Essential Maths for Physics Students Class B
Chapter 3 Vector Analysis

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⟨I⟩ Summary

1. Differential Operations on Scalar and Vector Fields

   (a) Definitions and Interpretations

   A scalar field in the plane or in space is a scalar function that assigns a scalar to every point in its domain. Similarly, a vector field in the plane or in space is a function that assigns a vector to every point in its domain.

   Let $\phi(x, y, z)$ be a scalar field which is a differentiable function of x, y, z and $\mathbf{F}(x, y, z) = P(x, y, z)\hat{\mathbf{i}} + Q(x, y, z)\hat{\mathbf{j}} + R(x, y, z)\hat{\mathbf{k}}$ be a vector field where P, Q, R are differentiable functions of x, y, z.

   i. Divergence: $\nabla \cdot \mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$.

   It measures the "outgoingness" of $\mathbf{F}$ from a given point.

   ii. Curl:

$$\nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$$

   It measures the "circulation" of $\mathbf{F}$ around a given point in a direction given by the right hand rule.

   iii. Laplacian:

$$\begin{aligned} \nabla^2 \phi &= \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}, \\ \nabla^2 \mathbf{F} &= \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}) \\ &= \nabla^2 P\,\hat{\mathbf{i}} + \nabla^2 Q\,\hat{\mathbf{j}} + \nabla^2 R\,\hat{\mathbf{k}} \end{aligned}$$

   (b) Important Formulas for the Del Operator ∇

$$\begin{aligned} \nabla \cdot (\phi \mathbf{F}) &= \phi(\nabla \cdot \mathbf{F}) + \nabla \phi \cdot \mathbf{F} \\ \nabla \times (\phi \mathbf{F}) &= \phi(\nabla \times \mathbf{F}) + (\nabla \phi) \times \mathbf{F} \\ \nabla \cdot (\mathbf{F} \times \mathbf{G}) &= \mathbf{G} \cdot (\nabla \times \mathbf{F}) + \mathbf{F} \cdot (\nabla \times \mathbf{G}) \\ \nabla \times (\mathbf{F} \times \mathbf{G}) &= (\mathbf{G} \cdot \nabla)\mathbf{F} - \mathbf{G}(\nabla \cdot \mathbf{F}) - (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F}(\nabla \cdot \mathbf{G}) \end{aligned}$$

$$\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{G}\cdot\nabla)\mathbf{F} + (\mathbf{F}\cdot\nabla)\mathbf{G} + \mathbf{G}\times(\nabla\times\mathbf{F}) + \mathbf{F}\times(\nabla\times\mathbf{G})$$
$$\nabla\times(\nabla\phi) = \mathbf{0}$$
$$\nabla\cdot(\nabla\times\mathbf{F}) = 0$$
$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$$

(c) Gradient, Divergence, and Curl in Curvilinear Coordinates

i. Cylindrical coordinates

$$\nabla T = \frac{\partial T}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial T}{\partial z}\hat{\boldsymbol{z}}$$
$$\nabla\cdot\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$
$$\nabla\times\mathbf{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{\boldsymbol{r}} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial \theta}\right]\hat{\boldsymbol{z}}$$

ii. Spherical coordinates

$$\nabla T = \frac{\partial T}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}$$
$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$
$$\nabla\times\mathbf{F} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(F_\phi\sin\theta) - \frac{\partial F_\theta}{\partial \phi}\right]\hat{\boldsymbol{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r}(rF_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

2. Line, Surface, and Volume Integrals

(a) Line Integrals

i. Definitions

Let C be a smooth oriented curve in space parametrized by $\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$ for $a \le t \le b$ with unit tangent vector $\hat{\mathbf{T}}(t)$.

If $f(x, y, z)$ is a continuous scalar function defined on the curve C, then the line integral of f along C with respect to arc length is

$$\begin{aligned}&\int_C f(x,y,z)\,ds\\&=\int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt\end{aligned}$$

If $\mathbf{F}(x,y,z) = P(x,y,z)\hat{\mathbf{i}} + Q(x,y,z)\hat{\mathbf{j}} + R(x,y,z)\hat{\mathbf{k}}$ is a continuous vector field defined on the curve C, then the line integral of $\mathbf{F}$ along C is

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\hat{\mathbf{T}}\,ds &= \int_a^b (\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt\\&=\int_C (P\,dx + Q\,dy + R\,dz)\end{aligned}$$

We can generalize the above definitions for the case that C is a piecewise-smooth curve.

ii. Independence of Path

If $\mathbf{F}$ is a continuous vector field on an open region D, then the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is said to be independent of path in D if it has the same value for any piecewise-smooth curve C lying in D with the same initial point A and terminal point B.

Alternatively, if a $\mathbf{F}$ is a continuous vector field on an open connected region D, then the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D if and only if $\mathbf{F}$ is a conservative vector field, i.e. there exists a potential function ϕ such that $\mathbf{F} = \nabla\phi$. In such case, the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \phi(x_B,y_B,z_B) - \phi(x_A,y_A,z_A)$$

for any piecewise-smooth curve C in D with initial point (x_A,y_A,z_A) and terminal point (x_B,y_B,z_B). Thus the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D if and only if it is equal to zero for every closed path C lying in D.

(b) Surface Integrals

If $f(x,y,z)$ is a continuous scalar field defined on a smooth surface S in space given by $z = g(x,y)$ whose projection onto the xy-plane is D, then the surface integral of f over S is

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\,dA$$

If $\mathbf{F}(x,y,z) = P(x,y,z)\hat{\mathbf{i}} + Q(x,y,z)\hat{\mathbf{j}} + R(x,y,z)\hat{\mathbf{k}}$ is a continuous vector field defined on a smooth oriented surface S in space given by $z = g(x,y)$ whose projection onto the xy-plane is D, then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)\,dA$$

where $\hat{\mathbf{n}}(x,y,z)$ is the unit normal vector to S.

(c) Volume Integrals

If $\mathbf{F}(x,y,z) = P(x,y,z)\hat{\mathbf{i}} + Q(x,y,z)\hat{\mathbf{j}} + R(x,y,z)\hat{\mathbf{k}}$ is a continuous vector field defined in a solid region E, then the volume integral of $\mathbf{F}$ over E is

$$\iiint_E \mathbf{F}\,dV = \left(\iiint_E P\,dV\right)\hat{\mathbf{i}} + \left(\iiint_E Q\,dV\right)\hat{\mathbf{j}} + \left(\iiint_E R\,dV\right)\hat{\mathbf{k}}$$

3. Fundamental Theorems of Vector Calculus

(a) Green's Theorem

Let C be a piecewise-smooth, simple closed curve in the plane with positive orientation and D be the region enclosed by C. If $P(x,y)$ and $Q(x,y)$ have continuous first-order partial derivatives on an open region that contains D, then

$$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA$$

(b) Divergence Theorem

Let E be a simple solid region bounded by a closed piecewise-smooth surface S that has a outward unit normal vector $\hat{\mathbf{n}}$. If $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region in space that contains E, then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_E \nabla\cdot\mathbf{F}\,dV$$

(c) Stokes' Theorem

Let S be an oriented piecewise-smooth surface with unit normal vector $\hat{\mathbf{n}}$ that is bounded by a piecewise-smooth, simple closed curve C with positive orientation. If $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region in space that contains S, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$$

⟨II⟩ Examples

1. Suppose that $\mathbf{F}(x, y) = x^2\hat{\mathbf{i}} + (y^2 - 4x)\hat{\mathbf{j}}$ represents the velocity field of a fluid in motion. For a small box centered at (x, y), determine whether the flow into the box is greater than, less than, or equal to the flow out of the box. (a) $(x, y) = (0, 0)$ and (b) $(x, y) = (1, 0)$.

Solution:

Recall that the divergence of the velocity field of a fluid at a point corresponds to the net flow of fluid per unit volume out of a small box centered at this point. For the given velocity field $\mathbf{F}(x, y)$,

$$\nabla \cdot \mathbf{F}(0,0) = \left.\frac{\partial(x^2)}{\partial x} + \frac{\partial(y^2 - 4x)}{\partial y}\right|_{(0,0)} = 2(x+y)|_{(0,0)} = 0$$

$$\nabla \cdot \mathbf{F}(1,0) = \left.\frac{\partial(x^2)}{\partial x} + \frac{\partial(y^2 - 4x)}{\partial y}\right|_{(1,0)} = 2(x+y)|_{(1,0)} = 2$$

As $\nabla \cdot \mathbf{F}(0,0) = 0$, the flow into the box is equal to the flow out of the box centered at $(0, 0)$. Besides, as $\nabla \cdot \mathbf{F}(1, 0) > 0$, the flow into the box is less than the flow out of the box centered at $(1, 0)$.

2. Gauss's law states that

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

where $\mathbf{E}$ is the electric field, ρ is the charge density, and ϵ_0 is the permittivity of free space. If $\mathbf{E}$ has a potential function $-\phi$, derive Poisson's equation:

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}.$$

Solution:

If $\mathbf{E}$ has a potential function $-\phi$, then $\mathbf{E} = -\nabla\phi$. Therefore,

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= -\nabla \cdot (\nabla\phi) \\
&= -\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{k}}\right) \\
&= -\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) \\
&= -\nabla^2\phi
\end{aligned}$$

Plugging it back into Gauss's law yields Poisson's equation.

3. In a steady flow, a fluid moves with velocity $\mathbf{u}$ and vorticity $\mathbf{w}$ through a medium with density ρ. Suppose there is an external force, such as gravity, with a potential function ϕ and the fluid pressure is given by the scalar function p. In this case, Euler's equation states that

$$\mathbf{w} \times \mathbf{u} + \frac{1}{2}\nabla u^2 = -\frac{1}{\rho}\nabla p - \nabla\phi.$$

Bernoulli's Theorem then says that

$$\frac{1}{2}u^2 + \phi + \frac{p}{\rho} = \text{constant}$$

at every point along a flow line. Use Euler's equation to derive Bernoulli's Theorem.

Solution:

The flow lines must be tangent to the velocity field. So we can start by finding the dot product of a vector function with velocity in order to compute the component of the function along a flow line. In this case, we take Euler's equation and find the dot product of each term with $\mathbf{u}$. Then we obtain

$$\begin{aligned}
&\mathbf{u} \cdot (\mathbf{w} \times \mathbf{u}) + \mathbf{u} \cdot \left(\frac{1}{2}\nabla u^2\right) = -\mathbf{u} \cdot \left(\frac{1}{\rho}\nabla p\right) - \mathbf{u} \cdot (\nabla\phi) \\
\Rightarrow\ &\mathbf{u} \cdot (\mathbf{w} \times \mathbf{u}) + \mathbf{u} \cdot \left(\frac{1}{2}\nabla u^2\right) + \mathbf{u} \cdot (\nabla\phi) + \mathbf{u} \cdot \left(\frac{1}{\rho}\nabla p\right) = 0
\end{aligned}$$

Notice that $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{u})=0$ since the cross product $\mathbf{w}\times\mathbf{u}$ is perpendicular to $\mathbf{u}$. All three remaining terms are gradients. So factoring out the scalar functions involved, we have

$$\mathbf{u}\cdot\nabla\left(\frac{1}{2}u^2+\phi+\frac{p}{\rho}\right)=0.$$

This says that the component of $\nabla(u^2/2+\phi+p/\rho)$ along $\mathbf{u}$ is zero in the direction of the tangent to the flow lines. This gives us the Bernoulli's Theorem that $u^2/2+\phi+p/\rho=$ constant at every point along a flow line.

4. Compute the work done by the force field $\mathbf{F}(x,y,z)=4y\hat{\mathbf{i}}+2xz\hat{\mathbf{j}}+3y\hat{\mathbf{k}}$ acting on an object as it moves along the helix C defined parametrically by $\mathbf{r}(t)=2\cos t\,\hat{\mathbf{i}}+2\sin t\,\hat{\mathbf{j}}+3t\,\hat{\mathbf{k}}$ from the point $(2,0,0)$ to the point $(-2,0,3\pi)$.

Solution:

The work done is given by the line integral of $\mathbf{F}$ along the helix C:

$$W=\int_C\mathbf{F}\cdot d\mathbf{r}=\int_C\left(4y\,dx+2xz\,dy+3y\,dz\right).$$

We have been already given the parametric equations for the curve, but not the range of t-values. Noting $z=3t$, we can determine that $(2,0,0)$ corresponds to $t=0$ and $(-2,0,3\pi)$ corresponds to $t=\pi$. Substituting in x, y, z and

$$dx=-2\sin t\,dt,\quad dy=2\cos t\,dt,\quad dz=3\,dt,$$

we obtain

$$\begin{aligned}
W&=\int_C(4y\,dx+2xz\,dy+3y\,dz)\\
&=\int_0^\pi\Big[\underbrace{4(2\sin t)}_{4y}\underbrace{(-2\sin t)}_{x'(t)}+\underbrace{2(2\cos t)(3t)}_{2xz}\underbrace{(2\cos t)}_{y'(t)}+\underbrace{3(2\sin t)}_{3y}\underbrace{(3)}_{z'(t)}\Big]dt\\
&=\int_0^\pi(-16\sin^2 t+24t\cos^2 t+18\sin t)dt\\
&=\int_0^\pi(18\sin t+8\cos 2t+12t\cos 2t-8+12t)dt\\
&=\left[-18\cos t+4\sin 2t+3\cos 2t+6t\sin 2t-8t+6t^2\right]_0^\pi\\
&=36-8\pi+6\pi^2
\end{aligned}$$

5. Compute the mass m of a rod with linear mass density $\lambda(x, y) = x$ in the shape of $y = x^2$ where $0 \leq x \leq 3$. (Hint: Use the formula $m = \int_C \lambda(x, y)\, ds$.)

   Solution:

   Taking $x = t$ as the parameter, we can write the parametric equations for the shape of the rod as $\mathbf{r}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}$ for $0 \leq t \leq 3$. So the arc length element ds is given by

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \sqrt{1^2 + (2t)^2}\, dt = \sqrt{4t^2 + 1}\, dt$$

   Hence, the mass of the rod is given by

$$m = \int_C \lambda(x, y) ds = \int_0^3 \underbrace{t}_{\lambda(x,y)} \underbrace{\sqrt{4t^2 + 1} dt}_{ds} = \left[\frac{1}{12}(4t^2 + 1)^{3/2}\right]_0^3 = \frac{(37^{3/2} - 1)}{12}$$

6. Show that the work done $W = \int_C \mathbf{F} \cdot d\mathbf{s}$ by the force field $\mathbf{F}(x, y) = (2xy - 3)\hat{\mathbf{i}} + (x^2 + 4y^3 + 5)\hat{\mathbf{j}}$ is independent of path, which implies that $\mathbf{F}$ is conservative. Then evaluate the work done by $\mathbf{F}$ for any curve C from $(-1, 2)$ to $(2, 3)$.

   Solution:

   The line integral of the vector field $\mathbf{F}$ is independent of path if and only if there exists a potential function ϕ such that $\mathbf{F} = \nabla\phi$. In other words, we look for a potential function $\phi(x, y)$ for which

$$\mathbf{F}(x, y) = (2xy - 3)\hat{\mathbf{i}} + (x^2 + 4y^3 + 5)\hat{\mathbf{j}} = \nabla\phi(x, y).$$

   Of course, this occurs when

$$\phi_x = 2xy - 3 \quad \text{and} \quad \phi_y = x^2 + 4y^3 + 5.$$

   Integrating the first of these two equations with respect to x gives

$$\phi(x, y) = \int (2xy - 3) dx = x^2 y - 3x + g(y)$$

   where $g(y)$ is some arbitrary function of y alone. Differentiating with respect to y, we obtain

$$\phi_y = x^2 + g'(y).$$

We already have an expression for ϕ_y in the above. Setting these two expressions equal, we get

$$\begin{aligned} x^2 + g'(y) &= x^2 + 4y^3 + 5 \\ \Rightarrow \ g'(y) &= 4y^3 + 5. \end{aligned}$$

Integrating this expression with respect to y gives

$$g(y) = y^4 + 5y + c$$

where c is a constant. So we have found that

$$\phi(x, y) = x^2 y - 3x + y^4 + 5y + c$$

is a potential function for $\mathbf{F}(x, y)$ where c is any constant. Hence, for any path C from $(-1, 2)$ to $(2, 3)$, the work done is equal to

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \phi(2, 3) - \phi(-1, 2) \\ &= [2^2(3) - 3(2) + 3^4 + 5(3) + c] - [(-1)^2 2 - 3(-1) + 2^4 + 5(2) + c] \\ &= 71 \end{aligned}$$

7. According to Coulomb's law, the electric force exerted by a unit charge at the origin on a charge q at the point (x, y, z) is

$$\mathbf{F}(x, y, z) = \frac{kq}{r^3}\boldsymbol{r}$$

where $\boldsymbol{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, $r = \sqrt{x^2 + y^2 + z^2}$, and k is a constant. Show that the work done by $\mathbf{F}$ in moving the charge q from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is equal to

$$W = \left(\frac{kq}{r_1} - \frac{kq}{r_2} \right)$$

where $r_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$ and $r_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}$.

Solution:

Notice that the Coulomb's force $\mathbf{F}$ is conservative since

$$\mathbf{F} = \frac{kq}{r^3}\boldsymbol{r} = \frac{kq}{r^2}\hat{\boldsymbol{r}} = \nabla\left(-\frac{kq}{r}\right)$$

It implies that the work done by the Coulomb's force $\mathbf{F}$ is independent of path. The work done by $\mathbf{F}$ in moving the charge q from P_1 to P_2 is thus given by

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{r_1}^{r_2} \frac{kq}{r^2}\hat{\mathbf{r}} \cdot dr\hat{\mathbf{r}} = \int_{r_1}^{r_2} \frac{kq}{r^2}\, dr = \left[-\frac{kq}{r}\right]_{r_1}^{r_2} = \left(\frac{kq}{r_1} - \frac{kq}{r_2}\right)$$

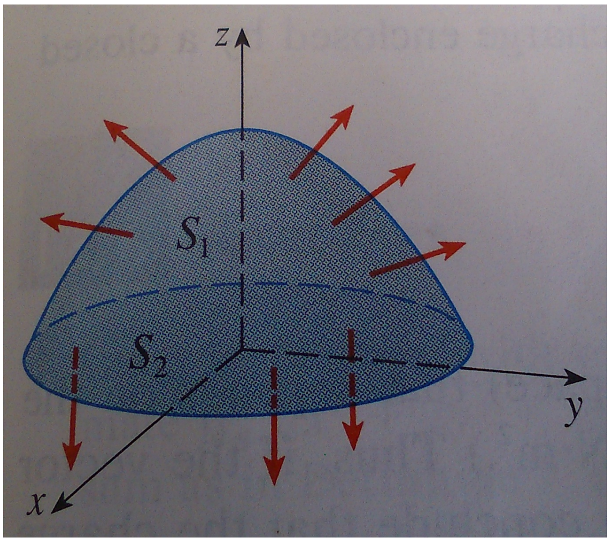
8. If a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flows through an oriented surface S with unit normal vector $\hat{\mathbf{n}}$, then the rate of flow through S is given by the surface integral

$$\iint_S \rho\mathbf{v} \cdot d\mathbf{S} = \iint_S \rho\mathbf{v} \cdot \hat{\mathbf{n}}\, dS.$$

Suppose a fluid with density $\rho(x, y, z) = 1$ flows with velocity $\mathbf{v} = y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. Find the rate of outward flow through the closed surface S enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution:

As shown in the figure below, we can regard the closed surface S to be composed of a parabolic top surface S_1 and a circular bottom surface S_2.



The parabolic top S_1 is defined by $z = g(x, y) = 1 - x^2 - y^2$ and its projection D onto the xy-plane is the disk given by $x^2 + y^2 \leq 1$. Then we have

$$\frac{\partial g}{\partial x} = -2x, \qquad \frac{\partial g}{\partial y} = -2y.$$

Therefore, the rate of outward flow through S_1 is given by

$$\begin{aligned}\iint_{S_1} \rho\mathbf{v}\cdot d\mathbf{S} &= \iint_D [-(y)(-2x)-(x)(-2y)+(1-x^2-y^2)]\,dA\\ &= \iint_D (1+4xy-x^2-y^2)dA\\ &= \int_0^{2\pi}\int_0^1 (1+4r^2\cos\theta\sin\theta - r^2)\,r\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^1 [r-(1-2\sin 2\theta)r^3]\,dr\,d\theta\\ &= \int_0^{2\pi}\left[\frac{1}{2}r^2-\frac{1}{4}(1-2\sin 2\theta)\,r^4\right]_0^1 d\theta\\ &= \int_0^{2\pi}\left(\frac{1}{4}+\frac{1}{2}\sin 2\theta\right)d\theta\\ &= \left[\frac{1}{4}\theta-\frac{1}{4}\cos 2\theta\right]_0^{2\pi}\\ &= \frac{\pi}{2}\end{aligned}$$

Note that the bottom disk S_2 has outward unit normal vector

$$\hat{\mathbf{n}} = -\hat{\mathbf{k}}$$

and it lies on the xy-plane. Therefore, the rate of outward flow through S_2 is given by

$$\iint_{S_2} \rho\mathbf{v}\cdot d\mathbf{S} = \iint_{S_2} \rho\mathbf{v}\cdot(-\hat{\mathbf{k}})\,dS = \iint_{S_2}(-z)\,dS = \iint_D 0\,dS = 0$$

since $z=0$ on S_2. Finally, we compute the rate of outward flow through S which is the sum of the outward flow rate over S_1 and S_2:

$$\iint_S \rho\mathbf{v}\cdot d\mathbf{S} = \iint_{S_1}\rho\mathbf{v}\cdot d\mathbf{S} + \iint_{S_2}\rho\mathbf{v}\cdot d\mathbf{S} = \frac{\pi}{2}+0 = \frac{\pi}{2}$$

9. If the temperature at a point (x,y,z) in a substance is represented by the function $T(x,y,z)$, then the rate of heat flow across the surface S in the substance is given by the surface integral

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -k\iint_S \nabla T\cdot\hat{\mathbf{n}}\,dS$$

where the vector field $\mathbf{F} = -k\nabla T$ is the heat flow and k is the conductivity of the substance. Suppose the temperature at the point (x, y, z) in a substance with conductivity $k = 2$ is

$$T(x, y, z) = 30 - \frac{1}{18}z^2.$$

Compute the rate of heat flow out of the sphere S defined by the equation $x^2 + y^2 + z^2 = 9$.

Solution:

The heat flow of the substance is

$$\mathbf{F} = -k\nabla T = -2\,\nabla\left(30 - \frac{1}{18}z^2\right) = -2\left(-\frac{1}{9}z\,\hat{\mathbf{k}}\right) = \frac{2}{9}z\,\hat{\mathbf{k}}$$

Since we want to compute the flow out of the sphere S, we need to find an outward unit normal vector to it. An outward normal vector to the sphere S is

$$\mathbf{n} = \nabla(x^2 + y^2 + z^2) = 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$$

So we take the outward unit normal vector

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{3}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$$

since $x^2 + y^2 + z^2 = 9$ on S. Therefore,

$$\mathbf{F} \cdot \hat{\mathbf{n}} = \left(\frac{2}{9}z\hat{\mathbf{k}}\right) \cdot \left[\frac{1}{3}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\right] = \frac{2}{27}z^2$$

Since the surface S is a sphere, we will use spherical coordinates for the surface integral. On the sphere S, $z = r\cos\theta = 3\cos\theta$ so that

$$\mathbf{F} \cdot \hat{\mathbf{n}} = \frac{2}{27}(3\cos\theta)^2 = \frac{2}{3}\cos^2\theta.$$

Moreover, the surface area element of the sphere S is

$$dS = r^2 \sin\theta\, d\theta\, d\phi = 9\sin\theta\, d\theta\, d\phi.$$

Thus the rate of flow out of the sphere S is

$$\begin{aligned}\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \int_0^{2\pi}\int_0^{\pi}\left(\frac{2}{3}\cos^2\theta\right)(9\sin\theta\, d\theta\, d\phi) \\ &= \int_0^{\pi} 6\cos^2\theta\sin\theta\, d\theta \int_0^{2\pi} d\phi = \left[-2\cos^3\theta\right]_0^{\pi}\left[\phi\right]_0^{2\pi} = 8\pi\end{aligned}$$

10. Use Green's Theorem to find the work done by the force $\mathbf{F} = x(x+y)\hat{\mathbf{i}} + xy^2\hat{\mathbf{j}}$ in moving a particle from the origin along the x-axis to $(1,0)$, then along the line segment to $(0,1)$, and then back to the origin along the y-axis.

    Solution:

    Let C be the path of the particle and D be the region bounded by C. Applying Green's Theorem yields the work done

$$\begin{aligned}
W &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\
&= \oint_C [x(x+y)\,dx + xy^2\,dy] \\
&= \iint_D \left\{ \frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}[x(x+y)] \right\} dA \\
&= \iint_D (y^2 - x)\,dA \\
&= \int_0^1 \int_0^{1-x} (y^2 - x)\,dy\,dx \\
&= \int_0^1 \left[\frac{1}{3}y^3 - xy \right]_0^{1-x} dx \\
&= \int_0^1 \left[\frac{1}{3}(1-x)^3 - x(1-x) \right] dx \\
&= \left[-\frac{1}{12}(1-x)^4 - \frac{1}{2}x^2 + \frac{1}{3}x^3 \right]_0^1 \\
&= -\frac{1}{12}
\end{aligned}$$

11. Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ of D are given by

$$\bar{x} = \frac{1}{2A}\oint_C x^2 dy, \qquad \bar{y} = -\frac{1}{2A}\oint_C y^2 dx$$

    where A is the area of D. Use this result to find the centroid of a quarter-circular region of radius a and constant density σ in the first quadrant.

Solution:

The centroid of an object is located at the same point as its center of mass if its density σ is constant. As the mass $m = \sigma A$ where A is the area of D, the formula for the centroid of D becomes:

$$\bar{x} = \frac{1}{m}\iint_D x\,dm = \frac{1}{A}\iint_D x\,dx\,dy,$$
$$\bar{y} = \frac{1}{m}\iint_D y\,dm = \frac{1}{A}\iint_D y\,dx\,dy.$$

Note that the centroid can be also computed from the line integrals on the boundary C using Green's theorem as follows.

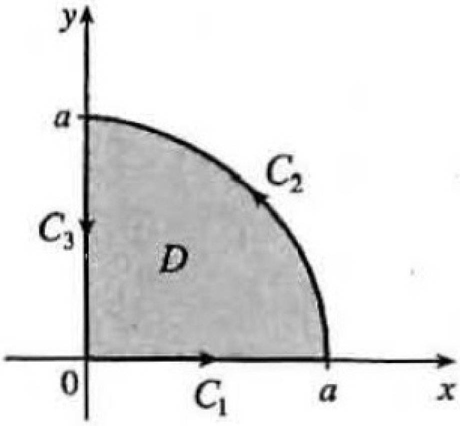
Applying Green's Theorem to the integral $\oint_C (P\,dx + Q\,dy) = \int_C x^2\,dy$, we have

$$P(x,y) = 0, \quad Q(x,y) = x^2, \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x$$

$$\therefore \ \frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\iint_D 2x\,dx\,dy = \frac{1}{A}\iint_D x\,dx\,dy = \bar{x}$$

Similarly, we can show that

$$-\frac{1}{2A}\oint_C y^2\,dx = \frac{1}{2A}\iint_D 2y\,dx\,dy = \frac{1}{A}\iint_D y\,dx\,dy = \bar{y}.$$



Next, we orient the quarter-circular region as shown in the above figure. The area of this region is $A = \pi a^2/4$. Using the above formula, we find that the coordinates of the centroid $(\bar{x}, \bar{y})$ are given by

$$\bar{x} = \frac{1}{\pi a^2/2}\oint_C x^2\,dy, \qquad \bar{y} = -\frac{1}{\pi a^2/2}\oint_C y^2\,dx$$

Note that the path $C = C_1 \cup C_2 \cup C_3$ where

$$\begin{aligned} &C_1\colon\ \mathbf{r}(t) = t\,\hat{\mathbf{i}}, \quad 0 \le t \le a,\\ &C_2\colon\ \mathbf{r}(t) = a\cos t\,\hat{\mathbf{i}} + a\sin t\,\hat{\mathbf{j}}, \quad 0 \le t \le \pi/2,\\ &C_3\colon\ \mathbf{r}(t) = (a-t)\,\hat{\mathbf{j}}, \quad 0 \le t \le a. \end{aligned}$$

So we have

$$\begin{aligned} \oint_C x^2\,dy &= \int_{C_1} x^2\,dy + \int_{C_2} x^2\,dy + \int_{C_3} x^2\,dy\\ &= \int_0^a (t^2)(0\,dt) + \int_0^{\pi/2} (a\cos t)^2(a\cos t\,dt) + \int_0^a (0^2)(-dt)\\ &= \int_0^{\pi/2} a^3\cos^3 t\,dt\\ &= a^3\int_0^{\pi/2}(1-\sin^2 t)\,d\sin t\\ &= a^3\left[\sin t - \frac{1}{3}\sin^3 t\right]_0^{\pi/2}\\ &= \frac{2}{3}a^3 \end{aligned}$$

Similarly,

$$\begin{aligned} \oint_C y^2\,dx &= \int_{C_1} y^2\,dx + \int_{C_2} y^2\,dx + \int_{C_3} y^2\,dx\\ &= \int_0^a (0^2)(dt) + \int_0^{\pi/2} (a\sin t)^2(-a\sin t\,dt) + \int_0^a (a-t)^2(0\,dt)\\ &= \int_0^{\pi/2}(-a^3\sin^3 t)\,dt\\ &= a^3\left[\cos t - \frac{1}{3}\cos^3 t\right]_0^{\pi/2}\\ &= -\frac{2}{3}a^3 \end{aligned}$$

Hence,

$$\bar{x} = \frac{1}{\pi a^2/2}\oint_C x^2\,dy = \frac{4a}{3\pi}, \qquad \bar{y} = -\frac{1}{\pi a^2/2}\oint_C y^2\,dz = \frac{4a}{3\pi}.$$

12. Use the Divergence Theorem and Maxwell's equation $\nabla \cdot \mathbf{B} = 0$ to show that $\iint_S \mathbf{B} \cdot d\mathbf{S} = 0$ for any closed surface S.

    Solution:

    Applying Divergence Theorem to $\iint_S \mathbf{B} \cdot d\mathbf{S}$ and using $\nabla \cdot \mathbf{B} = 0$, we have

    $$\iint_S \mathbf{B} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{B} \, dV = 0$$

    for any closed surface S that is the boundary of a solid region E. Such result implies that the flux of a magnetic field over any closed surface is equal to zero.

13. Suppose that the velocity field $\mathbf{v}$ of a fluid has a vector potential $\mathbf{w}$, i.e. $\mathbf{v} = \nabla \times \mathbf{w}$. Show that $\mathbf{v}$ is incompressible so that the flux of $\mathbf{v}$ across any closed surface S is equal to zero. Also show that if a closed surface S is partitioned into surfaces S_1 and S_2 (i.e. $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$), then the flux of $\mathbf{v}$ across S_1 is equal to the additive inverse of the flux of $\mathbf{v}$ across S_2.

    Solution:

    To show that $\mathbf{v}$ is incompressible, we consider

    $$\nabla \cdot \mathbf{v} = \nabla \cdot (\nabla \times \mathbf{w}) = 0$$

    since the divergence of the curl of any vector field is zero. Next, let S be a closed surface that is the boundary of a solid region E. Then, from the Divergence Theorem, we have

    $$\iint_S \mathbf{v} \cdot \mathbf{n} \, dS = \iiint_E \nabla \cdot \mathbf{v} \, dV = 0.$$

    Finally, since $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, we have

    $$\iint_{S_1} \mathbf{v} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{v} \cdot d\mathbf{S} = \iint_S \mathbf{v} \cdot d\mathbf{S} = 0$$
    $$\Rightarrow \iint_{S_1} \mathbf{v} \cdot d\mathbf{S} = -\iint_{S_2} \mathbf{v} \cdot d\mathbf{S}$$

    It implies that the flux of $\mathbf{v}$ across S_1 is equal to the additive inverse of the flux of $\mathbf{v}$ across S_2.

14. If the temperature function T has a continuous partial derivative with respect to time t, the heat flow out of the solid region E bounded by the surface S is given by

$$\iint_S (-k\nabla T)\cdot d\mathbf{S} = -\iiint_E \rho\sigma \frac{\partial T}{\partial t}\, dV$$

where ρ is the (constant) density and σ is the specific heat of the solid. Use the Divergence Theorem and the above equation to derive the heat equation

$$\frac{\partial T}{\partial t} = \alpha^2 \nabla^2 T,$$

where $\alpha^2 = k/(\rho\sigma)$ and $\nabla^2 T = \nabla\cdot(\nabla T)$ is the Laplacian of T.

Solution:

Applying the Divergence Theorem to the left-hand side of the given equation, we have

$$\begin{aligned}
&\iiint_E \nabla\cdot(-k\nabla T)\, dV = -\iiint_E \rho\sigma\frac{\partial T}{\partial t}\, dV \\
\Rightarrow\ &\iiint_E -k\nabla\cdot(\nabla T)\, dV + \iiint_E \rho\sigma\frac{\partial T}{\partial t}\, dV = 0 \\
\Rightarrow\ &\iiint_E \left(-k\nabla^2 T + \rho\sigma\frac{\partial T}{\partial t}\right) dV = 0
\end{aligned}$$

Observe that the only way for the above integral to be zero for every solid region E is the integrand equal to zero. That is to say,

$$\begin{aligned}
0 &= -k\nabla^2 T + \rho\sigma\frac{\partial T}{\partial t} \\
\Rightarrow\ \rho\sigma\frac{\partial T}{\partial t} &= k\nabla^2 T
\end{aligned}$$

Finally, dividing both sides by $\rho\sigma$ gives us the heat equation:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho\sigma}\nabla^2 T = \alpha^2\nabla^2 T.$$

15. In the case where the electric field $\mathbf{E}$ is constant and I represents the current enclosed by a positively oriented curve C, use the relationship

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = \frac{1}{\epsilon_0 c^2} I$$

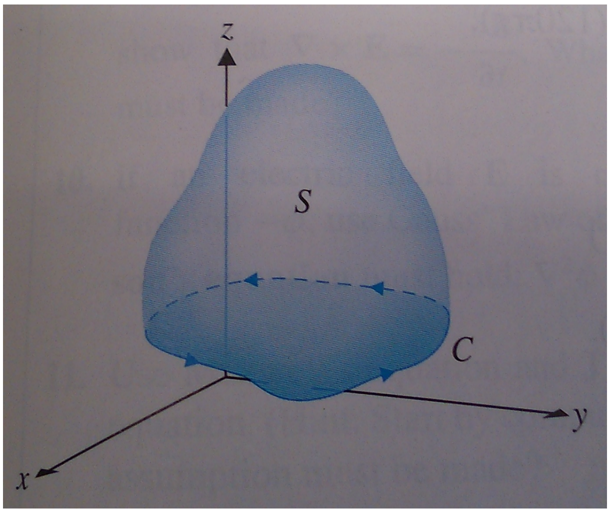
to derive Ampere's law:

$$\nabla \times \mathbf{B} = \frac{1}{\epsilon_0 c^2}\mathbf{J}$$

where $\mathbf{J}$ is the current density.

Solution:

Let S be any capping surface for the curve C, i.e. any positively oriented two-sided surface bounded by C (see below figure).



The enclosed current I is then related to the current density $\mathbf{J}$ by

$$I = \iint_S \mathbf{J} \cdot d\mathbf{S}.$$

By Stokes' Theorem, we can rewrite the line integral of $\mathbf{B}$ as

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S}.$$

Combining with the expression $\oint_C \mathbf{B} \cdot d\mathbf{r} = I/\epsilon_0 c^2$ yields

$$\iint_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \frac{1}{\epsilon_0 c^2} \iint_S \mathbf{J} \cdot d\mathbf{S}$$

$$\Rightarrow \quad \iint_S \left(\nabla \times \mathbf{B} - \frac{1}{\epsilon_0 c^2}\mathbf{J}\right) \cdot d\mathbf{S} = 0$$

Since this result holds for any capping surface S, it must be true that

$$\nabla \times \mathbf{B} = \frac{1}{\epsilon_0 c^2}\mathbf{J}.$$

16. An AC generator produces a voltage $V(t) = 120\sin(120\pi t)\,\mathrm{V}$. Determine the magnetic flux $\phi = \iint_S \mathbf{B}\cdot d\mathbf{S}$ through the surface S of the coil.

    Solution:

    The voltage $V(t)$ is given by

$$\oint_C \mathbf{E}\cdot d\mathbf{r} = 120\sin(120\pi t).$$

    Applying Stokes' Theorem to the left-hand side, we have

$$\iint_S (\nabla\times\mathbf{E})\cdot d\mathbf{S} = \oint_C \mathbf{E}\cdot d\mathbf{r} = 120\sin(120\pi t),$$

    where C is the positively oriented boundary of S. According to Faraday's law, we have

$$\iint_S \left(-\frac{\partial\mathbf{B}}{\partial t}\right)\cdot d\mathbf{S} = \iint_S (\nabla\times\mathbf{E})\cdot d\mathbf{S} = 120\sin(120\pi t).$$

    Assuming the integrand to be continuous, we can bring the derivative outside and obtain

$$-\frac{d}{dt}\left(\iint_S \mathbf{B}\cdot d\mathbf{S}\right) = 120\sin(120\pi t).$$

    Writing this in terms of the magnetic flux ϕ, we have

$$-\frac{d\phi}{dt} = 120\sin(120\pi t).$$

$$\Rightarrow\quad \phi(t) = \frac{1}{\pi}\cos(120\pi t) + c$$

    for some constant c.

⟨III⟩ Problems

1. Maxwell's equations relating the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ as they vary with time in a region containing no charge and no current can be stated as follows:

$$\operatorname{div}\mathbf{E} = 0, \quad \operatorname{curl}\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t},$$

$$\operatorname{div}\mathbf{B} = 0, \quad \operatorname{curl}\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}.$$

    where c is the speed of light in vacuum.

Use these equations to prove the followings:

(a) $\nabla \times (\nabla \times \mathbf{E}) = -\dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{E}}{\partial t^2}$

(b) $\nabla^2 \mathbf{E} = \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{E}}{\partial t^2}$

(c) $\nabla \times (\nabla \times \mathbf{B}) = -\dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{B}}{\partial t^2}$

(d) $\nabla^2 \mathbf{B} = \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{B}}{\partial t^2}$

2. For two-dimensional current flow, the velocity field $\mathbf{v} = v_x(x, y)\,\hat{\mathbf{i}} + v_y(x, y\,\hat{\mathbf{j}}$ is said to have a stream function g if $\partial g/\partial x = -v_y$ and $\partial g/\partial y = v_x$. Show that if $\mathbf{v}$ has a stream function and its components v_x and v_y have continuous partial derivatives, then $\nabla \cdot \mathbf{v} = 0$.

3. Compute the work done by the force field $\mathbf{F}(x, y, z) = xy\,\hat{\mathbf{i}} + 3z\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$ along the helix C defined parametrically by $\mathbf{r}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$ from $(1, 0, 0)$ to $(0, 1, \pi)$.

4. If $\lambda(x, y)$ denotes the linear mass density at the point (x, y) of a thin wire shaped in the shape of a curve C, then the moment of inertia I about an axis is given by
$$I = \int_C r_\perp^2\, \lambda(x, y)\, ds$$
where $r_\perp$ is the perpendicular distance from the point (x, y) to that axis.

   For a rod with linear mass density $\lambda(x, y) = y$ in the shape of the curve $y = 4 - x^2$ for $0 \le x \le 2$, compute the moment of inertia I of the rod about the x-axis. Note that in this case $r_\perp$ is the distance from the point (x, y) to the x-axis.

5. If $T(x, y)$ is the temperature function of a body, the line integral
$$\int_C (-k\nabla T) \cdot \hat{\mathbf{n}}\, ds$$
gives the rate of heat loss across the curve C where $\hat{\mathbf{n}}$ is the unit normal vector to C. For $T(x, y) = 60e^{y/50}$, compute the rate of heat loss across

the rectangle C with sides $x=-20, x=20, y=-5$, and $y=5$. Explain in terms of the temperature function why the line integral is zero along two sides of C.

6. The work done due to increase in the temperature of a gas from T_1 to T_2 is given by

$$W=\int_C\left(\frac{RT}{P}\,dP-R\,dT\right)$$

where R is a constant, T is temperature, P is pressure, and C is the path of (P,T) values as the change occurs. Compare the work done along the following two paths.

   (a) C_1 consists of the line segment from (P_1,T_1) to (P_1,T_2), followed by the line segment from (P_1,T_2) to (P_2,T_2);

   (b) C_2 consists of the line segment from (P_1,T_1) to (P_2,T_1), followed by the line segment from (P_2,T_1) to (P_2,T_2).

7. The circulation of a fluid with velocity field $\mathbf{v}$ around the closed path C is defined by $\Gamma=\oint_C \mathbf{v}\cdot d\mathbf{r}$. For inviscid flow,

$$\frac{d\Gamma}{dt}=\oint_C \mathbf{v}\cdot d\mathbf{v}.$$

Show that in this case

$$\frac{d\Gamma}{dt}=0.$$

This is known as Kelvin's Circulation Theorem which explains why small whirlpools in a stream stay coherent and move for periods of time.

8. The temperature T in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow outward from a sphere S of radius a with the origin at the center of the ball.

9. A fluid of constant density $\rho(x,y,z)=1500$ has velocity field $\mathbf{v}(x,y,z)=-y\hat{\mathbf{i}}+x\hat{\mathbf{j}}+2z\hat{\mathbf{k}}$. Find the rate of flow outward through the sphere $x^2+y^2+z^2=25$.

10. A particle starts at the point $(-2, 0)$, moves along the x-axis to $(2, 0)$, and then along the semicircle $y = \sqrt{4 - x^2}$ back to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = x\hat{\mathbf{i}} + (x^3 + 3xy^2)\hat{\mathbf{j}}$.

11. A plane lamina with constant density $\sigma(x, y) = \sigma$ occupies a region in the xy-plane bounded by a simple closed path C. Show that its moment of inertia about the x and y axes are

$$I_x = -\frac{\sigma}{3}\oint_C y^3\, dx, \qquad I_y = \frac{\sigma}{3}\oint_C x^3\, dy.$$

Hence, find the moment of inertia of a circular disk of radius a with constant density ρ about a diameter.

12. The integral form of Gauss' Law is

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\epsilon_0},$$

where $\mathbf{E}$ is the electric field, q is the total charge enclosed by the closed surface S, and ϵ_0 is the permittivity constant. Use Divergence Theorem to derive the differential form of Gauss's Law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0},$$

where ρ is the charge density.

13. According to Coulomb's Law, the electric field $\mathbf{E}$ due to a point charge q located at the origin is

$$\mathbf{E}(x, y, z) = \frac{kq}{r^3}\boldsymbol{r}$$

where $\boldsymbol{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, $r = \sqrt{x^2 + y^2 + z^2}$, and k is a constant. Using the Divergence Theorem, show that the electric flux of the electric field $\mathbf{E}$ through *any* closed surface S that encloses the origin is equal to

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi kq.$$

14. Suppose a fluid has velocity field $\mathbf{v}$ and density function ρ which has a continuous partial derivative with respect to time t. If there are no sources or sinks, the rate of change of the mass of fluid m contained in a solid region E is given by

$$\frac{dm}{dt} = \iiint_E \frac{d\rho}{dt}\, dV$$

In the absence of sources and sinks, the only way for change in mass inside the region E is the fluid flow across the boundary surface S. So we also have

$$\frac{dm}{dt} = -\iint_S \rho\mathbf{v} \cdot d\mathbf{S}.$$

Use these equations and the Divergence Theorem to derive the continuity equation:

$$\nabla \cdot (\rho\mathbf{v}) + \frac{\partial \rho}{\partial t} = 0.$$

15. Faraday showed that if $\mathbf{E}$ is the electric field and $\phi = \iint_S \mathbf{B} \cdot d\mathbf{S}$ is the magnetic flux for any capping surface S (i.e. any positively oriented open surface with boundary C), they are related as

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{d\phi}{dt}.$$

Use this equation to show that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

What mathematical assumption must be made in order to derive the above equation?

16. Use Stokes' Theorem to find the work done by the force field $\mathbf{F}(x, y, z) = yz\hat{\mathbf{i}} - xz\hat{\mathbf{j}} + z^3\hat{\mathbf{k}}$ on a particle when it is moved along the circle C that is the boundary of the part of the sphere $x^2 + y^2 + z^2 = 8$ lying inside the cone $z = \sqrt{x^2 + y^2}$ above the xy-plane and oriented counterclockwise as viewed from above.