Essential Maths for Physics Students Class B
Chapter 4 Linear Algebra

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⟨I⟩ Summary

1. Matrix

   (a) Definitions

   A matrix of size $m \times n$ is an array of quantities such as numbers or functions arranged in m rows and n columns which has the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{or} \quad \mathbf{A} = [a_{ij}]_{m\times n}.$$

   It's very often that a vector is represented by a $1 \times n$ or $n \times 1$ matrix. A $1 \times n$ matrix is called a row matrix while a $n \times 1$ matrix is called a column matrix. For a row matrix $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}$, the norm is the number defined by

$$\|\mathbf{u}\| = \sqrt{\overline{u_1}u_1 + \overline{u_2}u_2 + \ldots + \overline{u_n}u_n}$$

   where $\overline{u_1}$ is the complex conjugate of u_1. A row matrix is said to be normalized if its norm is equal to one. Similar definitions also hold for a column matrix $\mathbf{v}$.

   A $n \times n$ matrix is called a square matrix of order n. The trace of a $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is

$$\mathrm{Tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii},$$

   i.e. the sum of the diagonal elements of $\mathbf{A}$.

   A matrix, whether square or not, whose element are all zeros is called a zero matrix denoted as $\mathbf{0}$.

   (b) Algebra of Matrices

      i. Transpose

      The transpose of a $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ is the $n \times m$ matrix $\mathbf{A}^{\mathsf{T}} = [b_{ij}]$, where $b_{ij} = a_{ji}$.

For any matrices $\mathbf{A}$ and $\mathbf{B}$, and any scalar α,

- $(\mathbf{A}^\mathsf{T})^\mathsf{T} = \mathbf{A}$
- $(\alpha\mathbf{A})^\mathsf{T} = \alpha\mathbf{A}^\mathsf{T}$
- $(\mathbf{A} \pm \mathbf{B})^\mathsf{T} = \mathbf{A}^\mathsf{T} \pm \mathbf{B}^\mathsf{T}$
- $(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$

ii. Sum and difference

The sum or difference of two $m \times n$ matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ is the $m \times n$ matrix $\mathbf{C} = \mathbf{A} \pm \mathbf{B} = [c_{ij}]$ where $c_{ij} = a_{ij} \pm b_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

For any $m \times n$ matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$,

- $\mathbf{A} + \mathbf{0} = \mathbf{A}$
- $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$
- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

iii. Scalar multiple

The scalar multiple of a $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ by a scalar α is a $m \times n$ matrix $\mathbf{D} = \alpha\mathbf{A} = [d_{ij}]$ where $d_{ij} = \alpha a_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

For any $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, any $n \times n$ matrix $\mathbf{C}$, and any scalars α and β,

- $1\mathbf{A} = \mathbf{A}$
- $(\alpha\beta)\mathbf{A} = \alpha(\beta\mathbf{A})$
- $(\alpha \pm \beta)\mathbf{A} = \alpha\mathbf{A} \pm \beta\mathbf{A}$
- $\alpha(\mathbf{A} \pm \mathbf{B}) = \alpha\mathbf{A} \pm \alpha\mathbf{B}$
- $\mathrm{Tr}(\alpha\mathbf{C}) = \alpha\mathrm{Tr}(\mathbf{C})$

iv. Matrix Product

The product of a $m \times p$ matrix $\mathbf{A} = [a_{ij}]$ and a $p \times n$ matrix $\mathbf{B} = [b_{ij}]$ is the $m \times n$ matrix $\mathbf{C} = \mathbf{AB} = [c_{ij}]$ where $c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. In general, $\mathbf{AB} \neq \mathbf{BA}$.

For any $m \times p$ matrix $\mathbf{A}$, any $p \times n$ matrices $\mathbf{B}$ and $\mathbf{C}$, any $n \times n$ matrices $\mathbf{D}$ and $\mathbf{E}$, and any scalar α,

- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
- $(\mathbf{B} + \mathbf{C})\mathbf{D} = \mathbf{BD} + \mathbf{CD}$
- $\mathbf{A}(\mathbf{BD}) = (\mathbf{AB})\mathbf{D}$
- $\alpha(\mathbf{AB}) = (\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B})$
- $\mathrm{Tr}(\mathbf{DE}) = \mathrm{Tr}(\mathbf{ED})$

v. Scalar product

The scalar product (also called dot product) between two $n \times 1$ column vectors $\mathbf{u} = [u_i]$ and $\mathbf{v} = [v_i]$ is

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^\dagger \mathbf{v} = \overline{u_1} v_1 + \overline{u_2} v_2 + \ldots + \overline{u_n} v_n.$$

Two real vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

(c) Special Matrices

i. A $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is called an identity matrix $\mathbf{I}_n$ of order n if and only if $a_{ij} = \begin{cases} 1 & \text{for all } i = j, \\ 0 & \text{for all } i \neq j. \end{cases}$

ii. A $n \times n$ matrix $\mathbf{D} = [d_{ij}]$ is called a diagonal matrix if $d_{ij} = 0$ for all $i \neq j$.

iii. A square matrix $\mathbf{A}$ is said to be

- symmetric if and only if $\mathbf{A}^\mathsf{T} = \mathbf{A}$.
- skew-symmetric if and only if $\mathbf{A}^\mathsf{T} = -\mathbf{A}$.

iv. The conjugate of a matrix $\mathbf{A}$ is the matrix $\overline{\mathbf{A}}$ obtained by taking complex conjugates on each element of $\mathbf{A}$. The hermitian conjugate of a matrix $\mathbf{A}$ is $\mathbf{A}^\dagger = (\overline{\mathbf{A}})^\mathsf{T} = \overline{(\mathbf{A}^\mathsf{T})}$.

v. A square matrix $\mathbf{A}$ is said to be hermitian if and only if $\mathbf{A} = \mathbf{A}^\dagger$.

vi. A square matrix $\mathbf{A}$ is said to be unitary if $\mathbf{A}^\dagger \mathbf{A} = \mathbf{A}\mathbf{A}^\dagger = \mathbf{I}$.

(d) Determinant of a Square Matrix

i. The determinant of a $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is defined to be the number

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{vmatrix} = \sum_\sigma \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)}$$

where the sum is over all permutations σ of the set $\{1, 2, \ldots, n\}$ and $\operatorname{sgn}(\sigma)$ is the signature of the permutation σ which is defined as $+1$ if σ is even and -1 if σ is odd. For example,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{array}{l} a_{11}a_{22} \\ \quad -a_{12}a_{21}, \end{array} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} \\ +a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} \\ -a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}. \end{array}$$

ii. For any $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, and any scalar α,

- $\det \mathbf{A}^{\mathsf{T}} = \det \mathbf{A}$
- $\det \overline{\mathbf{A}} = \overline{\det \mathbf{A}}$
- $\det(\alpha \mathbf{A}) = \alpha^n \det \mathbf{A}$
- $\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$
- $|\det \mathbf{A}| = 1$ if $\mathbf{A}$ is unitary
- $\det \mathbf{I}_n = 1$ for any n

iii. The determinant of a matrix

- changes sign if two rows are interchanged.
- is unchanged if a constant multiple of one row is added to another row.

Similar properties hold for columns.

iv. The minor M_{ij} of the element a_{ij} of a square matrix $\mathbf{A}$ is the determinant of the submatrix of $\mathbf{A}$ obtained by deleting its ith row and jth column. The signed minor $C_{ij} = (-1)^{i+j} M_{ij}$ is called the cofactor of the element a_{ij}. Indeed, the determinant of a $n \times n$ matrix $\mathbf{A}$ can be expressed in terms of the cofactors as

$$\det \mathbf{A} = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \forall i, j = 1, 2, \ldots, n$$

v. The cofactor matrix of a $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is the $n \times n$ matrix $\operatorname{cof} \mathbf{A} = [C_{ij}]$ where C_{ij} is the cofactor of the element a_{ij} for all $i, j = 1, 2, \ldots, n$. The transpose of this cofactor matrix is called the adjoint of $\mathbf{A}$ which is denoted by $\operatorname{adj} \mathbf{A} = (\operatorname{cof} \mathbf{A})^{\mathsf{T}}$.

(e) Inverse of a Square Matrix

i. A square matrix $\mathbf{A}$ is said to be invertible or non-singular if and only if there exists a unique square matrix $\mathbf{B}$ of the same size such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$. In this case, $\mathbf{B}$ is called the inverse of $\mathbf{A}$ which is denoted by $\mathbf{A}^{-1}$.

ii. If the determinant of a square matrix $\mathbf{A}$ is non-zero, then $\mathbf{A}$ is invertible and its inverse

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \operatorname{adj} \mathbf{A}$$

For example, a 2×2 matrix $\mathbf{A} = [a_{ij}]$ is invertible if and only if $\det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21} \neq 0$. In such case,

$$\mathbf{A}^{-1} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

iii. If a $n \times n$ matrix $\mathbf{A}$ is invertible, then the equation $\mathbf{Ax} = \mathbf{b}$ has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ for any $n \times 1$ column matrix $\mathbf{b}$.

iv. For any $n \times n$ invertible matrices $\mathbf{A}$ and $\mathbf{B}$ and any scalar α,

- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(\mathbf{A}^{\mathsf{T}})^{-1} = (\mathbf{A}^{-1})^{\mathsf{T}}$
- $(\mathbf{A}^{\dagger})^{-1} = (\mathbf{A}^{-1})^{\dagger}$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $(\alpha\mathbf{A})^{-1} = (1/\alpha)\mathbf{A}^{-1}$
- $(\mathbf{A}^{n})^{-1} = (\mathbf{A}^{-1})^{n}$

(f) Eigenvalue Problems

i. A scalar λ is called an eigenvalue of a $n \times n$ matrix $\mathbf{A}$ if there exists non-zero $n \times 1$ column matrix $\mathbf{v}$ such that

$$\mathbf{Av} = \lambda\mathbf{v}$$

The column matrix $\mathbf{v}$ is called an eigenvector of $\mathbf{A}$ corresponding to λ. We call the above equation the eigenvalue problem of $\mathbf{A}$.

ii. It can be shown that the eigenvalues λ of a $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ are the roots of the characteristic equation

$$p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0$$

where $p(\lambda)$ is a nth degree polynomial in λ known as the characteristic polynomial of $\mathbf{A}$.

iii. If $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of a $n \times n$ matrix $\mathbf{A}$, then $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ are linearly independent.

iv. A square matrix that commutes with its Hermitian conjugate, i.e. $\mathbf{AA}^{\dagger} = \mathbf{A}^{\dagger}\mathbf{A}$, is called a normal matrix. For examples, Hermitian and unitary matrices are normal matrices. Let $\mathbf{A}$ be a normal square matrix.

- The eigenvalues of $\mathbf{A}^{\dagger}$ are the complex conjugates of the eigenvalues of $\mathbf{A}$.

- If $\mathbf{v}$ and $\mathbf{u}$ are eigenvectors of $\mathbf{A}$ corresponding to different eigenvalues λ and μ, then they are orthogonal, i.e. $\mathbf{v} \cdot \mathbf{u} = 0$.

v. Procedures for solving eigenvalue problems

- Compute the characteristic polynomial $p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$.
- Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ by solving the characteristic equation $p(\lambda) = 0$.
- For each λ_i, solve the matrix equation $\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i$ to find the corresponding eigenvectors $\mathbf{v}_i$.

(g) Diagonalization of Square Matrices

i. A square matrix $\mathbf{A}$ is said to be diagonalizable if it is similar to a diagonal matrix $\mathbf{D}$ of the same size, i.e. there exists an invertible matrix $\mathbf{P}$ such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}.$$

ii. An $n \times n$ matrix $\mathbf{A}$ is diagonalizable if and only if it has n linearly independent eigenvectors. More precisely, if $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ are linearly independent eigenvectors of $\mathbf{A}$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then we can define a $n \times n$ matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \ldots & \mathbf{v_n} \end{bmatrix}$$

such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$.

Thus $\mathbf{P}$ can be used to diagonalize the matrix $\mathbf{A}$. Moreover, the diagonalized matrix $\mathbf{D}$ has the same set of eigenvalues as $\mathbf{A}$.

iii. Procedures to diagonalize a $n \times n$ matrix $\mathbf{A}$

- Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$.
- Find the linearly independent eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ of $\mathbf{A}$.
- Construct the invertible matrix $\mathbf{P}$ and the diagonalized matrix $\mathbf{D}$ from the eigenvectors in step 2 as defined above.

2. System of Linear Algebraic Equations

(a) A system of linear algebraic equations (or a linear system) in the variables $x_1, x_2, \ldots, x_n$ has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$$

where $a_{11}, a_{12}, \ldots, a_{mn}$ and $b_1, b_2, \ldots, b_m$ are constants. If $b_1 = b_2 = \ldots = b_m = 0$, the system of equations is called homogeneous; otherwise, it is called inhomogeneous. It can be shown that a system of linear algebraic equations has either no solution or exactly one solution or infinitely many solutions.

(b) In matrix notation, the above linear system can be written as:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where $\mathbf{A} = [a_{ij}]$, $\mathbf{x} = [x_i]$, and $\mathbf{b} = [b_i]$. The matrix $\mathbf{A}$ is called the coefficient matrix of the system. Alternatively, we can write the system of equations as the augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$$

(c) The augmented matrix of two linear systems are called row equivalent if they have the same solution set. In fact, we can transform one matrix into a row-equivalent matrix by applying a sequence of elementary row operations:

- Interchange the ith and jth rows: $R_i \leftrightarrow R_j$.
- Multiply the ith row by a non-zero constant k: $R_i \rightarrow kR_i$.
- Add the non-zero constant k times the jth row to the ith row: $R_i \rightarrow R_i + kR_j$.

(d) A matrix is in reduced row echelon form if all the following conditions hold:

- Any rows consisting entirely of zeros are at the bottom.
- In each non-zero row, the first non-zero entry, called the leading entry, is in a column to the left of any leading entries below it.

- The leading entry in each non-zero row is a 1, called a leading 1.
- Each leading 1 is the only non-zero entry in its column.

If only the first two conditions hold, the matrix is in row echelon form. The process of applying elementary row operations to transform a matrix into row echelon form is called row reduction. We can always reduce a matrix into row echelon or reduced row echelon form.

(e) We can solve a system of linear algebraic equations by performing row reduction on the augmented matrix of the system using one of the following methods:

  i. Gaussian elimination
    - Row reduce the augmented matrix to row echelon form.
    - If the resulting system has solution, solve the equivalent system that corresponds to the row-reduced matrix using back substitution.

  ii. Gauss-Jordan elimination
    - Row reduce the augmented matrix to reduced row echelon form.
    - If the resulting system has solution, solve for the leading variables (i.e. the variables corresponding to the leading entries in the matrix) in terms of any free variables (i.e. the other variables).

3. System of Linear Differential Equations

(a) A system of linear differential equations in the variables $x_1(t)$, $x_2(t)$, …, $x_n(t)$ with constant coefficients can be written in the form

$$\begin{cases} p_{11}(D)x_1 + p_{12}(D)x_2 + \ldots + p_{1n}(D)x_n = b_1 \\ p_{21}(D)x_1 + p_{22}(D)x_2 + \ldots + p_{2n}(D)x_n = b_2 \\ \vdots \\ p_{n1}(D)x_1 + p_{n2}(D)x_2 + \ldots + p_{nn}(D)x_n = b_n \end{cases}$$

where $p_{ij}(D)$ are polynomial differential operators (i.e. polynomials of the differential operator $D \equiv d/dt$ with constant coefficients) and $b_i(t)$ are functions of t for all $i, j = 1, 2, \ldots, n$.

Let $\mathbf{A} = [p_{ij}(D)]$, $\mathbf{x} = [x_i]$, and $\mathbf{b} = [b_i]$. Then the system can be rewritten in terms of matrix as

$$\mathbf{Ax} = \mathbf{b}$$

(b) Let $\mathbf{x}_p(t)$ be a solution of the system of inhomogeneous linear differential equations $\mathbf{Ax} = \mathbf{b}$. If the associated homogeneous system $\mathbf{Ax} = \mathbf{0}$ has n linearly independent solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, …, $\mathbf{x}_n(t)$, then the general solution of the non-homogeneous system $\mathbf{Ax} = \mathbf{b}$ is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t) + \mathbf{x}_p(t)$$

where c_1, c_2, …, c_n are arbitrary constants.

⟨II⟩ Examples

1. In the special theory of relativity, the motion of a particle with energy E and momentum $\boldsymbol{p} = \langle p_x, p_y, p_z \rangle$ is described by the momentum-energy four-vector:

$$\mathbf{P} = \begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix}$$

where c is the speed of light in vacuum. The energy-momentum four-vector $\mathbf{P}$ in the inertial frame S and $\mathbf{P}'$ in the inertial frame S' that moves in the $+x$ direction at speed v with respect to S are related by

$$\mathbf{P}' = \mathbf{\Lambda P}$$

where $\mathbf{\Lambda}$ is the Lorentz transformation matrix defined by

$$\mathbf{\Lambda} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for $\gamma = 1/\sqrt{1 - v^2/c^2}$ and $\beta = v/c$.

(a) Find the inverse of $\mathbf{\Lambda}$. Hence, show that

$$E = \frac{(E' + vp'_x)}{\sqrt{1 - v^2/c^2}}, \quad p_x = \frac{(p'_x + vE'/c^2)}{\sqrt{1 - v^2/c^2}}, \quad p_y = p'_y, \quad p_z = p'_z.$$

(b) The rest mass of a particle is given by $m_0 = E^2/c^2 - (p_x^2 + p_y^2 + p_z^2)$, show that

$$m_0 = \mathbf{P}^\mathsf{T}\mathbf{g}\mathbf{P}.$$

where $\mathbf{g}$ is the Minkowski metric matrix defined by

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

that satisfies the relation

$$\mathbf{g} = \mathbf{\Lambda}^\mathsf{T}\mathbf{g}\mathbf{\Lambda}.$$

Hence, show that the rest mass of a particle m_0 is the same in any inertial reference frames.

Solution:

(a) The determinant and adjoint of $\mathbf{\Lambda}$ are

$$\det \mathbf{\Lambda} = \begin{vmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{vmatrix} = (1-\beta^2)\gamma^2 = 1$$

$$\text{adj } \mathbf{\Lambda} = (\text{cof } \mathbf{\Lambda})^\mathsf{T} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So the inverse of $\mathbf{\Lambda}$ is

$$\mathbf{\Lambda}^{-1} = \frac{1}{\det \mathbf{\Lambda}} \text{ adj } \mathbf{\Lambda} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix} = \mathbf{\Lambda}^{-1}\begin{bmatrix} E'/c \\ p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} E'/c \\ p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} \gamma(E'/c + \beta p'_x) \\ \gamma(\beta E'/c + p'_x) \\ p'_y \\ p'_z \end{bmatrix}$$

Hence we have

$$E = \frac{(E' + vp'_x)}{\sqrt{1 - v^2/c^2}}, \quad p_x = \frac{(p'_x + vE'/c^2)}{\sqrt{1 - v^2/c^2}}, \quad p_y = p'_y, \quad p_z = p'_z.$$

(b) We first compute $\mathbf{P}^\mathsf{T}\mathbf{g}\mathbf{P}$:

$$\begin{aligned}
\mathbf{P}^\mathsf{T}\mathbf{g}\mathbf{P} &= \begin{bmatrix} E/c & p_x & p_y & p_z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix} \\
&= \begin{bmatrix} E/c & p_x & p_y & p_z \end{bmatrix} \begin{bmatrix} E/c \\ -p_x \\ -p_y \\ -p_z \end{bmatrix} \\
&= E^2/c^2 - (p_x^2 + p_y^2 + p_z^2) \\
&= m_0
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathbf{\Lambda}^\mathsf{T}\mathbf{g}\mathbf{\Lambda} &= \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^\mathsf{T} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ \gamma\beta & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} \gamma^2(1-\beta^2) & 0 & 0 & 0 \\ 0 & -\gamma^2(1-\beta^2) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
&= \mathbf{g}
\end{aligned}$$

Therefore,

$$m'_0 = \mathbf{P}'^\mathsf{T}\mathbf{g}\mathbf{P}' = (\mathbf{\Lambda}\mathbf{P})^\mathsf{T}\mathbf{g}(\mathbf{\Lambda}\mathbf{P}) = \mathbf{P}^\mathsf{T}\mathbf{\Lambda}^\mathsf{T}\mathbf{g}\mathbf{\Lambda}\mathbf{P} = \mathbf{P}^\mathsf{T}\mathbf{g}\mathbf{P} = m_0$$

It implies that the rest mass of a particle m_0 is the same in any inertial reference frames.

2. In quantum mechanics, the state of a physical system is described by a state function $\psi(x)$ that is a vector in a complex vector space $\mathcal{H}$. Moreover, each physical observable O corresponds to a linear, Hermitian operator $\hat{O}$ and the possible outcomes in any measurement of the observable are the eigenvalues $O_1, O_2, O_3, \ldots$ that satisfy

$$\hat{O}\phi_i = O_i\phi_i$$

where ϕ_i is the eigenfunction with the corresponding eigenvalue O_i. It can be shown that the set of the eigenfunctions $\{\phi_i\}$ of a Hermitian operator spans the state space $\mathcal{H}$, i.e. any state function in $\mathcal{H}$ can be written as a linear combination of $\{\phi_i\}$. Thus, with respect to the basis vectors $\{\phi_i\}$, a state function can be represented by a column matrix $\boldsymbol{\psi}$ while an operator can be represented by a Hermitian square matrix $\mathbf{O}$. If a system is represented by a normalized state function $\boldsymbol{\psi}$, the expectation value of the observable associated with $\hat{O}$ is

$$\langle O \rangle = \boldsymbol{\psi}^{\dagger}\mathbf{O}\boldsymbol{\psi}.$$

Consider a physical system with a three-dimensional state space $\mathcal{H}$. Suppose with respect to a specific set of basis vectors the energy of the system E is represented by the matrix (in units of energy):

$$\mathbf{H} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Show that the matrix $\mathbf{H}$ is Hermitian, i.e. $\mathbf{H}^{\dagger} = \mathbf{H}$.

(b) What are the possible results when the energy of the system E is measured?

(c) A particle is in the state represented by

$$\boldsymbol{\psi} = \frac{1}{\sqrt{3}} \begin{bmatrix} i \\ -i \\ i \end{bmatrix}$$

Find the expectation values $\langle E \rangle$ and $\langle E^2 \rangle$. Hence, determine $\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2}$.

Solution:

(a) The Hermitian conjugate of $\mathbf{H}$ is

$$\mathbf{H}^\dagger = \overline{\mathbf{H}}^\mathsf{T} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^\mathsf{T} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \mathbf{H}$$

So the matrix $\mathbf{H}$ is Hermitian.

(b) The possible results for the measurement of energy are the eigenvalues of $\mathbf{H}$ which can be found by solving the characteristic equation:

$$\det(\mathbf{H} - \lambda \mathbf{I}) = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \; [(2-\lambda)^2 - 1](3-\lambda) = 0$$

$$\Rightarrow \; (3-\lambda)^2(1-\lambda) = 0$$

$$\Rightarrow \; \lambda = 1 \quad \text{or} \quad \lambda = 3$$

Thus the possible energies found from measurement are $E_1 = 1$ and $E_2 = 3$ (in units of energy).

(c) First notice that the given state function ψ is normalized since

$$\|\boldsymbol{\psi}\|^2 = \boldsymbol{\psi}^\dagger \boldsymbol{\psi} = \left(\frac{1}{\sqrt{3}}\right)^2 \begin{bmatrix} -i & i & -i \end{bmatrix} \begin{bmatrix} i \\ -i \\ i \end{bmatrix} = \frac{1}{3}[(-i)i + i(-i) + (-i)i] = 1$$

Therefore, the expectation value of E is given by

$$\begin{aligned} \langle E \rangle &= \boldsymbol{\psi}^\dagger \mathbf{H} \boldsymbol{\psi} \\ &= \left(\frac{1}{\sqrt{3}}\right)^2 \begin{bmatrix} -i & i & -i \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} i \\ -i \\ i \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} -i & i & -i \end{bmatrix} \begin{bmatrix} i \\ -i \\ 3i \end{bmatrix} \\ &= \frac{1}{3}[(-i)i + i(-i) + (-i)(3i)] \\ &= \frac{5}{3} \end{aligned}$$

Similarly, the expectation value of E^2 is given by

$$\begin{aligned}
\langle E^2 \rangle &= \boldsymbol{\psi}^\dagger \mathbf{H}^2 \boldsymbol{\psi} \\
&= \left(\frac{1}{\sqrt{3}}\right)^2 \begin{bmatrix} -i & i & -i \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^2 \begin{bmatrix} i \\ -i \\ i \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} -i & i & -i \end{bmatrix} \begin{bmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} i \\ -i \\ i \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} -i & i & -i \end{bmatrix} \begin{bmatrix} i \\ -i \\ 9i \end{bmatrix} \\
&= \frac{1}{3} [(-i)i + i(-i) + (-i)(9i)] \\
&= \frac{11}{3}
\end{aligned}$$

$$\therefore \; \Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2} = \sqrt{\frac{11}{3} - \left(\frac{5}{3}\right)^2} = \frac{2\sqrt{2}}{3}$$

3. Suppose the state space for a system with orbital quantum number $\ell = 1$ is spanned by the set of the common eigenfunctions $\{Y_{1,1}, Y_{1,0}, Y_{1,-1}\}$ of the operators $\hat{L}^2$ and $\hat{L}_z$, where $Y_{\ell,m}$ denotes the eigenfunction with the eigenvalues of $\hat{L}^2$ and $\hat{L}_z$ equal $\ell(\ell+1)\hbar^2$ and $m\hbar$ respectively. In terms of this set of basis vectors, the orbital angular momentum operators are represented by the 3×3 matrices:

$$\mathbf{L_x} = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; \mathbf{L_y} = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \; \mathbf{L_z} = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

   (a) Show that the commutator $[\mathbf{L_x}, \mathbf{L_y}] = \mathbf{L_x}\mathbf{L_y} - \mathbf{L_y}\mathbf{L_x} = i\hbar\mathbf{L_z}$.

   (b) If the state of the system is represented by

$$\boldsymbol{\psi} = \begin{bmatrix} a & b & c \end{bmatrix}^\mathsf{T}$$

   where $|a|^2 + |b|^2 + |c|^2 = 1$, calculate the expectation values $\langle L_x \rangle$, $\langle L_y \rangle$, and $\langle L_z \rangle$ in terms of a, b, and c.

Solution:

(a) The left hand side of the given equation is

$$\begin{aligned}
[\mathbf{L_x}, \mathbf{L_y}] &= \mathbf{L_x}\mathbf{L_y} - \mathbf{L_y}\mathbf{L_x} \\
&= \frac{\hbar^2}{2}\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right) \\
&= \frac{\hbar^2}{2}\left(\begin{bmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{bmatrix}\right) \\
&= \hbar^2 \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix} \\
&= i\hbar\mathbf{L_z}
\end{aligned}$$

which is equal to the right hand side of this equation.

(b) As $\|\boldsymbol{\psi}\| = \sqrt{|a|^2 + |b|^2 + |c|^2} = 1$, the expectation values are given by

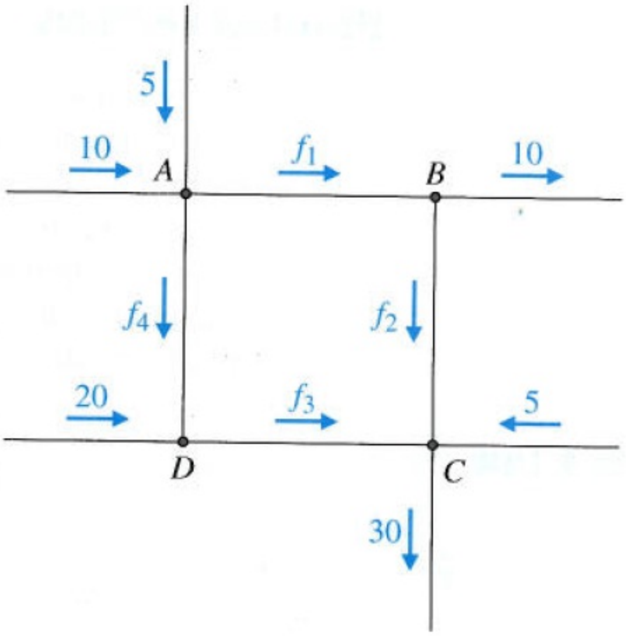
$$\begin{aligned}
\langle L_x \rangle &= \boldsymbol{\psi}^\dagger \mathbf{L_x} \boldsymbol{\psi} \\
&= \frac{\hbar}{\sqrt{2}}\begin{bmatrix} \overline{a} & \overline{b} & \overline{c} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \\
&= \frac{\hbar}{\sqrt{2}}\begin{bmatrix} \overline{a} & \overline{b} & \overline{c} \end{bmatrix}\begin{bmatrix} b \\ a+c \\ b \end{bmatrix} \\
&= \frac{\hbar}{\sqrt{2}}(a\overline{b} + \overline{a}b + b\overline{c} + \overline{b}c)
\end{aligned}$$

$$\begin{aligned}
\langle L_y \rangle &= \boldsymbol{\psi}^\dagger \mathbf{L_y} \boldsymbol{\psi} \\
&= \frac{\hbar}{\sqrt{2}}\begin{bmatrix} \overline{a} & \overline{b} & \overline{c} \end{bmatrix}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \\
&= \frac{\hbar}{\sqrt{2}}\begin{bmatrix} \overline{a} & \overline{b} & \overline{c} \end{bmatrix}\begin{bmatrix} -ib \\ i(a-c) \\ ib \end{bmatrix}
\end{aligned}$$

$$= \frac{i\hbar}{\sqrt{2}}(a\bar{b} - \bar{a}b + b\bar{c} - \bar{b}c)$$

$$\begin{aligned}
\langle L_z \rangle &= \boldsymbol{\psi}^{\dagger}\mathbf{L_z}\boldsymbol{\psi} \\
&= \hbar \begin{bmatrix} \bar{a} & \bar{b} & \bar{c} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\
&= \hbar \begin{bmatrix} \bar{a} & \bar{b} & \bar{c} \end{bmatrix} \begin{bmatrix} a \\ 0 \\ -c \end{bmatrix} \\
&= \hbar(a\bar{a} - c\bar{c})
\end{aligned}$$

4. A common application of linear systems is the study of the flow of some quantity through a network subjected to certain constraints. Graphically, a network is represented as a finite number of nodes connected by a series of directed edges known as branches or arcs. Each branch is labeled with a flow that represents the amount of some quantity flowing along that branch in the indicated direction. The basic assumption of network flow is that the total flow into a network equals the total flow out of the network and the total flow into each node equals the total flow out of the node. So the network flow can be described by a set of linear algebraic equations. Below figure shows a network of water pipes where flow is measured in liters per minute. Determine the possible flows through this network.



Solution:

Setting the flow in equal to the flow out at each node, we obtain

$$\begin{array}{ll} \text{Node A:} & 15 = f_1 + f_4 \\ \text{Node B:} & f_1 = f_2 + 10 \\ \text{Node C:} & f_2 + f_3 + 5 = 30 \\ \text{Node D:} & f_4 + 20 = f_3 \end{array} \quad \Rightarrow \quad \begin{cases} f_1 + f_4 = 15 \\ f_1 - f_2 = 10 \\ f_2 + f_3 = 25 \\ f_3 - f_4 = 20 \end{cases}$$

To solve this system of linear algebraic equations, we apply Gauss-Jordan elimination to row reduce the augmented matrix of this system:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 15 \\ 1 & -1 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & 25 \\ 0 & 0 & 1 & -1 & 20 \end{bmatrix} \xrightarrow{-R_2 + R_1 \to R_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 15 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 0 & 25 \\ 0 & 0 & 1 & -1 & 20 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & 0 & 0 & 1 & 15 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & -1 & 20 \\ 0 & 0 & 1 & -1 & 20 \end{bmatrix} \xrightarrow{R_4 - R_3 \to R_4} \begin{bmatrix} 1 & 0 & 0 & 1 & 15 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & -1 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

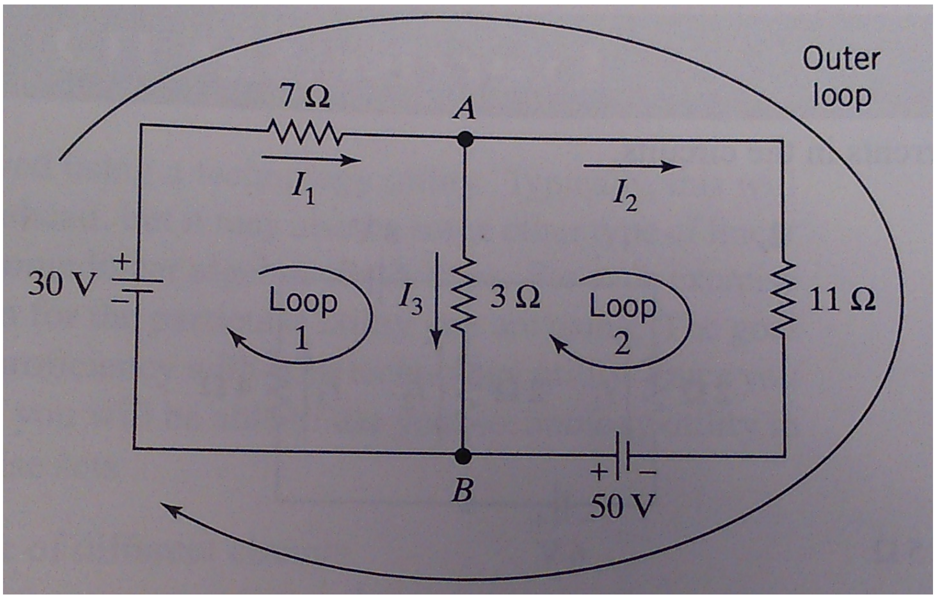
We can see that there are infinitely many solutions as there is one free variable f_4. If we let $f_4 = t$ where t can assume any real value, we obtain

$$\begin{cases} f_1 = 15 - t \\ f_2 = 5 - t \\ f_3 = 20 + t \\ f_4 = t \end{cases}$$

This set of equations describe all possible flows and allows us to analyze this network.

5. The current flow in an electrical network is governed by a pair of laws called Kirchhoff's laws:
   - Kirchhoff's current law: The sum of the currents flowing into any point equals to the sum of the currents flowing out from this point.
   - Kirchhoff's voltage law: The algebraic sum of the voltage drops around any closed loop is zero.

Using Kirchoff's law, we can set up a system of linear algebraic equations which allows us to determine the currents in an electrical network. As an example, consider an electrical network composed of two voltage sources and three resistors as shown in the figure below.



(a) Using Kirchhoff's laws, show that the unknown currents I_1, I_2, and I_3 are governed by the system of linear algebraic equations:

$$\begin{cases} I_1 - I_2 - I_3 = 0 \\ 7I_1 + 3I_3 = 30 \\ 11I_2 - 3I_3 = 50 \end{cases}$$

(b) Hence, find the unknown currents I_1, I_2, and I_3.

Solution:

(a) Applying Kirchhoff's current law to points A and B yields

$$\begin{aligned} I_1 &= I_2 + I_3, \quad \text{(Point } A\text{)} \\ I_2 + I_3 &= I_1. \quad \text{(Point } B\text{)} \end{aligned}$$

Since these equations both simplify to the same equation

$$I_1 - I_2 - I_3 = 0,$$

we need two more equations to determine I_1, I_2, and I_3. We will obtain them using Kirchhoff's voltage law.

To apply Kirchhoff's voltage law, we choose arbitrary direction for the current flow in each closed loop and adopt the following conventions:

- A voltage drop occurs at a resistor if the current passing through the resistor is in the same direction as the current in a loop; otherwise, a voltage rise occurs at the resistor.
- A voltage drop occurs at a power source if the current in the loop flows from the +ve terminal to the −ve terminal of the power source; otherwise, a voltage rise occurs at the power source.

Applying Kirchhoff's voltage law to loops 1 and 2 in the given figure with the indicated directions chosen yields

$$\begin{aligned} -7I_1 - 3I_3 + 30 &= 0, \quad \text{(Loop 1)} \\ -11I_2 + 3I_3 + 50 &= 0. \quad \text{(Loop 2)} \end{aligned}$$

Combining the above results, we obtain the system of linear algebraic equations:

$$\begin{cases} I_1 - I_2 - I_3 = 0 \\ 7I_1 + 3I_3 = 30 \\ 11I_2 - 3I_3 = 50 \end{cases}$$

(b) To solve the system of equations obtained in (a), we apply Gaussian elimination to row reduce the augmented matrix of this system:

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 7 & 0 & 3 & 30 \\ 0 & 11 & -3 & 50 \end{bmatrix} \xrightarrow{R_2 - 7R_1 \to R_2} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 7 & 10 & 30 \\ 0 & 11 & -3 & 50 \end{bmatrix}$$

$$\xrightarrow{7R_3 - 11R_2 \to R_3} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 7 & 10 & 30 \\ 0 & 0 & -131 & 20 \end{bmatrix}$$

Then we find I_1 I_2, and I_3 using back substitution as follows:

$$\begin{aligned} -131I_3 = 20 \quad &\Rightarrow \; I_3 = -\frac{20}{131} \text{ (A)} \\ 7I_2 + 10I_3 = 30 \quad &\Rightarrow \; I_2 = \frac{30}{7} - \frac{10}{7}I_3 = \frac{590}{131} \text{ (A)} \\ I_1 - I_2 - I_3 = 0 \quad &\Rightarrow \; I_1 = I_2 + I_3 = \frac{570}{131} \text{ (A)} \end{aligned}$$

6. A systematic method for balancing chemical equations is to set up a vector equation that describes the numbers of atoms of each type involved in a reaction. When propane gas burns, the propane (C_3H_8) combines with oxygen (O_2) to form carbon dioxide (CO_2) and water (H_2O), according to the equation of the form

$$(x_1)\mathrm{C_3H_8} + (x_2)\mathrm{O_2} \to (x_3)\mathrm{CO_2} + (x_4)\mathrm{H_2O}$$

Balance this chemical equation using the vector equation approach.

Solution:

To "balance" this equation, one must find the whole numbers x_1 x_2, x_3, and x_4 such that the total numbers of carbon (C), hydrogen (H), and oxygen (O) atoms on the left match the corresponding numbers of atoms on the right because atoms are neither created or destroyed in a chemical reaction.

Since the given chemical equation involves only three types of atoms (carbon, hydrogen, and oxygen), we construct a vector in $\mathbb{R}^3$ for each reactant and product in the equation that lists the number of "atoms per molecule" as follows:

$$\mathrm{C_3H_8}: \begin{bmatrix}3\\8\\0\end{bmatrix}, \quad \mathrm{O_2}: \begin{bmatrix}0\\0\\2\end{bmatrix}, \quad \mathrm{CO_2}: \begin{bmatrix}1\\0\\2\end{bmatrix}, \quad \mathrm{H_2O}: \begin{bmatrix}0\\2\\1\end{bmatrix} \begin{array}{l}\leftarrow \text{ Carbon}\\ \leftarrow \text{ Hydrogen}\\ \leftarrow \text{ Oxygen}\end{array}$$

To balance the given chemical equation, the coefficients x_1 x_2, x_3, and x_4 must satisfy:

$$x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} = x_3\begin{bmatrix}1\\0\\2\end{bmatrix} + x_4\begin{bmatrix}0\\2\\1\end{bmatrix}$$

To solve for x_1 x_2, x_3, and x_4, move all the terms to the left:

$$x_1\begin{bmatrix}3\\8\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\end{bmatrix} + x_3\begin{bmatrix}-1\\0\\-2\end{bmatrix} + x_4\begin{bmatrix}0\\-2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Next, we apply Gaussian elimination to perform row reduction on the augmented matrix of this system:

$$\begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 8 & 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{3R_3 - 8R_1 \to R_3} \begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 8 & -6 & 0 \end{bmatrix}$$

Then we find x_1 x_2, x_3, and x_4 using back substitution:

$$8x_3 - 6x_4 = 0 \quad \Rightarrow \; x_3 = \frac{3}{4}x_4$$

$$2x_2 - 2x_3 - x_4 = 0 \quad \Rightarrow \; x_2 = x_3 + \frac{1}{2}x_4 = \frac{5}{4}x_4$$

$$3x_1 - x_3 = 0 \quad \Rightarrow \; x_1 = \frac{1}{3x_3} = \frac{1}{4}x_4$$

Since the coefficients in a chemical equations must be integers, we take $x_4 = 4$ such that $x_1 = 1$, $x_2 = 5$, and $x_3 = 3$. So the balanced equation is

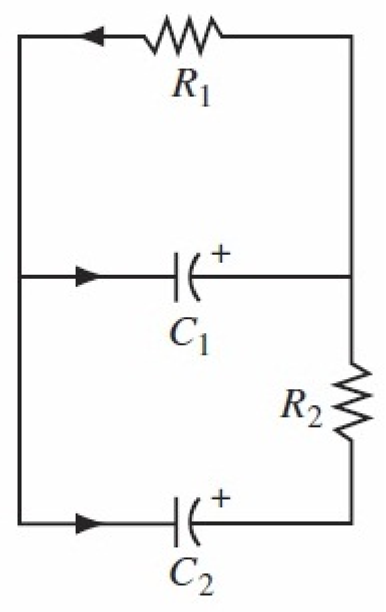
$$\mathrm{C_3H_8} + 5\mathrm{O_2} \to 3\mathrm{CO_2} + 4\mathrm{H_2O}.$$

This equation would be balanced if, for example, each coefficient were doubled. For most purposes, however, scientists prefer to use a balanced equation whose coefficients are the smallest possible whole numbers.

7. The figure on the top of next page shows an electrical circuit which is described by the system of linear differential equations:

$$\begin{cases} \dfrac{dx_1}{dt} = -\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)\dfrac{1}{C_1}x_1 + \dfrac{1}{R_2C_1}x_2 \\ \dfrac{dx_2}{dt} = \dfrac{1}{R_2C_2}x_1 - \dfrac{1}{R_2C_2}x_2 \end{cases}$$

where $x_1(t)$ and $x_2(t)$ are the voltages across the two capacitors at time t. Suppose $R_1 = 1\,\Omega$, $R_2 = 2\,\Omega$, $C_1 = 1\,\mathrm{F}$, $C_2 = 0.5\,\mathrm{F}$ with an initial voltage of 5 volts on capacitor C_1 and 4 volts on capacitor C_2. Find formulas for $x_1(t)$ and $x_2(t)$ which describe how the voltages change over time.



Solution:

Plugging in the given parameters and using the operator $D \equiv d/dt$, the given system of linear differential equations becomes:

$$\begin{cases} Dx_1 = -\dfrac{3}{2}x_1 + \dfrac{1}{2}x_2 \\ Dx_2 = x_1 - x_2 \end{cases} \Rightarrow \begin{cases} \left(D + \dfrac{3}{2}\right) x_1 - \dfrac{1}{2}x_2 = 0 \\ -x_1 + (D+1)x_2 = 0 \end{cases}$$

The augmented matrix of this system is then given by

$$\left[\begin{array}{cc|c} D + \frac{3}{2} & -\frac{1}{2} & 0 \\ -1 & D+1 & 0 \end{array}\right]$$

Applying Gaussian elimination to row reduce the augmented matrix, we obtain

$$\left[\begin{array}{ccc} D + \frac{3}{2} & -\frac{1}{2} & 0 \\ -1 & D+1 & 0 \end{array}\right] \xrightarrow[\to R_2]{R_2(D+3/2)+R_1} \left[\begin{array}{ccc} D + \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & D^2 + \frac{5D}{2} + 1 & 0 \end{array}\right]$$

Using back substitution, we obtain

$$\frac{d^2x_2}{dt^2} + \frac{5}{2}\frac{dx_2}{dt} + x_2 = 0 \quad \Rightarrow \; x_2(t) = c_1 e^{-2t} + c_2 e^{-t/2}$$

$$\begin{aligned}
\frac{dx_1}{dt} + \frac{3}{2}x_1 - \frac{1}{2}c_1 e^{-2t} - \frac{1}{2}c_2 e^{-t/2} = 0 \\
\Rightarrow\ \frac{d}{dt}(x_1 e^{3t/2}) = \frac{1}{2}c_1 e^{-t/2} + \frac{1}{2}c_2 e^{t} \\
\Rightarrow\ x_1(t) = -c_1 e^{-2t} + \frac{1}{2}c_2 e^{-t/2}
\end{aligned}$$

where c_1 and c_2 are arbitrary constants.

Comparing with the initial conditions $x_1(0) = 5\,\mathrm{V}$ and $x_2(0) = 4\,\mathrm{V}$ yields

$$x_1(0) = -c_1 + \frac{1}{2}c_2 = 5, \qquad x_2(0) = c_1 + c_2 = 4$$
$$\Rightarrow\ c_1 = -2, \qquad c_2 = 6$$

Hence, we find that

$$x_1(t) = 2e^{-2t} + 3e^{-t/2}, \qquad x_2(t) = -2e^{-2t} + 6e^{-t/2}.$$

8. A particle is moving in a planar force field with its position vector $\mathbf{x}(t) = x_1(t)\hat{\mathbf{i}} + x_2(t)\hat{\mathbf{j}}$ governed by the system of linear differential equations:

$$\begin{cases} \dfrac{dx_1}{dt} = 4x_1 - 5x_2 \\ \dfrac{dx_2}{dt} = -2x_1 + x_2 \end{cases}$$

Find a solution of this system that satisfies the initial conditions $x_1(0) = 2.9$ and $x_2(0) = 2.6$ (in SI units).

Solution:

The given system of linear DEs can be rewritten in matrix form as

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

where $\quad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 2.9 \\ 2.6 \end{bmatrix}.$

Assume the solution of this equation to be of the form $\mathbf{x} = \mathbf{v}e^{\lambda t}$ where λ

is a undetermined constant and $\mathbf{v}$ is a constant non-zero column vector. Plugging this solution back into the matrix equation, we have

$$\begin{aligned} \lambda\mathbf{v} &= \mathbf{A}\mathbf{v} \\ \Rightarrow\ (\mathbf{A}-\lambda\mathbf{I})\mathbf{v} &= \mathbf{0} \end{aligned}$$

It implies that λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ is the corresponding eigenvector. To find the eigenvalue λ, we solve the characteristic equation

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & -5 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow\ (6-\lambda)(1+\lambda) = 0$$

$$\Rightarrow\ \lambda = 6 \quad \text{or} \quad \lambda = -1$$

For the eigenvalue $\lambda = \lambda_1 = 6$, the matrix equation becomes

$$\begin{bmatrix} -2 & -5 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

which implies $2v_1 + 5v_2 = 0$. So the eigenvector for such case is

$$\mathbf{v}_1 = c_1 \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

where c_1 is an arbitrary constant.

For the eigenvalue $\lambda = \lambda_2 = -1$, the matrix equation becomes

$$\begin{bmatrix} 5 & -5 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

which implies $v_1 - v_2 = 0$. So the eigenvector for such case is

$$\mathbf{v}_2 = c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where c_2 is an arbitrary constant.

Notice that $\mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{v}_2 e^{\lambda_2 t}$ are linearly independent solutions of the matrix equation $d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$. Thus the general solution of this matrix equation is

$$\mathbf{x}(t) = \mathbf{v}_1 e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 5 \\ -2 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

for any constants c_1 and c_2. To find the solution that satisfies the initial conditions $\mathbf{x}(0)$, we seek for the values of c_1 and c_2 that "match" with $\mathbf{x}(0)$, i.e.

$$c_1 \begin{bmatrix} 5 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.9 \\ 2.6 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 5 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2.9 \\ 2.6 \end{bmatrix}$$

Solving this equation yields $c_1 = 3/70$ and $c_2 = 94/35$. Hence, the solution of the system that satisfies the given initial conditions is

$$\mathbf{x}(t) = \frac{3}{70} \begin{bmatrix} 5 \\ -2 \end{bmatrix} e^{6t} + \frac{94}{35} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

Alternative approach:

Let λ_1, λ_2 and $\mathbf{v_1}$, $\mathbf{v_2}$ be the eigenvalues and the corresponding eigenvectors of the matrix $\mathbf{A}$ (which we have found above).

We first diagonalize $\mathbf{A}$ by setting

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$$

where $\quad \mathbf{P} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -2 & 1 \end{bmatrix} \quad$ and $\quad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}.$

Next, let $y_1(t)$ and $y_2(t)$ be the functions that satisfies the relation

$$\mathbf{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \mathbf{P}^{-1}\mathbf{x}$$

Then we have

$$\frac{d\mathbf{y}}{dt} = \mathbf{P}^{-1}\frac{d\mathbf{x}}{dt} = \mathbf{P}^{-1}\mathbf{A}\mathbf{x} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{y} = \mathbf{D}\mathbf{y}$$

So $\mathbf{y}(t)$ is governed by the uncoupled system of linear DEs

$$\begin{cases} \dfrac{dy_1}{dt} = 6y_1 \\ \dfrac{dy_2}{dt} = -y_2 \end{cases}$$

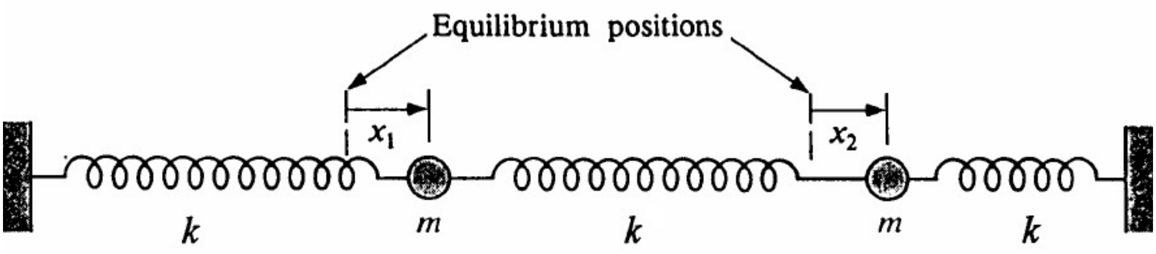
which has solution $y_1(t) = c_1 e^{6t}$ and $y_2(t) = c_2 e^{-t}$ for arbitrary constants c_1 and c_2.

Hence, the general solution of the matrix equation $d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = \mathbf{P}\mathbf{y}(t) = \begin{bmatrix} 5c_1e^{6t} + c_2e^{-t} \\ -2c_1e^{6t} + c_2e^{-t} \end{bmatrix} = c_1 \begin{bmatrix} 5 \\ -2 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

Then we can again find the solution that satisfies the given initial conditions by matching with the initial conditions.

9. Suppose two identical masses m, which are free to slide over a frictionless horizontal surface, are attached to one another and to two immovable walls by means of three identical light horizontal springs of spring constant k, as shown in the figure below. Let $x_1(t)$ and $x_2(t)$ be the displacements of the left and right masses where $x_1 = x_2 = 0$ corresponds to the equilibrium configuration in which all the springs are unextended.



   (a) Show that the displacements $x_1(t)$ and $x_2(t)$ satisfy the following system of linear differential equations:

$$\begin{cases} m\dfrac{d^2x_1}{dt^2} = -kx_1 + k(x_2 - x_1), \\ m\dfrac{d^2x_2}{dt^2} = -k(x_2 - x_1) + k(-x_2). \end{cases}$$

   (b) To find the common frequency ω such that all masses vibrate at this frequency when the system is vibrating, we can write the displacement of the masses as

$$x_1(t) = c_1 \exp(i\omega t), \qquad x_2(t) = c_2 \exp(i\omega t)$$

   for some constants c_1 and c_2.
   Show that ω satisfies the characteristic equation

$$\mathbf{A}\mathbf{x} = \omega^2\mathbf{x}$$

   where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{bmatrix}$.

   Hence, find the possible values for the common frequency ω.

Solution:

(a) If the left and right masses have displacements $x_1(t)$ and $x_2(t)$ respectively, then the extensions of the left, middle, and right springs are x_1, $x_2 - x_1$, and $-x_2$, respectively. Thus the equations of motion of the two masses are

$$\begin{cases} m\dfrac{d^2x_1}{dt^2} = -kx_1 + k(x_2 - x_1), \\ m\dfrac{d^2x_2}{dt^2} = -k(x_2 - x_1) + k(-x_2). \end{cases}$$

(b) We first rewrite the system of differential equations obtained in (a):

$$\begin{cases} \dfrac{d^2x_1}{dt^2} + \dfrac{2k}{m}x_1 - \dfrac{k}{m}x_2 = 0, \\ \dfrac{d^2x_2}{dt^2} - \dfrac{k}{m}x_1 + \dfrac{2k}{m}x_2 = 0. \end{cases}$$

It can be expressed in matrix notation as

$$\frac{d^2\mathbf{x}}{dt^2} + \mathbf{A}\mathbf{x} = 0$$

where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{bmatrix}$, and $\omega_0 = \sqrt{\dfrac{k}{m}}$.

Substituting $x_1(t) = C_1 \exp(i\omega t)$ and $x_2(t) = C_2 \exp(i\omega t)$ into the above equation, we obtain

$$\begin{aligned} &(i\omega)^2\mathbf{x} + \mathbf{A}\mathbf{x} = 0 \\ \Rightarrow\ &\mathbf{A}\mathbf{x} = \omega^2\mathbf{x} \end{aligned}$$

It implies that the vector $\mathbf{x}(t)$ must be the eigenvector of the matrix $\mathbf{A}$ with corresponding eigenvalue ω^2. To find the eigenvalue ω^2, we solve the characteristic equation

$$\det(\mathbf{A} - \omega^2\mathbf{I}) = \begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{vmatrix} = 0$$

$$\begin{aligned} &\Rightarrow\ (\omega_0^2 - \omega^2)(3\omega_0^2 - \omega^2) = 0 \\ &\Rightarrow\ \omega = \omega_0 \quad \text{or} \quad \omega = \sqrt{3}\omega_0 \end{aligned}$$

(Here we have neglected the negative roots since a negative frequency oscillation is equivalent to an oscillation with an equal and positive frequency but an equal and opposite phase.) It is thus apparent that the dynamical system shown in the given figure has two unique frequencies of oscillation: $\omega = \omega_0 = \sqrt{k/m}$ and $\omega = \sqrt{3}\omega_0 = \sqrt{3k/m}$.

⟨III⟩ Problems

1. Consider the matrix

$$\mathbf{L} = \begin{bmatrix} \gamma & 0 & \gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   where $\gamma = 1/\sqrt{1-\beta^2}$ and $\beta \in [0, 1)$.

   (a) Find the inverse of $\mathbf{L}$.

   (b) Show that $\mathbf{L}$ is a Lorentz transformation, i.e. it satisfies the relation

$$\mathbf{g} = \mathbf{L}^{\mathsf{T}}\mathbf{g}\mathbf{L}$$

   where $\mathbf{g}$ is the Minkowski metric matrix defined by

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

   (c) Define the hyperbolic angle ϕ by the relation $\cosh\phi = (e^{\phi} + e^{-\phi})/2 = \gamma$. Show that $\mathbf{L}$ can be rewritten as

$$\mathbf{L} = \begin{bmatrix} \cosh\phi & 0 & \sinh\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\phi & 0 & \cosh\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Also show that $\tanh\phi = \beta$ and $e^{\phi} = \sqrt{(1+\beta)/(1-\beta)}$.

2. Consider a physical system whose energy E is represented by the matrix

$$\mathbf{H} = \hbar\omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

while two other observables A and B are represented by the matrices

$$\mathbf{A} = \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 2\alpha \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2\beta & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \beta & 0 \end{bmatrix}$$

where α, β, and ω are real, non-zero numbers.

(a) Find the eigenvalues and eigenvectors of $\mathbf{A}$ and $\mathbf{B}$.

(b) Suppose the system is in the state described by the state vector

$$\boldsymbol{\psi} = c_1\boldsymbol{\phi}_1 + c_2\boldsymbol{\phi}_2 + c_3\boldsymbol{\phi}_3$$

where c_1, c_2, and c_3 are (complex) constants and

$$\boldsymbol{\phi}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\mathsf{T}, \quad \boldsymbol{\phi}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\mathsf{T}, \quad \boldsymbol{\phi}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\mathsf{T}.$$

(i) Find the relationship between c_1, c_2, and c_3 so that $\boldsymbol{\psi}$ is normalized to unity.

(ii) Suppose $\boldsymbol{\psi}$ is normalized to unity. Determine the expectation values of E, A, and B in terms of c_1, c_2, and c_3.

(iii) What are the possible values of the energy E that can be obtained in a measurement if the system is in the state described by the vector $\boldsymbol{\psi}$?

3. In terms of the basis vectors of the space spanned by the common eigenfunctions of the operators $\hat{S}^2$ and $\hat{S}_z$ with spin quantum number $s = 1/2$, the spin angular momentum operators are represented by the matrices

$$\mathbf{S}_\mathrm{x} = \frac{\hbar}{2}\boldsymbol{\sigma_x}, \quad \mathbf{S}_\mathrm{y} = \frac{\hbar}{2}\boldsymbol{\sigma_y}, \quad \mathbf{S}_\mathrm{z} = \frac{\hbar}{2}\boldsymbol{\sigma_z},$$

where $\boldsymbol{\sigma_x}$, $\boldsymbol{\sigma_y}$, and $\boldsymbol{\sigma_z}$ are the Pauli matrices defined by

$$\boldsymbol{\sigma_x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma_y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \boldsymbol{\sigma_z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(a) Verify that $\boldsymbol{\sigma_x^2} = \boldsymbol{\sigma_y^2} = \boldsymbol{\sigma_z^2} = \mathbf{I}$.

(b) Calculate the commutators $[\mathbf{S}_\mathrm{x}, \mathbf{S}_\mathrm{y}]$, $[\mathbf{S}_\mathrm{y}, \mathbf{S}_\mathrm{z}]$, and $[\mathbf{S}_\mathrm{z}, \mathbf{S}_\mathrm{x}]$. (Recall that the commutator $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$.)

(c) Calculate the anticommutator $\{\mathbf{S}_\mathrm{x}, \mathbf{S}_\mathrm{y}\} = \mathbf{S}_\mathrm{x}\mathbf{S}_\mathrm{y} + \mathbf{S}_\mathrm{y}\mathbf{S}_\mathrm{x}$.

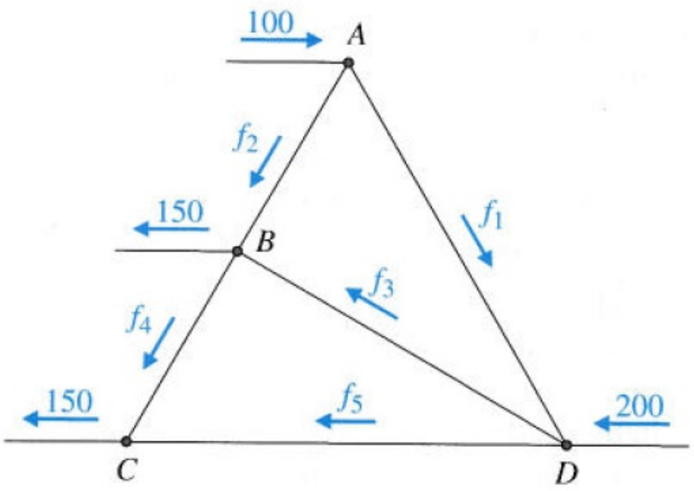
4. A network of irrigation ditches is shown in the figure below, where flows are measured in thousands of liters per day.

   (a) Set up and solve a system of linear algebraic equations to find the possible flows $f_1, f_2, \dots, f_5$.

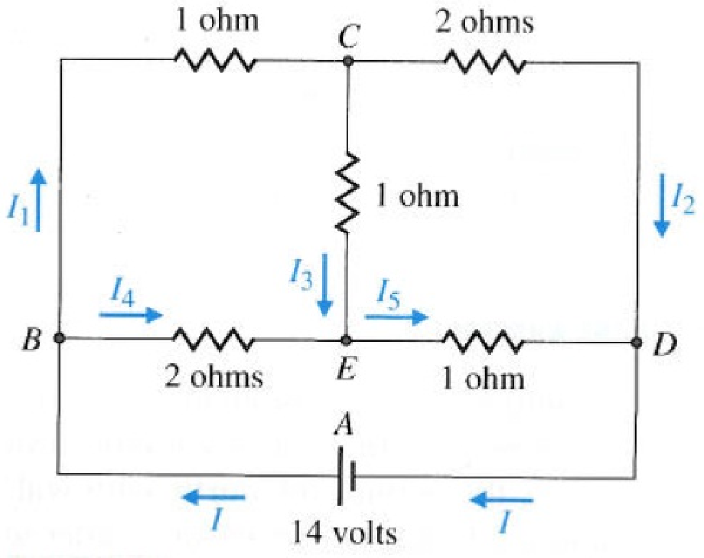
   (b) Suppose the ditch DC is closed. What range of flow will need to be maintained through the ditch DB?

   (c) From below figure, it is clear that the ditch DB cannot be closed. (Why not?) How does your solution in part (a) show this?

   (d) From your solution in part (a), determine the minimum and maximum flows through the ditch DB.



5. Using Kirchhoff's laws, find the currents $I_1, I_2, \dots, I_5$ in the electrical network as shown below.

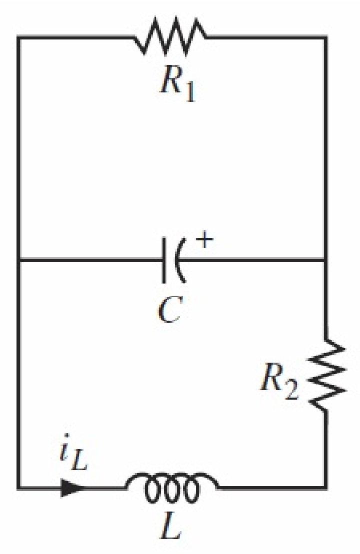


6. At a high temperature, sodium cyanide (NaCN) is produced by the reaction between sodium carbonate (Na_2CO_3), carbon (C), and nitrogen gas (N_2) in the presence of iron as a catalyst according to the equation

$$(x_1)\mathrm{Na_2CO_3} + (x_2)\mathrm{C} + (x_3)\mathrm{N_2} \to (x_4)\mathrm{NaCN} + (x_5)\mathrm{CO}.$$

Balance this chemical equation using the vector equation approach.

7. Below figure shows an RLC electrical circuit with $R_1 = 1\,\Omega$, $R_2 = 0.125\,\Omega$, $C = 0.2\,\mathrm{F}$, $L = 0.125\,\mathrm{H}$.



It can be shown that the circuit is described by the system of linear differential equations:

$$\begin{cases} \dfrac{di_L}{dt} = -\dfrac{R_2}{L} i_L - \dfrac{1}{L} v_C, \\ \dfrac{dv_C}{dt} = \dfrac{1}{C} i_L - \dfrac{1}{R_1 C} v_C, \end{cases}$$

where $i_L(t)$ is the current passing through the inductor L and $v_C(t)$ is the voltage drop across the capacitor C at time t. Find formulas for $i_L(t)$ and $v_C(t)$ if the initial current through the inductor is 0 amp and initial voltage across the capacitor is 15 volts.

8. Consider a general nonhomogeneous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{b}(t)$$

where $\mathbf{A}$ is a constant coefficient $n \times n$ matrix with real eigenvalues and n linearly independent eigenvectors $\mathbf{v}_1(t)$, $\mathbf{v}_2(t)$, …, $\mathbf{v}_n(t)$. Substituting the transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} \quad \text{with } \mathbf{P} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \dots & \mathbf{v_n} \end{bmatrix},$$

the above system can be transformed into the uncoupled system

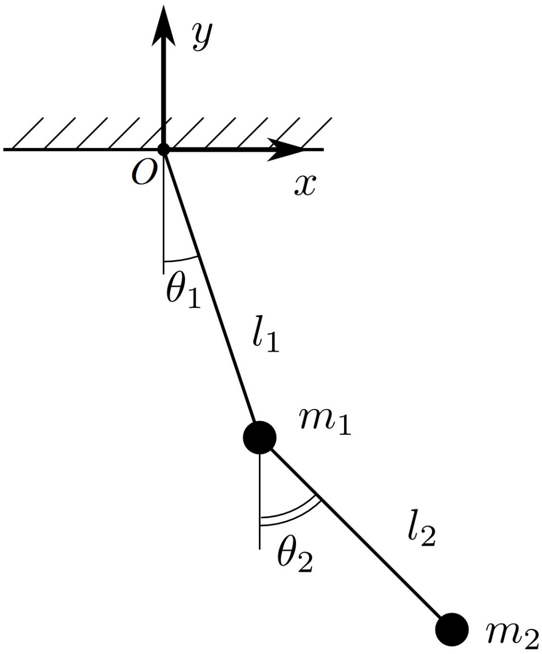
$$\frac{d\mathbf{y}}{dt} = \mathbf{D}\mathbf{y} + \mathbf{P}^{-1}\mathbf{b}(t)$$

where $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a $n \times n$ diagonal matrix and $\mathbf{P}^{-1}\mathbf{b}(t)$ is an n-element column vector depending on the non-homogeneous term $\mathbf{b}(t)$.

Using this approach, find the position of the particle $\mathbf{x}(t) = x_1(t)\hat{\mathbf{i}} + x_2(t)\hat{\mathbf{j}}$ moving in a planar force field governed by the nonhomogeneous system:

$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1 - 4x_2 + (2t - 1), \\ \dfrac{dx_2}{dt} = -x_1 + x_2 + \sin t. \end{cases}$$

9. The figure on the next page shows a double pendulum which is consisted of two simple pendulums of masses m_1 and m_2 and lengths l_1 and l_2, respectively. Assume that the angular displacement are so small that $\sin\theta_1 \approx \theta_1$ and $\sin\theta_2 \approx \theta_2$.



(a) Show that the equations of motion of the double pendulum are

$$(m_1+m_2)l_1\frac{d^2\theta_1}{dt^2}+m_2l_2\frac{d^2\theta_2}{dt^2}\cos(\theta_2-\theta_1)-m_2l_2\left(\frac{d\theta_2}{dt}\right)^2\sin(\theta_2-\theta_1)$$
$$=-(m_1+m_2)g\sin\theta_1$$

$$l_1\frac{d^2\theta_1}{dt^2}\cos(\theta_2-\theta_1)+l_1\left(\frac{d\theta_1}{dt}\right)^2\sin(\theta_2-\theta_1)+l_2\frac{d^2\theta_2}{dt^2}=-g\sin\theta_2$$

(b) Using the results of part (a) with the terms involving $(d\theta_1/dt)^2$ and $(d\theta_2/dt)^2$ neglected and replacing $\cos(\theta_1-\theta_2)$ by 1, it can be shown that the motion of this mechanical system is described by the system of linear differential equations:

$$\begin{cases} (m_1+m_2)l_1\dfrac{d^2\theta_1}{dt^2}+m_2l_2\dfrac{d^2\theta_2}{dt^2}=-(m_1+m_2)g\theta_1, \\ l_1\dfrac{d^2\theta_1}{dt^2}+l_2\dfrac{d^2\theta_2}{dt^2}=-g\theta_2. \end{cases}$$

Assume the solution to be of the form

$$\theta_1(t)=C_1\exp(i\omega t),\qquad \theta_2(t)=C_2\exp(i\omega t).$$

Show that the possible values for the parameter ω are

$$\omega=\sqrt{\frac{(m_1+m_2)(l_1+l_2)g\pm\sqrt{(m_1+m_2)[m_1(l_1-l_2)^2+m_2(l_1+l_2)^2]}g}{2m_1l_1l_2}}$$