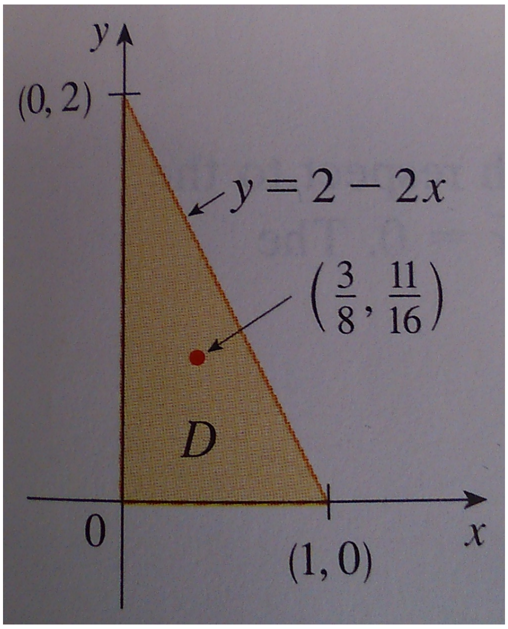
**Solution:**

From the given figure, we can see that the total charge $Q$ is given by

$$\begin{aligned} Q &= \iint_D \sigma(x,y)\,dA \\ &= \int_0^1 \int_{1-x}^1 xy\,dy\,dx \\ &= \int_0^1 x \left[\frac{y^2}{2}\right]_{1-x}^1 dx \\ &= \frac{1}{2}\int_0^1 (2x^2 - x^3)\,dx \\ &= \frac{1}{2}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1 \\ &= \frac{5}{24} \quad \text{(in C)} \end{aligned}$$

10. Find the mass and center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$, $(0,2)$ if the density function is $\sigma(x,y) = 1 + 3x + y$.

**Solution:**




As shown in the above figure, the equation of the upper boundary of the lamina triangular is $y = 2 - 2x$. So its mass is

$$
\begin{aligned}
m &= \iint_D \sigma(x,y)\,dA \\
&= \int_0^1 \int_0^{2-2x} (1+3x+y)\,dy\,dx \\
&= \int_0^1 \left[ y + 3xy + \frac{y^2}{2} \right]_0^{2-2x} dx \\
&= 4\int_0^1 (1-x^2)\,dx \\
&= 4\left[ x - \frac{x^3}{3} \right]_0^1 \\
&= \frac{8}{3}
\end{aligned}
$$

The coordinates $(\bar{x}, \bar{y})$ of the center of mass of the lamina are given by

$$
\begin{aligned}
\bar{x} &= \frac{1}{m}\iint_D x\,\sigma(x,y)\,dA \\
&= \frac{3}{8}\int_0^1 \int_0^{2-2x} (x+3x^2+xy)\,dy\,dx \\
&= \frac{3}{8}\int_0^1 \left[ xy + 3x^2y + \frac{xy^2}{2} \right]_0^{2-2x} dx \\
&= \frac{3}{2}\int_0^1 (x-x^3)\,dx \\
&= \frac{3}{2}\left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 \\
&= \frac{3}{8}
\end{aligned}
$$

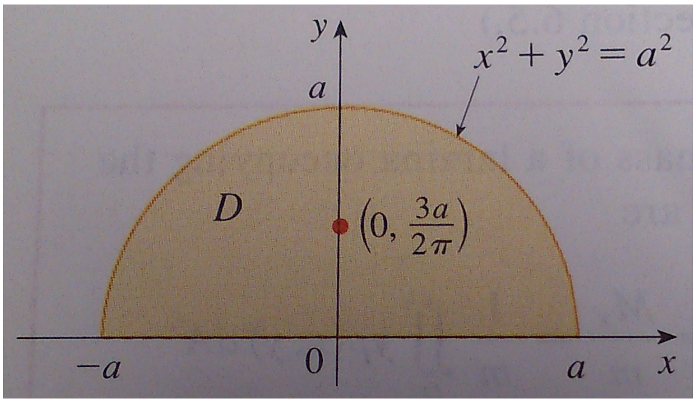
$$
\begin{aligned}
\bar{y} &= \frac{1}{m}\iint_D y\,\sigma(x,y)\,dA \\
&= \frac{3}{8}\int_0^1 \int_0^{2-2x} (y+3xy+y^2)\,dy\,dx \\
&= \frac{3}{8}\int_0^1 \left[ \frac{y^2}{2} + \frac{3xy^2}{2} + \frac{y^3}{3} \right]_0^{2-2x} dx \\
&= \frac{1}{4}\int_0^1 (7-9x-3x^2+5x^3)\,dx
\end{aligned}
$$

$$= \frac{1}{4}\left[7x - \frac{9x^2}{2} - x^3 + \frac{5x^4}{4}\right]_0^1$$
$$= \frac{11}{16}$$

11. The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina using double integral in polar coordinates.

**Solution:**

Let's place the lamina as the upper half of the circle $x^2 + y^2 = a^2$ with center at the origin (see below figure). Then the distance from a point $(x, y)$ to the center of the circle is $\sqrt{x^2 + y^2}$. So the density function is $\sigma(x, y) = K\sqrt{x^2 + y^2}$ for some constant $K$.




In polar coordinates, $\sqrt{x^2 + y^2} = r$ and the region $D$ is described by $0 \le r \le a, 0 \le \theta \le \pi$. Thus, the mass of the lamina is

$$m = \iint_D \sigma(x, y)\, dA$$
$$= \iint_D K\sqrt{x^2 + y^2}\, dA$$
$$= \int_0^{\pi} \int_0^{a} (Kr)\, r\, dr\, d\theta$$
$$= K \int_0^{a} r^2\, dr \int_0^{\pi} d\theta$$
$$= K \left[\frac{r^3}{3}\right]_0^a [\theta]_0^{\pi}$$
$$= \frac{1}{3} K\pi a^3$$

Both the lamina and the density function are symmetric with respect to the $y$-axis. So the center of mass must lie on the $y$-axis, i.e. $\bar{x} = 0$. Besides, the $y$-coordinate of the center of mass is given by

$$\begin{aligned}
\bar{y} &= \frac{1}{m} \iint_D y\, \sigma(x, y)\, dA \\
&= \frac{3}{K\pi a^3} \int_0^{\pi} \int_0^{a} r \sin\theta (Kr)\, r dr d\theta \\
&= \frac{3}{\pi a^3} \int_0^{\pi} \sin\theta\, d\theta \int_0^{a} r^3\, dr \\
&= \frac{3}{\pi a^3} [-\cos\theta]_0^{\pi} \left[\frac{r^4}{4}\right]_0^{a} \\
&= \frac{3a}{2\pi}
\end{aligned}$$

12. Find the moments of inertia about the $x$-axis $I_x$, about the $y$-axis $I_y$, and about the origin $I_0$ of a homogeneous disk $D$ with density $\sigma(x, y) = \sigma$, center at the origin, and radius $a$.

**Solution:**

The boundary of $D$ is the circle $x^2 + y^2 = a^2$ and in polar coordinates $D$ is described by $0 \le r \le a, 0 \le \theta \le 2\pi$. Let's compute $I_0$ first:

$$\begin{aligned}
I_0 &= \iint_D (x^2 + y^2)\, \sigma\, dA \\
&= \sigma \int_0^{2\pi} \int_0^{a} (r^2)\, r\, dr\, d\theta \\
&= \sigma \int_0^{2\pi} d\theta \int_0^{a} r^3\, dr \\
&= \sigma\, [\theta]_0^{2\pi} \left[\frac{r^4}{4}\right]_0^{a} \\
&= \frac{\pi}{2} \sigma a^4
\end{aligned}$$

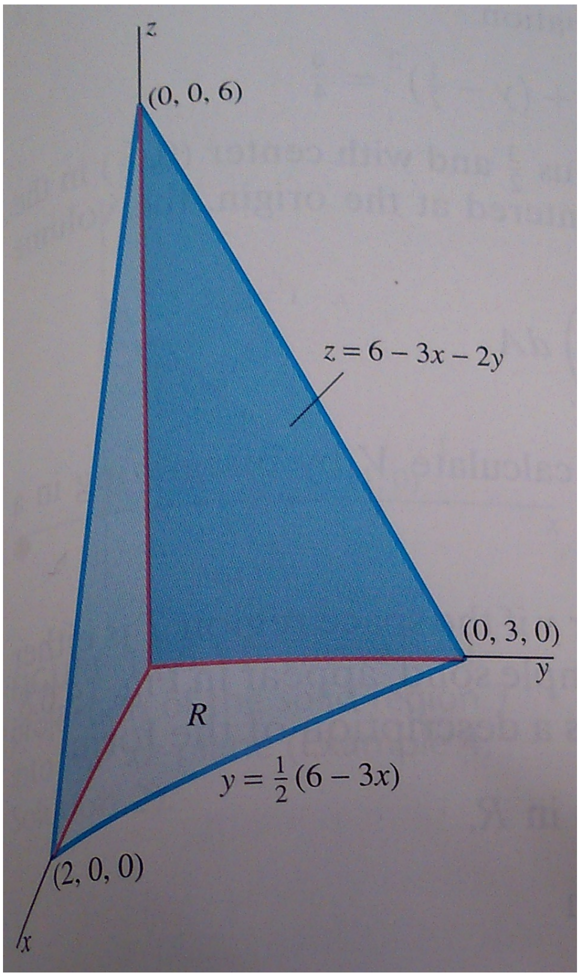
Instead of computing $I_x$ and $I_y$ directly, we use the fact that $I_x + I_y = I_0$ and $I_x = I_y$ due to the symmetry of the disk $D$. Hence,

$$I_x = I_y = I_0/2 = \pi\sigma a^4/4$$

13. A pyramid $T$ is bounded by the coordinate planes and the plane $z = 6 - 3x - 2y$ in the first octant. Use triple integration to find the mass $m$ of $T$ if its mass density function is $\rho(x, y, z) = z$.

**Solution:**

The pyramid $T$ is bounded below by the $xy$-plane and above by the plane $z = 6 - 3x - 2y$. Its base is the plane region $R$ bounded by the $x$- and $y$-axes and the line $y = (6 - 3x)/2$ (see below figure).




Therefore, the mass $m$ of the pyramid $T$ is given by

$$
\begin{aligned}
m &= \iiint_T \rho(x, y, z)\, dV \\
&= \int_0^2 \int_0^{(6-3x)/2} \int_0^{6-3x-2y} z\, dz\, dy\, dx \\
&= \int_0^2 \int_0^{(6-3x)/2} \left[\frac{z^2}{2}\right]_0^{6-3x-2y} dy\, dx
\end{aligned}
$$

$$= \frac{1}{2}\int_0^2 \int_0^{(6-3x)/2} (6-3x-2y)^2 dy\, dx$$
$$= \frac{1}{2}\int_0^2 \left[-\frac{1}{6}(6-3x-2y)^3\right]_0^{(6-3x)/2} dx$$
$$= \frac{1}{12}\int_0^2 (6-3x)^3 dx$$
$$= \frac{1}{12}\left[-\frac{1}{12}(6-3x)^4\right]_0^2$$
$$= 9$$

It can be shown that the coordinates of the centroid $(\bar{x}, \bar{y}, \bar{z})$ of the pyramid $T$ are given by

$$\bar{x} = \frac{1}{m}\iiint_T x\,\rho(x,y,z)\,dV = \frac{1}{9}\int_0^2 \int_0^{(6-3x)/2} \int_0^{6-3x-2y} xz\,dz\,dy\,dx = \frac{2}{5}$$
$$\bar{y} = \frac{1}{m}\iiint_T y\,\rho(x,y,z)\,dV = \frac{1}{9}\int_0^2 \int_0^{(6-3x)/2} \int_0^{6-3x-2y} yz\,dz\,dy\,dx = \frac{3}{5}$$
$$\bar{z} = \frac{1}{m}\iiint_T z\,\rho(x,y,z)\,dV = \frac{1}{9}\int_0^2 \int_0^{(6-3x)/2} \int_0^{6-3x-2y} z^2\,dz\,dy\,dx = \frac{12}{5}$$

14. Use triple integration to compute the volume of the region $T$ bounded by the parabolic cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$. Also find the center of mass of $T$ if it has constant density $\rho = 1$.

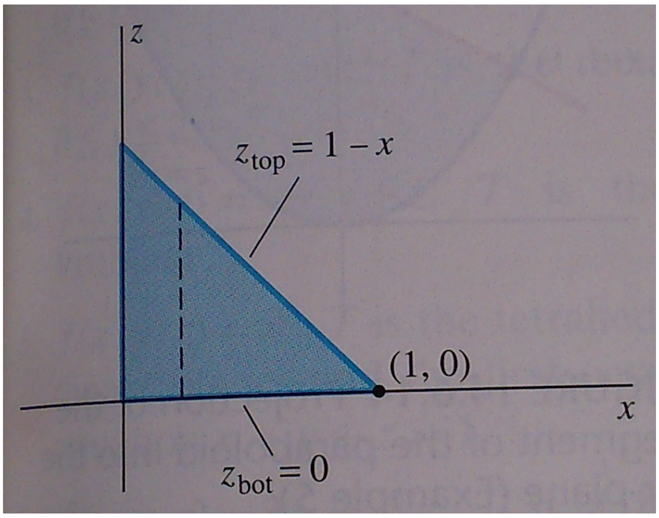
**Solution:**

The integral for the volume $V$ of the region $T$ can be evaluated in any order. Here we shows three ways to do so.

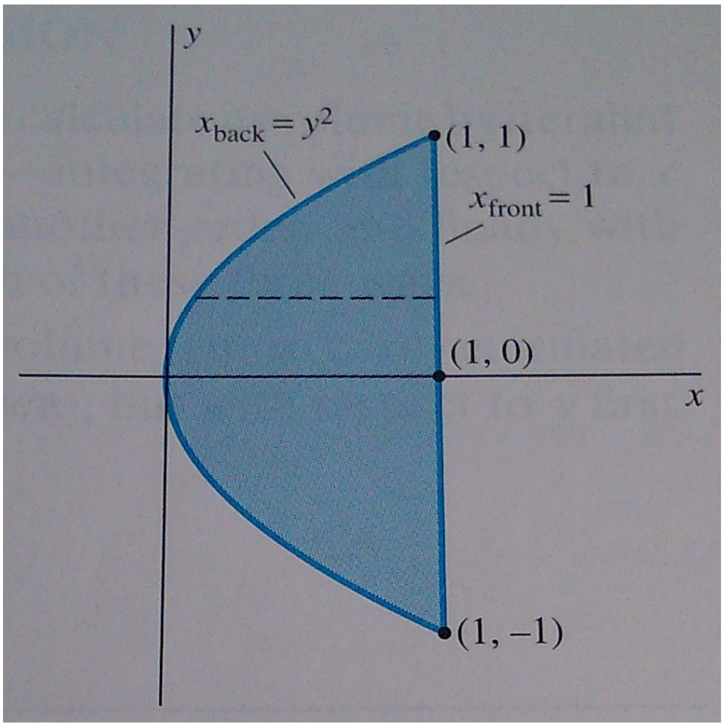
Method 1: The projection of $T$ on the $xz$-plane is the triangle bounded by the coordinate axes and the line $x + z = 1$ (see the figure on the next page). So the volume of $T$ is given by

$$V = \iiint_T dV$$
$$= \int_0^1 \int_0^{1-x} \int_{-\sqrt{x}}^{\sqrt{x}} dy\,dz\,dx$$
$$= 2\int_0^1 \int_0^{1-x} \sqrt{x}\,dz\,dx$$

$$
\begin{aligned}
&= 2\int_0^1 \sqrt{x}(1-x)\,dx \\
&= 2\left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^1 \\
&= \frac{8}{15}
\end{aligned}
$$



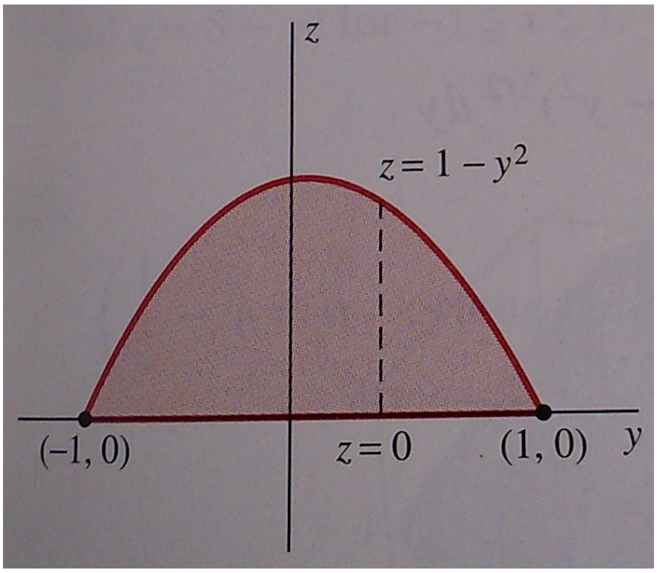

Method 2: The projection of $T$ on the $xy$-plane is the region bounded by the parabola $x = y^2$ and the line $x = 1$ (see below figure).



So the volume of $T$ is given by

$$
\begin{aligned}
V &= \iiint_T dV \\
&= \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{1-x} dz\, dx\, dy \\
&= \int_{-1}^{1} \int_{y^2}^{1} (1-x)\, dx\, dy \\
&= \int_{-1}^{1} \left[ x - \frac{x^2}{2} \right]_{y^2}^{1} dy \\
&= \int_{-1}^{1} \left( \frac{1}{2} - y^2 + \frac{y^4}{2} \right) dy \\
&= \left[ \frac{y}{2} - \frac{y^3}{3} + \frac{y^5}{10} \right]_{-1}^{1} \\
&= \frac{8}{15}
\end{aligned}
$$

Method 3: The projection of $T$ on the $yz$-plane is the triangle bounded by the $y$-axis and the parabola $z = 1 - y^2$ (see below figure).




So the volume of $T$ is given by

$$V = \iiint_T dV = \int_{-1}^{1} \int_{0}^{1-y^2} \int_{y^2}^{1-z} dx\, dz\, dy = \ldots = \frac{8}{15}$$

Next we calculate the center of mass of $T$. Since the region $T$ is symmetric with respect to the $xz$-plane, its center of mass lies in this plane and so $\bar{y} = 0$. We compute $\bar{x}$ and $\bar{z}$ by first integrating with respect to $y$:

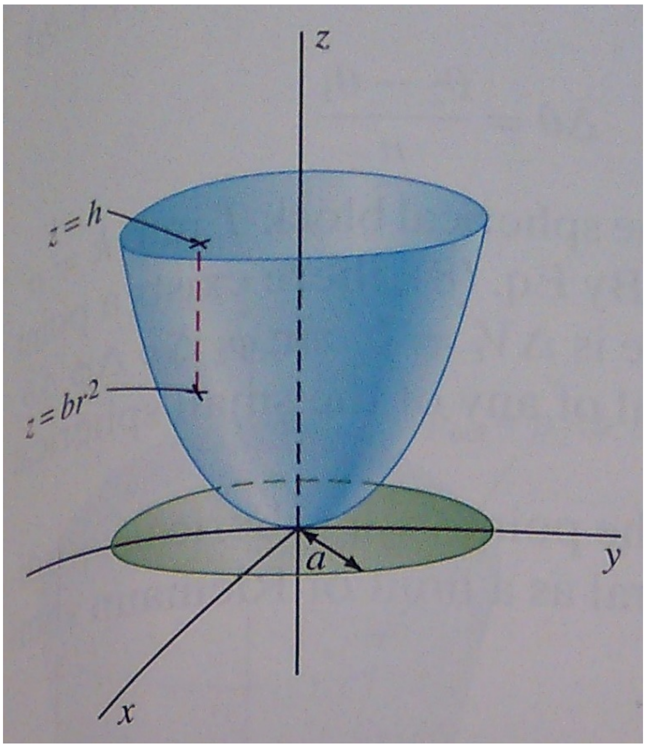
$$\begin{aligned}
\bar{x} &= \frac{1}{\rho V} \iiint_T x\,\rho\,dV \\
&= \frac{15}{8} \int_0^1 \int_0^{1-x} \int_{-\sqrt{x}}^{\sqrt{x}} x\,dy\,dz\,dx \\
&= \frac{15}{4} \int_0^1 \int_0^{1-x} x^{3/2}\,dz\,dx \\
&= \frac{15}{4} \int_0^1 x^{3/2}(1-x)\,dx \\
&= \frac{15}{4} \left[\frac{2}{5}x^{5/2} - \frac{2}{7}x^{7/2}\right]_0^1 \\
&= \frac{3}{7}
\end{aligned}$$

$$\begin{aligned}
\bar{z} &= \frac{1}{\rho V} \iiint_T z\,\rho\,dV \\
&= \frac{15}{8} \int_0^1 \int_0^{1-x} \int_{-\sqrt{x}}^{\sqrt{x}} z\,dy\,dz\,dx \\
&= \frac{15}{4} \int_0^1 \int_0^{1-x} \sqrt{x}z\,dzdx \\
&= \frac{15}{4} \int_0^1 \sqrt{x}\left[\frac{z^2}{2}\right]_0^{1-x} dx \\
&= \frac{15}{8} \int_0^1 \sqrt{x}(1-x)^2 dx \\
&= \frac{15}{8} \left[\frac{2}{3}x^{3/2} - \frac{4}{5}x^{5/2} + \frac{2}{7}x^{7/2}\right]_0^1 \\
&= \frac{2}{7}
\end{aligned}$$

15. By triple integration in cylindrical coordinates, find the volume and center of mass of the solid $T$ with constant density $\rho$ that is bounded by the paraboloid $z = b(x^2 + y^2)$ $(b > 0)$ and the plane $z = h$ $(h > 0)$.

**Solution:**

As shown in the figure below, we can find the radius $a$ of the circular top of $T$ by equating $z = b(x^2 + y^2) = br^2$ and $z = h$. This gives $a = \sqrt{h/b}$ which is the radius of the circle over which the solid lies.




Using cylindrical coordinates, the volume of the solid $T$ is then given by

$$
\begin{aligned}
V &= \iiint_T dV \\
&= \int_0^{2\pi} \int_0^a \int_{br^2}^h r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^a r(h - br^2)\,dr\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^a (hr - br^3)\,dr \\
&= [\theta]_0^{2\pi} \left[\frac{hr^2}{2} - \frac{br^4}{4}\right]_0^a \\
&= \frac{\pi}{2} a^2 h
\end{aligned}
$$

By symmetry, the center of mass of the solid $T$ lies on the $z$-axis. So all that remains is to compute $\bar{z}$:

$$\begin{aligned}
\bar{z} &= \frac{1}{\rho V} \iiint_T z\,\rho\,dV \\
&= \frac{2}{\pi a^2 h} \int_0^{2\pi} \int_0^a \int_{br^2}^h rz\,dz\,dr\,d\theta \\
&= \frac{2}{\pi a^2 h} \int_0^{2\pi} \int_0^a r \left[\frac{z^2}{2}\right]_{br^2}^h dr\,d\theta \\
&= \frac{2}{\pi a^2 h} \int_0^{2\pi} \int_0^a r \left(\frac{h^2}{2} - \frac{b^2 r^4}{2}\right) dr\,d\theta \\
&= \frac{2}{\pi a^2 h} \int_0^{2\pi} d\theta \int_0^a \left(\frac{h^2 r}{2} - \frac{b^2 r^5}{2}\right) dr \\
&= \frac{2}{\pi a^2 h} [\theta]_0^{2\pi} \left[\frac{h^2 r^2}{4} - \frac{b^2 r^6}{12}\right]_0^a \\
&= \frac{2}{3} h
\end{aligned}$$

16. A solid sphere $T$ with constant density $\rho$ is bounded by the spherical surface described by the equation $r = a$. Use triple integration in spherical coordinates to compute its mass and its moment of inertia around the $z$-axis.

**Solution:**

The points of the sphere $T$ are described by the inequalities: $0 \leq r \leq a,\ 0 \leq \theta \leq \pi,\ 0 \leq \phi \leq 2\pi$. So its mass is given by:

$$\begin{aligned}
m &= \iiint_T \rho\,dV \\
&= \rho \int_0^{2\pi} \int_0^{\pi} \int_0^a r^2 \sin\theta\,dr\,d\theta\,d\phi \\
&= \rho \left(\int_0^a r^2\,dr\right) \left(\int_0^{\pi} \sin\theta\,d\theta\right) \left(\int_0^{2\pi} d\phi\right) \\
&= \rho \left[\frac{r^3}{3}\right]_0^a [-\cos\theta]_0^{\pi} [\phi]_0^{2\pi} \\
&= \frac{4\pi}{3} \rho a^3
\end{aligned}$$

The distance from any point $(r, \theta, \phi)$ of the sphere to the $z$-axis is $r \sin\theta$. So its moment of inertia around that axis is

$$
\begin{aligned}
I_z &= \iiint_T r^2 \sin^2\theta \, \rho \, dV \\
&= \rho \int_0^{2\pi} \int_0^{\pi} \int_0^{a} r^4 \sin^3\theta \, dr \, d\theta \, d\phi \\
&= \rho \left( \int_0^a r^4 \, dr \right) \left( \int_0^{\pi} \sin^3\theta \, d\theta \right) \left( \int_0^{2\pi} d\phi \right) \\
&= \frac{\rho a^5}{5} \left[ \frac{r^5}{5} \right]_0^a \left[ \frac{1}{3}\cos^3\theta - \cos\theta \right]_0^{\pi} [\phi]_0^{2\pi} \\
&= \frac{8\pi}{15} \rho a^5 = \frac{2}{5} m a^2
\end{aligned}
$$

## ⟨III⟩ Problems

1. The wave height $h$ in the open sea depends on the speed of the wind $v$ and the length of time $t$ that the wind has been blowing at that speed. The values of the function $h = f(v, t)$ are recorded in feet in the table below.

| | Duration $t$ (hours) | | | | | | |
|---|---|---|---|---|---|---|---|
| $v$ (knots) | 5 | 10 | 15 | 20 | 30 | 40 | 50 |
| 10 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 20 | 5 | 7 | 8 | 8 | 9 | 9 | 9 |
| 30 | 9 | 13 | 16 | 17 | 18 | 19 | 19 |
| 40 | 14 | 21 | 25 | 28 | 31 | 33 | 33 |
| 50 | 19 | 29 | 36 | 40 | 45 | 48 | 50 |
| 60 | 24 | 37 | 47 | 54 | 62 | 67 | 69 |

(a) What are the meanings of the partial derivatives $\partial h/\partial v$ and $\partial h/\partial t$?

(b) Estimate the values of $f_v(40, 15)$ and $f_h(40, 15)$.

(c) What appears to be the value of $\lim_{t\to\infty} \partial h/\partial t$?

2. Show that the functions

$$u_1(x,t) = \sin(x - at) + \ln(x + at) \quad \text{and} \quad u_2(x,t) = \frac{t}{(a^2t^2 - x^2)}$$

are solutions of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

3. Show that the function

$$u(x,y,t) = \frac{1}{4\pi kt} \exp\left[-\frac{(x^2 + y^2)}{4kt}\right]$$

satisfies the two-dimensional heat equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

4. If $R$ is the total resistance of three resistors with resistances $R_1$, $R_2$, $R_3$ connected in parallel, then

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

If the resistances are measured in ohms as $R_1 = 25\,\Omega$, $R_2 = 40\,\Omega$, and $R_3 = 50\,\Omega$, with a possible error of 0.5% in each quantity, estimate the maximum error in the calculated value of $R$.

5. In a simple electrical circuit, the voltage $V$ is slowly decreasing as the battery wears out while the resistance $R$ is slowly increasing as the resistor heats up. Use Ohm's law $V = IR$ to find the rate of change of the current $I$ at the moment $R = 400\,\Omega$, $I = 0.08\,\text{A}$, $dV/dt = -0.01\,\text{V/s}$, and $dR/dt = 0.03\,\Omega/\text{s}$.

6. Suppose that over a certain region of space the electrical potential $V$ is given by

$$V(x,y,z) = 5x^2 - 3xy + xyz.$$

(a) Find the rate of change of $V$ at $P(3,4,5)$ in the direction of the vector $\mathbf{v} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$.

(b) In which direction does $V$ change most rapidly at $P$?

(c) What is the maximum rate of change of $V$ at $P$?

7. (a) Show that

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

where the symbol $(\partial w/\partial u)_v$ denotes the partial derivative of $w$, which is regarded as a function of the independent variables $u$ and $v$, with respect to $u$.

(b) Verify the result of part (a) for the equation

$$F(P, V, T) = PV - nRT = 0$$

($n$ and $R$ are constants), which expresses the ideal gas law.

8. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.

9. Electric charge is distributed over the unit disk $x^2 + y^2 \leq 1$ so that the charge density at the point $(x, y)$ is $\sigma(x, y) = 1 + x^2 + y^2$ (measured in coulombs per square meter). Find the total charge on the disk.

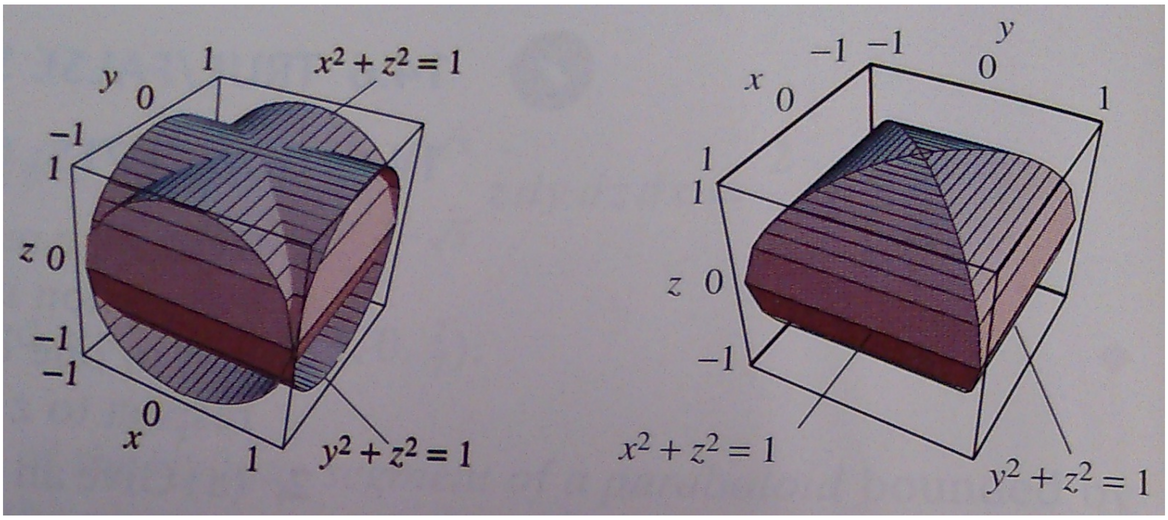
10. Find the mass and center of mass of the lamina occupying the triangular region $D$ with vertices $(0, 0)$, $(2, 1)$, $(0, 3)$ which has mass density function $\sigma(x, y) = x + y$.

11. A lamina occupies the region $D$ that is bounded by the parabola $x = y^2$ and the line $y = x - 2$. Find its mass and center of mass if the mass density function $\sigma(x, y) = x$.

12. By double integration in polar coordinates, find the moment of inertia $I_x$, $I_y$, $I_z$ for a lamina that occupies the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant. Assume that the density at any point is proportional to the square of its distance from the origin.

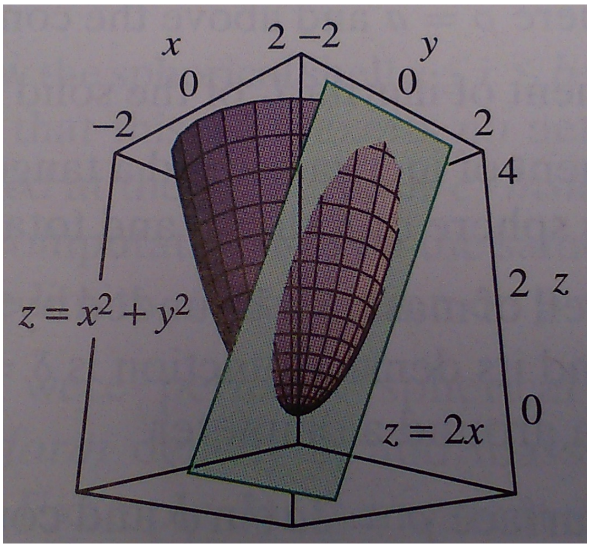
13. Find the center of mass of the first-octant region that is interior to the

two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ as shown in the figure below. Assume the region has uniform density $\rho = 1$.




14. The cube bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$ has density $\rho = kz$ at the point $(x, y, z)$ ($k$ is a positive constant). Find its center of mass and its moment of inertia around the $z$-axis.

15. By triple integration in cylindrical coordinates, find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 2x$ (see below figure).




16. By triple integration in spherical coordinates, find the moment of inertia around the $z$-axis of the region that lies inside both the cylinder $x^2 + y^2 =$

$a^2$ and the sphere $x^2 + y^2 + z^2 = 4a^2$. Assume the region has uniform density $\rho = 1$.