The University of Hong Kong

Department of Physics

Summer Enrichment Classes
Essential Mathematics for Physics Study

June 2025

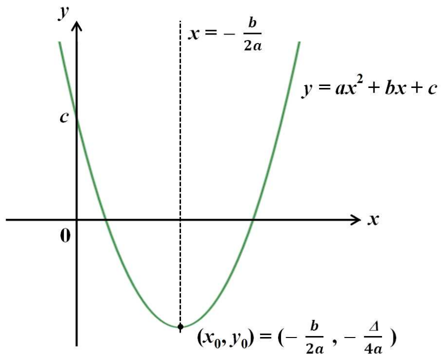
Contents

Chapter 1

Quadratic Equations

1.1 Introduction

A quadratic equation has the form $a\,x^2+b\,x+c=0$, where a, b and c are real coefficients. The equation has two roots α and β, where the sum and product of them can be expressed in terms of the coefficients: $\alpha+\beta=-b/a$ and $\alpha\beta=c/a$ respectively. The equation has two distinct real roots when the discriminant $\Delta>0$, where $\Delta=b^2-4ac$. The roots are equal when $\Delta=0$. If $\Delta<0$, the roots become imaginary.



A quadratic equation $a\,x^2+b\,x+c=0$ has two roots and they are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Given a quadratic function $y=a\,x^2+b\,x+c$, it represents a parabola on the 2-D plane with y-intercept at $(0,c)$. Completing the square of the above function, we have

$$y=a\,(x+\frac{b}{2a})^2-\frac{b^2-4\,ac}{4\,a}$$

The condition for $y>0$ or $y<0$ for all real values of x is $\Delta<0$. The parabola has a maximum if $a<0$ and it has a minimum if $a>0$. The extremum of the parabola is

$-\Delta/(4a)$ while the line of symmetry of the parabola is $x = -b/(2a)$. Alternatively, if we let the coordinates of the vertex be (x_0, y_0), then

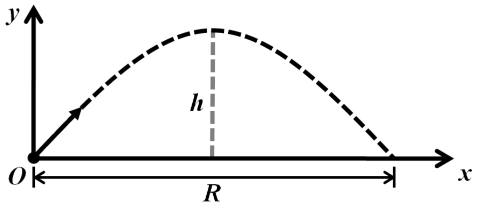
$$(x_0, y_0) = (-\frac{b}{2\,a}, -\frac{\Delta}{4\,a}) = (x_0,\, c - a\,x_0^2) \tag{1.1}$$

Example 1.1

(a) Find the formula for a quadratic function $y = f(x)$ with x-intercepts $(\alpha, 0)$ and $(\beta, 0)$ and y-intercept $(0, c)$, where α, β, and c are non-zero numbers.

(b) A particle is projected from a point O on the horizontal floor, as shown in the figure. The range of the projectile is R and the maximum height that the particle can reach is h. Show that the equation of trajectory of the particle is

$$\frac{y}{h} = \frac{4x}{R}\left(1 - \frac{x}{R}\right)$$



Solution

(a) Let the formula be $y = A\,(x - \alpha)\,(x - \beta)$ for which $(\alpha, 0)$ and $(\beta, 0)$ satisfy the equation. Plugging the y-intercept into the formula which gives $c = A\,(-\alpha)\,(-\beta)$. So $A = \dfrac{c}{\alpha\,\beta}$. Hence, the formula is $y = \dfrac{c}{\alpha\,\beta}\,(x - \alpha)\,(x - \beta)$.

(b) The projectile intersection the floor at $x = 0$ and $x = R$. So the trajectory equation is given by

$$y = A\,(x - 0)\,(x - R)\,,$$

where A is a constant to be determined. Simply, we can write

$$y = A\,x\,(x - R)\,,$$

Knowing that the trajectory is symmetric about $x = \dfrac{R}{2}$, then $(\dfrac{R}{2}\,, h)$ satisfies the above equation. So

$$\begin{aligned} h &= A\left(\frac{R}{2}\right)\left(\frac{R}{2} - R\right) \\ A &= -\frac{4h}{R^2} \end{aligned}$$

Therefore, we get $y = -\frac{4h}{R^2}\left[x\left(x-R\right)\right]$, then

$$\frac{y}{h} = \frac{4x}{R}\left(1 - \frac{x}{R}\right)$$

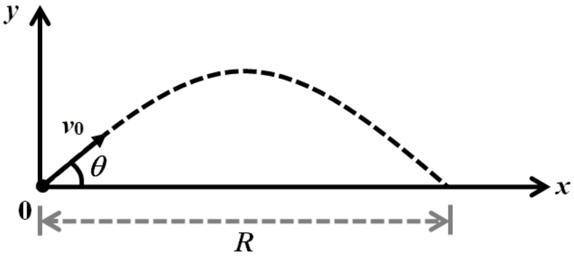
■

Example 1.2 A particle is thrown on a horizontal plane with an initial speed v_0. Show that the sum of all possible angles of projection is $\pi/2$, if these projection angles result the same range.

Solution

Let the projection angle be θ. The horizontal and vertical displacements of the particle at time t are

$$\begin{cases} x = (v_0 \cos\theta)\, t \\ y = (v_0 \sin\theta)\, t - \frac{1}{2} g t^2 \end{cases}$$



Eliminate the time t, we obtain the trajectory equation

$$y = x\tan\theta - \frac{g}{2v_0^2\cos^2\theta}\, x^2 \tag{1.2}$$

The above equation can be rewritten as $y = x\tan\theta - \frac{g\,x^2}{2v_0^2}\left(1+\tan^2\theta\right)$. The particle hits the floor when the particle is located at $x = R$ and $y = 0$. Hence, we obtain a quadratic equation of $\tan\theta$, where

$$gR\tan^2\theta - 2v_0^2\tan\theta + gR = 0 \tag{1.3}$$

Let $\tan\theta_1$ and $\tan\theta_2$ be the roots of the equation, we have $\tan\theta_1\,\tan\theta_2 = gR/(gR) = 1$. Hence, $\theta_1 + \theta_2 = \pi/2$.

Remark:

We have used the identity $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

Alternative method:

Dividing both sides of equation (1.3) by $\tan^2\theta$ and set $\theta = \theta_1$, we have

$$gR\cot^2\theta_1 - 2v_0^2\cot\theta_1 + gR = 0\,,$$

which implies $gR\tan^2(\frac{\pi}{2} - \theta_1) - 2v_0^2\tan(\frac{\pi}{2} - \theta_1) + gR = 0$. Thus, $\theta_2 = \frac{\pi}{2} - \theta_1$ satisfies equation (1.3), i.e. $\theta_1 + \theta_2 = \frac{\pi}{2}$. Another method is shown in example 2.2. ■

Example 1.3 A particle is projected with speed v_0 and the angle of projection is θ. Obtain the maximum height that it can reach.

Solution

The trajectory equation is given by $y = x\tan\theta - \dfrac{g}{2v_0^2\cos^2\theta}\,x^2$. Rearrange the equation, it becomes $\dfrac{g}{2v_0^2\cos^2\theta}\,x^2 - x\tan\theta + y = 0$. This equation has real roots in x if $\Delta = (-\tan\theta)^2 - 4\left(\dfrac{g}{2\,v_0^2\cos^2\theta}\right)y \geq 0$. Hence, we have

$$y \leq \frac{v_0^2\sin^2\theta}{2g}$$

■

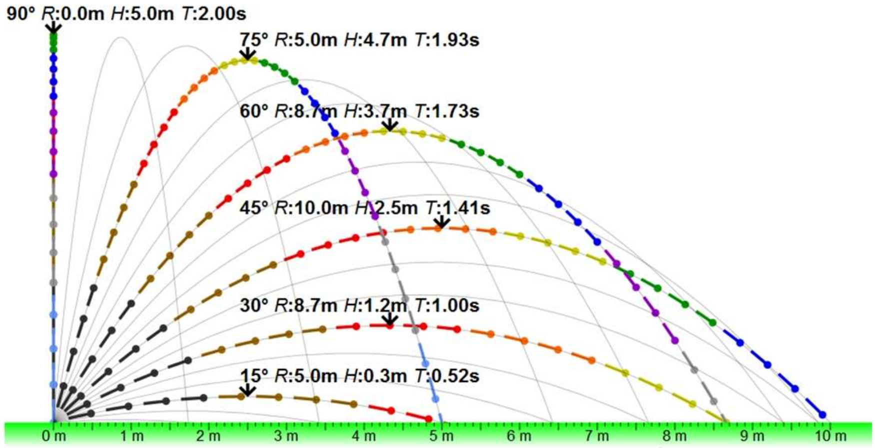
Example 1.4 A rifle can fire a shell with speed v in any direction. Show that a shell can reach any target within a paraboloid

$$2gv^2z = v^4 - g^2r^2,$$

where z is the height of the target and r is the horizontal distance of the target from the rifle.

Solution

Let's take a look on a numerical example before we investigate the general case. The following figure shows the trajectories when the initial speed is $v_0 = 10$ m/s and the angle of projection varies for every 5°. The particle cannot reach any point outside the paraboloid formed by these trajectories.



Let's proceed the general study. The horizontal and vertical displacements of the shell at time t are

$$\begin{cases} x = (v\cos\theta)\,t \\ z = (v\sin\theta)\,t - \dfrac{1}{2}gt^2 \end{cases}$$

So, we have $z = x\tan\theta - \dfrac{g\,x^2}{2v^2}\,(1+\tan^2\theta)$. Rewrite x by r, where r is the horizontal distance measured from the origin in all directions. We obtain $z = r\tan\theta - \dfrac{g\,r^2}{2v^2}\,(1+\tan^2\theta)$. Rearrange it, we have $\dfrac{gr^2}{2v^2}\,\tan^2\theta - r\,\tan\theta + (z + \dfrac{g\,r^2}{2v^2}) = 0$. It is a quadratic equation in $\tan\theta$. For real roots in the equation, we have

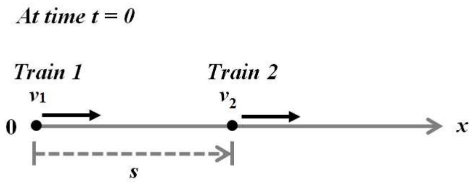
$$r^2 - \frac{4gr^2}{2v^2}\,(z + \frac{gr^2}{2v^2}) \geq 0\,,$$

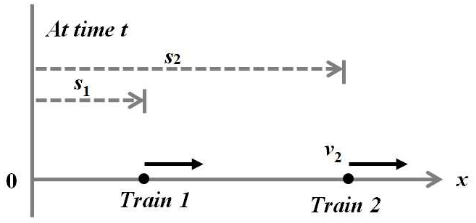
which implies $2gv^2z \leq v^4 - g^2r^2$. Therefore, the shell is bounded by the paraboloid $2gv^2z = v^4 - g^2r^2$. ■

Example 1.5 A train of speed v_1 moves along a straight railway and it discovers that there is another train moving in front of it and at a distance s apart from it. The second train has a constant speed v_2 moving along the same direction on the same railway, where $v_1 > v_2$. At this instant, the first train decelerates with magnitude d such that they never collide. Find the condition of d.

Solution

Let us construct the coordinate system and fix the origin of it at the location of train 1 when train 1 just discovered train 2. The x-axis is along the motion of trains. Let s_1 and s_2 be the coordinates of train 1 and train 2 at time t.





$$\begin{cases} s_1 = v_1\,t - \dfrac{1}{2}\,d\,t^2 \\ s_2 = s + v_2\,t \end{cases}$$

If the trains never collide, $s_1 < s_2$. In other words, $v_1 t - \dfrac{1}{2}\,d\,t^2 < s + v_2\,t$. The inequality becomes $\dfrac{1}{2}\,d\,t^2 - (v_1 - v_2)\,t + s > 0$ and it is true for all real values of t when

$$\Delta = (v_1 - v_2)^2 - 4\,(\frac{1}{2}\,d)\,(s) < 0.$$

Therefore, the magnitude of the deceleration $d > \dfrac{(v_1 - v_2)^2}{2s}$. ■

Exercise 1.1 Sketch the graph of $y = 2x^2 - 5x + 2$.

Exercise 1.2 Find the general solutions of $\cos 2\theta = \dfrac{\sqrt{3}}{2}$. Hence, find $\cos\dfrac{\pi}{12}\,\cos\dfrac{13\pi}{12}$ without using a calculator.

Exercise 1.3 Obtain the condition that the equations $x^2+bx+c=0$ and $x^2+px+q=0$ should have a common root.

Exercise 1.4 Obtain the extrema of y, where $y = \dfrac{x^2+2x+1}{x^2-x+1}$ and x is a real number.

Exercise 1.5 A particle is projected at the origin to reach a point (x, y), where x and y are the horizontal and vertical distance measured from the origin. Find the least speed of projection such that it occurs. Find also the corresponding angle of projection.

1.2 Kinetic Energy of a Particle

Consider a particle of mass m moving from position A to position B under the influence of a force $\vec{F}$. The work done by the force is given by $\displaystyle\int_A^B \vec{F}\cdot \mathrm{d}\vec{r} = \int_A^B m\,\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\cdot \mathrm{d}\vec{r} = \int_A^B m\,\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\cdot \vec{v}\,\mathrm{d}t = \int_A^B m\,\vec{v}\cdot \mathrm{d}\vec{v} = \int_A^B m\,\mathrm{d}\left(\frac{\vec{v}\cdot\vec{v}}{2}\right) = \int_A^B \mathrm{d}\left(\frac{mv^2}{2}\right) = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$. The quantity $T = \dfrac{1}{2}\,mv^2$ is the kinetic energy of the particle. Thus, the work done by the force is the change in kinetic energy of the particle.

1.3 Potential Energy of a Spring-mass System

A particle of mass m is attached to one end of an elastic spring. The next end of the spring is fixed. When the particle is displaced by an applied force $\vec{F}$ such that the spring becomes stretched, a restoring force $\vec{F_r}$ exerts in the spring. According to Hooke's law, the restoring force in the spring is governed by $\vec{F_r} = -k\,\vec{x}$, where $\vec{F_r} = -\vec{F}$ and $\vec{x}$ is the displacement vector of the particle from its equilibrium position. Thus, the work done on the spring by the applied force is $\displaystyle\int_0^{\vec{x}} \vec{F}\cdot \mathrm{d}\vec{x} = \int_0^{x} kx\,\mathrm{d}x = \frac{1}{2}\,kx^2$. This is the elastic potential energy stored in the spring.

1.4 Electrical Energy Stored in a Capacitor

When a capacitor has charges Q stored in it and the potential difference across it is V, we can write $Q = VC$, where C is the capacitance. The work done to charge up an uncharged capacitor of capacitance C is $\displaystyle\int_0^{Q} V\mathrm{d}q = \int_0^{Q} \frac{q}{C}\,\mathrm{d}q = \frac{Q^2}{2C}$. This is the electrical energy stored in the capacitor.

1.5 Resonance of a Driven Oscillation

A particle of mass m is connected to a massless spring of force constant k. An external dring force $F_0\,\cos\omega t$ is applied to the particle. If the motion is damped by a frictional force which is linearly proportional to the velocity of the particle, then the total force on the particle is

$$F = -kx - b\,\dot{x} + F_0\cos\omega t$$

Thus, the equation of motion of the particle becomes $m\ddot{x} + b\dot{x} + kx = F_0\,\cos\omega t$ or we can write

$$\ddot{x} + 2\,\beta\,\dot{x} + \omega^2 x = A\,\cos\omega t\,,$$

where $A = \dfrac{F_0}{m}$, $\omega^2 = \dfrac{k}{m}$, and $\beta = \dfrac{b}{2\,m}$. The complete solution is $x = x_c(t) + x_p(t)$. After significant time has gone, the complementary solution $x_c(t)$ fades out and the final solution is given by the particular solution

$$x_p(t) = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\,\omega^2\beta^2}}\,\cos(\omega t - \delta)$$

Hence, the amplitude of $x_p(t)$ is given by

$$D = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\,\omega^2\beta^2}}$$

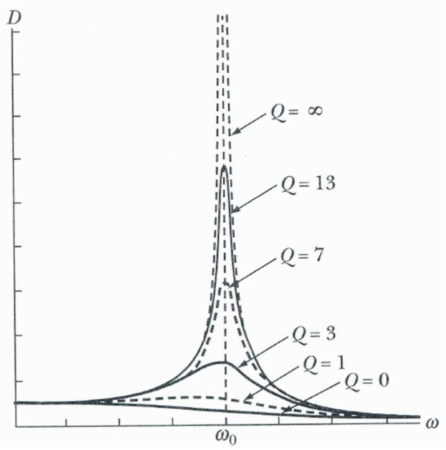
The maximum value of D is obtained when $\omega = \omega_R$, where ω_R is the resonant frequency of the oscillation and $\omega_R^2 \equiv \omega_0^2 - 2\,\beta^2$. In fact, the value of ω_R can be easily obtained by completing the square. Note that

$$\begin{aligned}
(\omega_0^2 - \omega^2)^2 + 4\,\omega^2\beta^2 &= \omega^4 + 2\,(2\,\beta^2 - \omega_0^2)\,\omega^2 + \omega_0^4 \\
&= [\omega^2 + 2\,\beta^2 - \omega_0^2]^2 - (2\,\beta^2 - \omega_0^2)^2 + \omega_0^4 \\
&= [\omega^2 + 2\,\beta^2 - \omega_0^2]^2 + 4\,\beta^2\,(\omega_0^2 - \beta^2)
\end{aligned}$$

Obviously, the above expression is a minimum and thus D attains its maximum value when $\omega = \omega_R$. Hence,

$$D_{\max} = \frac{A}{2\,\beta\,\sqrt{\omega_0^2 - \beta^2}} = \frac{A}{\sqrt{\omega_0^4 - \omega_R^4}}$$

A plot of D against ω is shown in the figure below, where the quality factor is defined as $Q \equiv \dfrac{\omega_R}{2\beta}$.



Remark: One can also obtain ω_R and $D_{\max}$ by the results stated in equation (1.1).

Chapter 2

Trigonometry

2.1 Useful Formulae in Trigonometry

$\sin^2 A + \cos^2 A = 1$	$\tan A = \dfrac{\sin A}{\cos A}$
$\sec^2 A = 1 + \tan^2 A$	$\sec A = \dfrac{1}{\cos A}$
$\csc^2 A = 1 + \cot^2 A$	$\csc A = \dfrac{1}{\sin A}$
	$\cot A = \dfrac{1}{\tan A}$

$\sin 0^\circ = 0$	$\sin 30^\circ = \dfrac{1}{2}$	$\sin 45^\circ = \dfrac{1}{\sqrt{2}}$	$\sin 60^\circ = \dfrac{\sqrt{3}}{2}$
$\cos 0^\circ = 1$	$\cos 30^\circ = \dfrac{\sqrt{3}}{2}$	$\cos 45^\circ = \dfrac{1}{\sqrt{2}}$	$\cos 60^\circ = \dfrac{1}{2}$
$\tan 0^\circ = 0$	$\tan 30^\circ = \dfrac{1}{\sqrt{3}}$	$\tan 45^\circ = 1$	$\tan 60^\circ = \sqrt{3}$

$$\begin{array}{lll} \sin 90^\circ = 1 & \sin 180^\circ = 0 & \sin 270^\circ = -1 \\ \cos 90^\circ = 0 & \cos 180^\circ = -1 & \cos 270^\circ = 0 \\ \tan 90^\circ = +\infty & \tan 180^\circ = 0 & \tan 270^\circ = -\infty \end{array}$$

$$\sin(A+B) = \sin A\, \cos B + \cos A\, \sin B$$
$$\sin(A-B) = \sin A\, \cos B - \cos A\, \sin B$$
$$\cos(A+B) = \cos A\, \cos B - \sin A\, \sin B$$
$$\cos(A-B) = \cos A\, \cos B + \sin A\, \sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\, \tan B}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\, \tan B}$$

$$\sin 2A = 2\, \sin A\, \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\, \sin^2 A = 2\, \cos^2 A - 1$$
$$\tan 2A = \frac{2\, \tan A}{1 - \tan^2 A}$$
$$\sin 3A = 3\, \sin A - 4\, \sin^3 A$$
$$\cos 3A = 4\, \cos^3 A - 3\, \cos A$$
$$\tan 3A = \frac{3\, \tan A - \tan^3 A}{1 - 3\, \tan^2 A}$$

$$\sin^2 A = \frac{1}{2}\,(1 - \cos 2A)$$

$$\cos^2 A = \frac{1}{2}\,(1 + \cos 2A)$$

$$\sin A + \sin B = 2\,\sin\left(\frac{A+B}{2}\right)\,\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\,\cos\left(\frac{A+B}{2}\right)\,\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\,\cos\left(\frac{A+B}{2}\right)\,\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\,\sin\left(\frac{A+B}{2}\right)\,\sin\left(\frac{A-B}{2}\right)$$

$$\sin A\,\cos B = \frac{1}{2}\,\{\sin(A+B) + \sin(A-B)\}$$

$$\cos A\,\cos B = \frac{1}{2}\,\{\cos(A+B) + \cos(A-B)\}$$

$$\sin A\,\sin B = -\frac{1}{2}\,\{\cos(A+B) - \cos(A-B)\}$$

Example 2.1 Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ from the first principle, if (a) $y = \sin x$ and (b) $y = \tan x$.

Solution

(a) Using the fact that $\sin A - \sin B = 2\,\cos\left(\frac{A+B}{2}\right)\,\sin\left(\frac{A-B}{2}\right)$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{2\,\cos(x+\frac{h}{2})\,\sin\frac{h}{2}}{h} \\
&= 2\lim_{h\to 0}\cos(x+\frac{h}{2})\,\lim_{\frac{h}{2}\to 0}\left(\frac{1}{2}\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right) \\
&= (2\,\cos x)\left(\frac{1}{2}\right)(1) \\
&= \cos x
\end{aligned}$$

(b) Using the fact that $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\,\tan B}$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h} \\
&= \lim_{h\to 0}\frac{(\tan h)\,[1+\tan(x+h)\,\tan x]}{h} \\
&= \lim_{h\to 0}\frac{\tan h}{h}\,\lim_{h\to 0}[1+\tan(x+h)\,\tan x] \\
&= (1)\,[1+\tan^2 x] \\
&= \sec^2 x
\end{aligned}$$

■

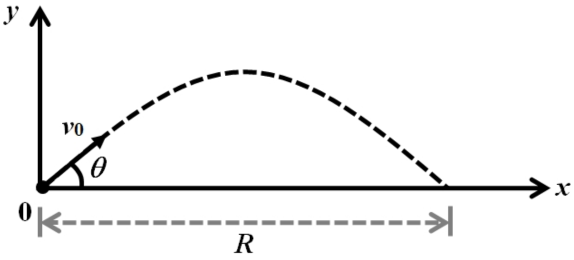
Exercise 2.1 Evaluate $\cos 36^\circ$ without using a calculator.

Example 2.2 A particle is thrown on a horizontal plane with an initial speed v_0. Show that the sum of all possible angles of projection is $\pi/2$, if these projection angles result the same range.

Solution

Let the projection angle be θ. The horizontal and vertical displacements of the particle at time t are

$$\begin{cases} x = (v_0\,\cos\theta)\,t \\ y = (v_0\,\sin\theta)\,t - \frac{1}{2}gt^2 \end{cases}$$



Eliminate the time t, we obtain the trajectory equation

$$y = x\tan\theta - \frac{g}{2v_0^2\cos^2\theta}\,x^2 \tag{2.1}$$

When the particle hits the floor, $x = R$ and $y = 0$, where R is the range of flight. Equation (2.1) gives

$$R = \frac{2\,v_0^2\,\sin\theta\cos\theta}{g} = \frac{v_0^2\,\sin 2\,\theta}{g}$$

where the identity $\sin 2\theta = 2\,\sin\theta\,\cos\theta$ has been applied.

Let θ_1 and θ_2 be the pair of angles of projection such that the same range is obtained, then we have

$$\sin 2\,\theta_1 = \sin 2\,\theta_2$$

Thus, $2\,\theta_2 = \pi - 2\,\theta_1$ gives $\theta_1 + \theta_2 = \dfrac{\pi}{2}$. ■

Example 2.3 Suppose that there are numerous particles projected simultaneously with the same initial speed v_0 at the origin at time $t = 0$ and the projections are in all directions and in the same vertical plane. What is the shape of the curve that the particles lie on at time $t > 0$?

Solution

Let the projection angle of any one of the particles be θ. The horizontal and vertical displacements of this particle at time t are

$$\begin{cases} x = (v_0\,\cos\theta)\,t \\ y = (v_0\,\sin\theta)\,t - \dfrac{1}{2}\,gt^2 \end{cases}$$

Rearrange the above equations, we obtain

$$\begin{cases} x^2 = v_0^2\,t^2\,\cos^2\theta \\ (y + \dfrac{1}{2}gt^2)^2 = v_0^2\,t^2\,\sin^2\theta \end{cases}$$

By considering the identity $\sin^2\theta + \cos^2\theta = 1$, we obtain an equation without θ

$$x^2 + (y + \frac{1}{2}gt^2)^2 = v_0^2\,t^2$$

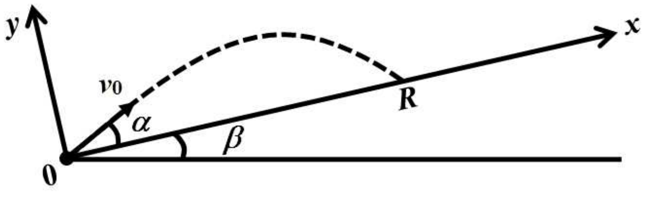
The above equation states that if numerous particles are projected simultaneously with the same initial speed v_0 at the origin at time $t = 0$, then at time $t > 0$ the particles lie on a circle. In particular, the circle is centered at $(0, -\frac{1}{2}gt^2)$ and has a radius $v_0\,t$. ■

Example 2.4 A particle is thrown uphill with speed v_0 on an inclined plane. The angle of projection measured from the inclined plane is α. The elevation angle of the inclined plane measured from the horizontal is β. Find the range of the particle on the inclined plane. Show further that if α_1 and α_2 are the possible angles for the same range, then $\alpha_1 + \alpha_2 + \beta = \dfrac{\pi}{2}$.

Solution

Construct the coordinate system where the origin is located at the point of projection and the x-axis and y-axis are along and normal to the inclined plane respectively. The positive directions of x and y are uphill and above the inclined plane respectively. At time t, the displacements of the particle along the x and y axes are

$$\begin{cases} x = (v_0 \cos\alpha)\, t - \dfrac{1}{2}\,(g \sin\beta)\, t^2 \\[2ex] y = (v_0 \sin\alpha)\, t - \dfrac{1}{2}\,(g \cos\beta)\, t^2 \end{cases}$$



When the particle hits the plane, the particle has coordinates $(R, 0)$. The second equation gives $v_0 \sin\alpha = \dfrac{1}{2}\, gt \cos\beta$ and thus the required time $t = \dfrac{2v_0\,\sin\alpha}{g\cos\beta}$. Substituting the time expression into the first equation, we have the range

$$\begin{aligned} R &= (v_0\,\cos\alpha)\left(\frac{2\,v_0\,\sin\alpha}{g\cos\beta}\right) - \frac{1}{2}\,(g\,\sin\beta)\left(\frac{4\,v_0^2\sin^2\alpha}{g^2\,\cos^2\beta}\right) \\ &= \frac{2\,v_0^2\sin\alpha}{g\,\cos^2\beta}\,[\cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta] \\ &= \frac{2\,v_0^2\sin\alpha}{g\,\cos^2\beta}\,[\cos(\alpha+\beta)] \\ &= \frac{v_0^2}{g\,\cos^2\beta}\,[\sin(2\,\alpha+\beta) - \sin\beta] \end{aligned}$$

Let α_1 and α_2 be the possible angles which achieve the same uphill range. We have $2\alpha_2 + \beta = \pi - (2\alpha_1 + \beta)$ which gives $\alpha_1 + \alpha_2 + \beta = \pi/2$.

Remarks

The range R on the inclined plane is a maximum when $2\alpha + \beta = \pi/2$ which implies $\alpha = \dfrac{\pi}{4} - \dfrac{\beta}{2}$. Hence, the maximum range is

$$\frac{v_0^2}{g\,(1 + \sin\beta)}$$

■

Exercise 2.2 Refer to example 2.4, obtain the maximum range of the projectile if the particle is thrown downhill with the same speed.

Example 2.5 Given that $\sin A + \sin B = 1$ and $\cos A + \cos B = 1$, evaluate $\sin A + \cos A$.

Solution

Let's label the equations first.

$$\begin{aligned} \sin A + \sin B &= 1 \qquad (2.2) \\ \cos A + \cos B &= 1 \qquad (2.3) \end{aligned}$$

Then

$$\begin{aligned} (\sin A + \sin B)^2 &= \sin^2 A + 2\,\sin A \sin B + \sin^2 B \;=\; 1 \qquad (2.4) \\ (\cos A + \cos B)^2 &= \cos^2 A + 2\,\cos A \cos B + \cos^2 B \;=\; 1 \qquad (2.5) \end{aligned}$$

Subtracting (Eq. 2.4) from (Eq. 2.5), it gives

$$\cos 2A + \cos 2B + 2\,\cos(A + B) = 0 \qquad (2.6)$$

Thus, we obtain $\cos(A + B)\,[\cos(A - B) + 1)] = 0$. However, $\cos(A - B) + 1 = 0$ is rejected because it implies $A - B = \pi$, that is $A = \pi + B$. Substituting it into the sum $\sin A + \sin B = -\sin B + \sin B = 0$ which contradicts with Eq. (2.2). Now, we have $\cos(A + B) = 0$ which gives $A + B = \dfrac{\pi}{2}$, that is $B = \dfrac{\pi}{2} - A$. From Eq. (2.2) $\sin A + \sin B = 1$ which implies $\sin A + \sin\left(\dfrac{\pi}{2} - A\right) = 1$, then we get $\sin A + \cos A = 1$.

■

Exercise 2.3 Show that if $\sin C = \cos A + \cos B$ in $\triangle ABC$, then $\cos A\,\cos B = 0$.

Exercise 2.4 A particle is to be projected with a given speed u so that its horizontal range is R, where $R < u^2/g$. Show that if t and t' are the times of flight corresponding to the two possible angles of projection with the horizontal, then tt' is a constant. Find the constant.

Exercise 2.5 A particle is projected at an angle α to the horizontal from the foot of an inclined plane of elevated angle β. If the particle strikes the plane along the horizontal, show that $\tan\alpha = 2\,\tan\beta$. Show also that $\tan\alpha = 2\,\tan\beta + \cot\beta$, if the particle strikes the plane normally.

2.2 Sides and Area of a Triangle

A triangle ABC having corresponding sides a, b and c has the following properties.
The Sine law:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r,$$

where r is radius of the circle to inscribe the triangle.
The Cosine law:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2\,bc\,\cos A \\ b^2 &= c^2 + a^2 - 2\,ca\,\cos B \\ c^2 &= a^2 + b^2 - 2\,ab\,\cos C \end{aligned}$$

The corresponding angles are related in the following ways.

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2\,bc} \\ \cos B &= \frac{c^2 + a^2 - b^2}{2\,ca} \\ \cos C &= \frac{a^2 + b^2 - c^2}{2\,ab} \end{aligned}$$

The Area of ΔABC:

$$\mathcal{A} = \frac{1}{2}\,ab\,\sin C = \frac{1}{2}\,bc\,\sin A = \frac{1}{2}\,ca\,\sin B$$

The area can also be expressed in terms of the sides of ΔABC,

$$\mathcal{A} = \sqrt{s\,(s-a)\,(s-b)\,(s-c)},$$

where $s = (a+b+c)/2$. This is the well-known Heron's formula.

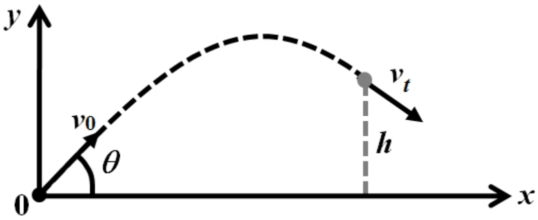
Exercise 2.6 By considering the scalar products of two vectors, derive the Cosine law.

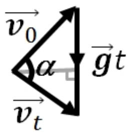
Exercise 2.7 Derive the Heron's formula by using the Cosine law.

Example 2.6 A particle is thrown with velocity $\vec{v}_0$ at point O on a horizontal plane. Find the maximum horizontal distance travelled by the particle when it is at a height h above the point of projection.

Solution

Denote $\vec{v}_t$ as the instantaneous velocity of the particle when it reaches a height h at time t. The horizontal displacement at the instant is x. Let θ be the angle of projection and α be the angle between $\vec{v}_0$ and $\vec{v}_t$. The magnitudes of $\vec{v}_0$ and $\vec{v}_t$ are v_0 and v_t respectively. The vectors $\vec{v}_0$ and $\vec{v}_t$ depict a triangle which has an area $\mathcal{A}$, where $\mathcal{A} = \frac{1}{2}\, v_0\, v_t\, \sin\alpha$. On the other hand, the area $\mathcal{A}$ can be expressed in an alternative form, e.g. $\mathcal{A} = \frac{1}{2}\,(gt)\,(v_0\,\cos\theta)$, because $\vec{v}_t = \vec{v}_0 + \vec{g}\,t$.





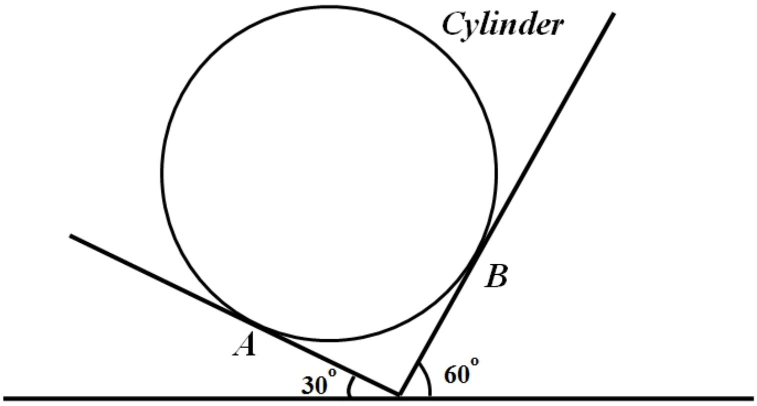
In the projectile, the horizontal distance travelled by the particle is x, where $x = (v_0\,\cos\theta)\,t$. Hence, we have $\mathcal{A} = \frac{1}{2}\,(g\,x)$. When x has a maximum value, the area $\mathcal{A}$ becomes the greatest. Refer to the previous discussion on $\mathcal{A}$, where $\mathcal{A} = \frac{1}{2}\, v_0\, v_t\, \sin\alpha$, it becomes the greatest when $\alpha = \pi/2$, since v_0 and v_t are constants and $v_t^2 = v_0^2 - 2gh$.

Therefore, we obtain $gx_{\text{max}} = v_0\, v_t$, which implies

$$x_{\text{max}} = \frac{v_0\, v_t}{g} = \frac{v_0}{g}\sqrt{v_0^2 - 2gh}$$

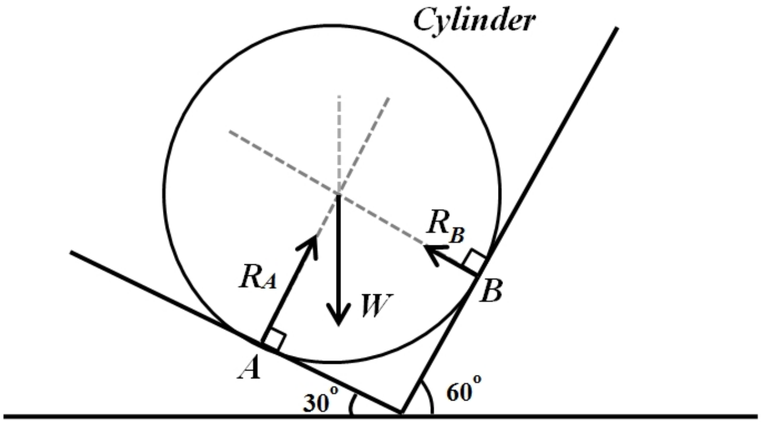
■

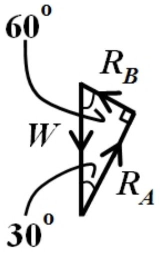
Example 2.7 A cylinder of mass 50 kg is resting on a smooth surface which are inclined at 30° and 60° to horizontal as shown in the figure. Determine the reaction at contact A and B.



Solution

Denote W, R_A and R_B as the weight of the cylinder and the reaction force acting on the cylinder at points A and B respectively. The cylinder is at equilibrium, then the magnitudes of W, R_A, and R_B form the sides of a triangle as shown in the right figure below.





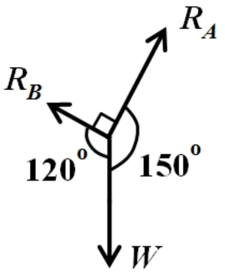
Use Sine rule, we have

$$\frac{W}{\sin 90^\circ} = \frac{R_A}{\sin 60^\circ} = \frac{R_B}{\sin 30^\circ}$$

Since $W = (50)(9.81) = 490.5\,\text{N}$, then $R_A = 424.79\,\text{N}$, and $R_B = 245.25\,\text{N}$.

Remark:

The three forces: W, R_A, and R_B can be shifted to form a vector diagram as shown below.



For equilibrium of three coplanar forces, Lami's theorem states that

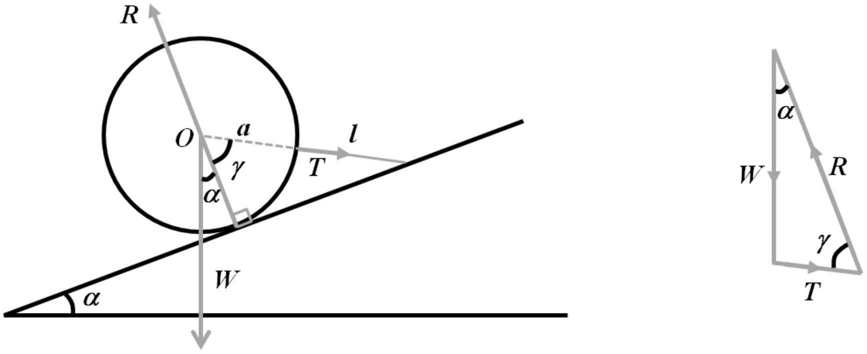
$$\frac{W}{\sin 90^\circ} = \frac{R_A}{\sin 120^\circ} = \frac{R_B}{\sin 150^\circ}$$

which give the same answers. ■

Example 2.8 A sphere, of radius a and weight W, rests on a smooth inclined plane supported by a string of length l with one end attached to a point on the surface of the sphere and the other end fastened to a point on the plane. If the angle of inclination of the plane to the horizontal is α, and the string is normal to the surface of sphere, prove that the tension of the string T is

$$\frac{W\,(a+l)\,\sin\alpha}{\sqrt{l^2+2al}}$$

Solution



The sphere is at equilibrium, thus the three forces acting on the sphere can be used to form a triangle with sides W, R and T. Using the Sine law, we have

$$\begin{aligned} \frac{T}{\sin\alpha} &= \frac{W}{\sin\gamma} \\ T &= \frac{W\,\sin\alpha}{\sin\gamma} \end{aligned}$$

Now, $\cos\gamma = \dfrac{a}{a+l}$ gives $\sin\gamma = \dfrac{\sqrt{l^2+2al}}{a+l}$. Hence,

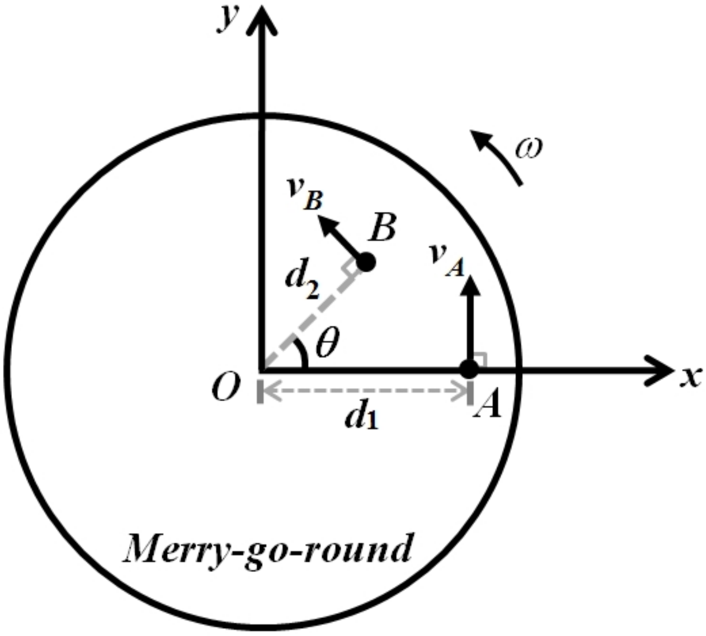
$$T = \frac{W\,(a+l)\,\sin\alpha}{\sqrt{l^2+2al}}$$

■

Example 2.9 A merry-go-round is rotating uniformly with angular speed ω about its center O. Show that B is rotating with an angular speed ω relative to A, where A and B are two observers located at two arbitrary points on the merry-go-round.

Solution

Without loss of generosity, let's consider the instant when OA lies on the x-axis and OB makes an angle θ with the x-axis, where $OA = d_1$ and $OB = d_2$.



The tangential speeds of A and B are

$$\begin{cases} \vec{v}_A = d_1\,\omega\,\hat{j} \\ \vec{v}_B = d_2\,\omega\,(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \end{cases}$$

As a review to the circular motions, one should notice that $|\vec{v}_A| = d_1\,\omega$ and $|\vec{v}_B| = d_2\,\omega$, where $\vec{v}_A \perp \overrightarrow{OA}$ and $\vec{v}_B \perp \overrightarrow{OB}$. The velocity of B with respect to A is $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$ and thus

$$\vec{v}_{BA} = -d_2\,\omega\,\sin\theta\,\hat{i} + \omega\,(d_2\,\cos\theta - d_1)\,\hat{j}$$

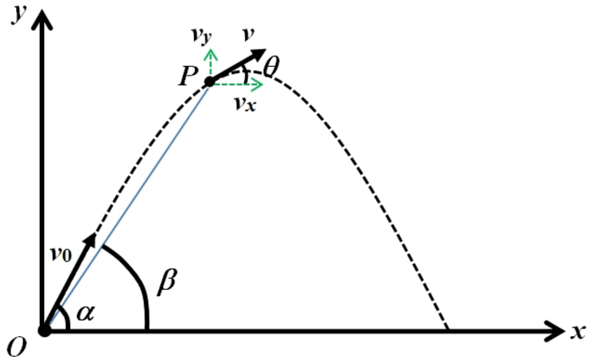
The magnitude of the relative velocity vector is

$$\begin{aligned} |\vec{v}_{BA}|^2 &= d_2^2\,\omega^2\,\sin^2\theta + \omega^2\,(d_2\,\cos\theta - d_1)^2 \\ &= \omega^2\,(d_1^2 + d_2^2 - 2d_1 d_2\,\cos\theta) \\ &= \omega^2\,d^2\,, \end{aligned}$$

where $d = AB$. Hence, we obtain $|\vec{v}_{BA}| = \omega d$. ■

Example 2.10 A particle is projected at an angle of elevation α from a point O on a horizontal plane. When the particle is travelling upwards and passes through a point P, its velocity vector makes an elevated angle θ to the horizontal. If the line PO is at an angle β to the horizontal, show that $\tan\alpha$, $\tan\beta$, and $\tan\theta$ are in arithmetic progression.

Solution



Consider the displacement of the particle along x and y directions respectively.

$$\begin{cases} x = (v_0\,\cos\alpha)\,t \\ y = (v_0\,\sin\alpha)\,t - \dfrac{1}{2}\,gt^2 \end{cases}$$

we have

$$\frac{y}{x} = \tan\beta = \frac{(v_0 \sin\alpha)\,t - \dfrac{1}{2}gt^2}{v_0\,\cos\alpha)\,t} = \tan\alpha - \frac{gt}{2\,v_0\,\cos\alpha}$$

Consider the velocity of the particle at P as v, then the components of it along x and y directions are

$$\begin{cases} v_x = (v_0\,\cos\alpha) \\ v_y = v_0\,\sin\alpha - gt \end{cases}$$

we have

$$\frac{v_y}{v_x} = \tan\theta = \frac{v_0 \sin\alpha - gt}{v_0\,\cos\alpha)} = \tan\alpha - \frac{gt}{v_0\,\cos\alpha}$$

Thus $\tan\alpha$, $\tan\beta$, and $\tan\theta$ form an arithmetic progression, where the common difference is $-\dfrac{gt}{2\,v_0\,\cos\alpha}$. The answer is also true when the particle is falling. ■

Exercise 2.8 Two smooth uniform spheres of equal masses but of radii 2 m and 3 m are placed in a fixed hemispherical bowl of radius 8 m. Show that the spheres reach their equilibrium positions when the line joing the centers of the spheres makes an angle θ with the vertical, where $\tan\theta = 48/11$.

2.3 A Useful Trick

Given a function $y = a\sin\theta + b\cos\theta$, where a and b are positive constants. To obtain the extremum of y, one can rewrite y as $r\sin(\theta+\alpha)$, where $r = \sqrt{a^2+b^2}$ and $0 < \alpha = \tan^{-1}(b/a) < \pi/2$ or one can rewrite y as $r\cos(\theta-\alpha)$, where $0 < \alpha = \tan^{-1}(a/b) < \pi/2$. Similar techniques apply to the cases when the signs of a and b have other combinations.

Example 2.11 Let $y = e^{ax}\,\sin(bx+c)$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = r\,e^{ax}\,\sin(bx+c+\phi)$, where $r = \sqrt{a^2+b^2}$ and $\tan\phi = \dfrac{b}{a}$. Hence, find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and deduce the general formula of $\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}$, where n is a positive integer.

Solution:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= b\,e^{ax}\,\cos(bx+c) + a\,e^{ax}\,\sin(bx+c)\\ &= e^{ax}\,[b\,\cos(bx+c) + a\,\sin(bx+c)]\end{aligned}$$

Let $a = r\,\cos\phi$ and $b = r\,\sin\phi$, then $r = \sqrt{a^2+b^2}$. Thus

$$\begin{aligned}b\,\cos(bx+c) + a\,\sin(bx+c) &= r\,[\sin\phi\,\cos(bx+c) + \cos\phi\,\sin(bx+c)]\\ &= r\,\sin(bx+c+\phi)\end{aligned}$$

Hence, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = r\,e^{ax}\,\sin(bx+c+\phi)$$

Differentiating both sides with respect to x again, we get

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= r\,[b\,e^{ax}\,\cos(bx+c+\phi) + a\,e^{ax}\,\sin(bx+c+\phi)]\\ &= r\,e^{ax}\,[b\,\cos(bx+c+\phi) + a\,\sin(bx+c+\phi)]\\ &= r^2\,e^{ax}\,\sin(bx+c+2\phi)\end{aligned}$$

Similarly, we have $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n} = r^n\, e^{ax}\, \sin(bx + c + n\phi)$. ■

Example 2.12 Find the greatest and least values of $\sin^2 x + 2\sin x\, \cos x + 3\, \cos^2 x$.

Solution

Let $y = \sin^2 x + 2\sin x\, \cos x + 3\, \cos^2 x$, then

$$\begin{aligned} y &= \sin 2x + 2\, \cos^2 x + 1 && (\sin 2A = 2\, \sin A\, \cos A) \\ &= \sin 2x + \cos 2x + 2 && (\cos^2 A = \frac{1}{2}\, (1 + \cos 2A)) \\ &= \sqrt{2}\, \sin\left(2x + \frac{\pi}{4}\right) + 2 \end{aligned}$$

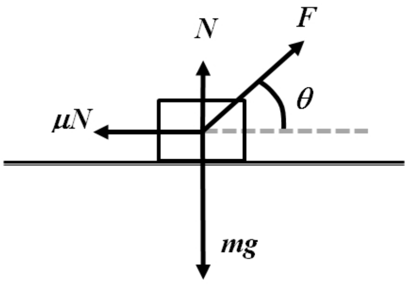
So, the greastest value of y is $2 + \sqrt{2}$ and the least value of y is $2 - \sqrt{2}$. ■

Example 2.13 An object of mass m is at rest on a rough table which has coefficient of static friction μ. Find the minimum force to move the object.

Solution

Let F be the applied force which has an elevated angle θ. The normal reaction force on the mass is N. When the mass is just to move, the equations of motion along and normal the horizontal are

$$\begin{cases} F\cos\theta = \mu\, N \\ F\sin\theta + N = mg \end{cases}$$



Eliminating N in the above equations, we have $F\sin\theta + \dfrac{F\cos\theta}{\mu} = mg$. Hence, we can write

$$\begin{aligned} F &= \frac{mg\mu}{\mu\, \sin\theta + \cos\theta} \\ &= \frac{mg\mu}{\sqrt{\mu^2+1}\,[\sin\theta\cos\alpha + \cos\theta\sin\alpha]} \\ &= \frac{mg\mu}{\sqrt{\mu^2+1}\, \sin(\theta+\alpha)} \end{aligned}$$

The least applied force $F_{\min}$ is obtained if we put $\sin(\theta + \alpha) = 1$. Therefore,

$$F_{\min} = \frac{mg\mu}{\sqrt{\mu^2 + 1}}.$$

■

Exercise 2.9 Find the greatest and least values of $\frac{x}{2} - 2 + \sqrt{4 - x^2}$.

Exercise 2.10 Find the greatest and least values of $\frac{\sin x + 1}{\cos x - 2}$.

Exercise 2.11 A man with a boat wishes to cross a river of width a from a point O to a point P on the opposite bank at a distance b downstream. The stream flows with constant speed u and the boat has a constant speed v in still water. Express v in terms of α, u, a and b, where α is an acute angle formed by the steering direction of the boat pointing upstream and the line of width of the river. Find the least speed of the boat in still water and the direction of it. What is the corresponding time required?

Chapter 3

Differentiation

3.1 Basic Ideas and the Extremum

The derivative of a function $f(x)$ is denoted by $f'(x)$ which represents the slope of the function at x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Generally, the geometrical properties of $f(x)$ can be shown by the derivatives of f.

- A function $f(x)$ is said to be increasing in the interval $a < x < b$ if $f'(x) > 0$ for any $a < x < b$.
- A function $f(x)$ is said to be decreasing in the interval $a < x < b$ if $f'(x) < 0$ for any $a < x < b$.
- If a function $f(x)$ has a local maximum at x_0 and it is twice differentiable there, then $f'(x_0) = 0$ and $f''(x_0) < 0$.
- If a function $f(x)$ has a local minimum at x_0 and it is twice differentiable there, then $f'(x_0) = 0$ and $f''(x_0) > 0$.

While doing the computation of derivatives, there is a useful relation to facilitate the calculations, it is the **chain rule**:
If f is a function of u, and u is a function of x, then we have

$$\frac{\mathrm{d}f(u)}{\mathrm{d}x} = \frac{\mathrm{d}f(u)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

We can also write it again as

$$\frac{\mathrm{d}}{\mathrm{d}x}f(u(x)) = [f'(u(x))]\,[u'(x)].$$

The **total differential** of a differentiable function $y = f(x)$ is

$$\mathrm{d}y = f'(x)\,\mathrm{d}x,$$

where the derivative $f'(x)$ links up the change in y due to the change in x.

Example 3.1 An object moving along a straight line has an acceleration a, where a is the derivative of velocity with time. Write down an expression of a without time.

Solution

From the definition of a, we have

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right) v\,,$$

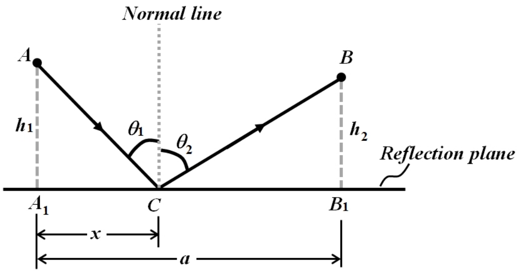
where v and x are the velocity and displacement of the object at time t. And, t does not appear explicitly in the last expression. ■

Example 3.2 Consider light passing a medium and is reflected by a plane. Use Fermat's principle to derive the law of reflection: $\theta_1 = \theta_2$, where θ_1 and θ_2 are the angle of incidence and the angle of reflection respectively.

Fermat's Principle: Light travels by the path that takes the least amount of time.

Solution

Let's think about a beam which passes through point A in a medium and is reflected at point C from the plane. The reflected beam passes through point B, as shown in the figure. The points of projection from A and B onto the reflection plane are A_1 and B_1 respectively. Denote the vertical distances that A and B make with the plane as h_1 and h_2 respectively. Let $A_1B_1 = a$ and $A_1C = x$, where $0 \leq x \leq a$.



If v is the speed of light in the medium, the total travelling time for paths AC and CB is given by

$$T(x) = \frac{AC}{v} + \frac{CB}{v} = \frac{\sqrt{h_1^2 + x^2}}{v} + \frac{\sqrt{h_2^2 + (a-x)^2}}{v}$$

The first and second derivative of T with respect to x are

$$\begin{aligned} T'(x) &= \frac{1}{v} \cdot \frac{x}{\sqrt{h_1^2 + x^2}} - \frac{1}{v} \cdot \frac{a-x}{\sqrt{h_2^2 + (a-x)^2}} \\ T''(x) &= \frac{1}{v} \cdot \frac{h_1^2}{(h_1^2 + x^2)^{3/2}} + \frac{1}{v} \cdot \frac{h_2^2}{[h_2^2 + (a-x)^2]^{3/2}} > 0 \end{aligned}$$

The turning point of $T(x)$ can be obtained when one solves the equation $T'(x) = 0$. Since $T''(x)$ is positive, the turning point of $T(x)$ is a minimum value which is the minimum time stated in Fermat's principle. Let's work it out and set $T'(x) = 0$, Thus,

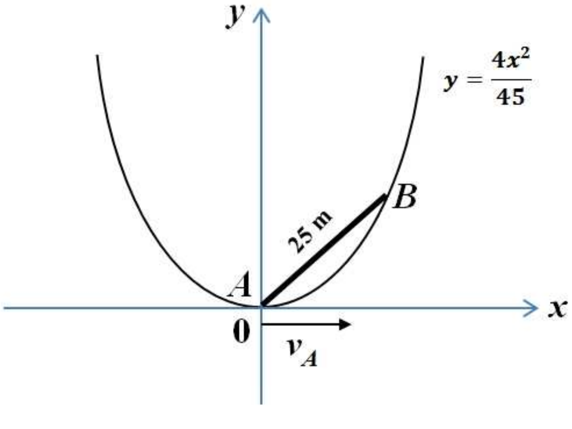
$$\begin{aligned} \frac{1}{v_1} \cdot \frac{x}{\sqrt{h_1^2 + x^2}} &= \frac{1}{v_1} \cdot \frac{a-x}{\sqrt{h_2^2 + (a-x)^2}} \\ \sin\theta_1 &= \sin\theta_2 \\ \theta_1 &= \theta_2 \end{aligned}$$

The last equation is the law of reflection of light. ■

Exercise 3.1 Use Fermat's principle to obtain the law of refraction for light.

Exercise 3.2 If $x^y + y^x = a^b$, find $\mathrm{d}y/\mathrm{d}x$.

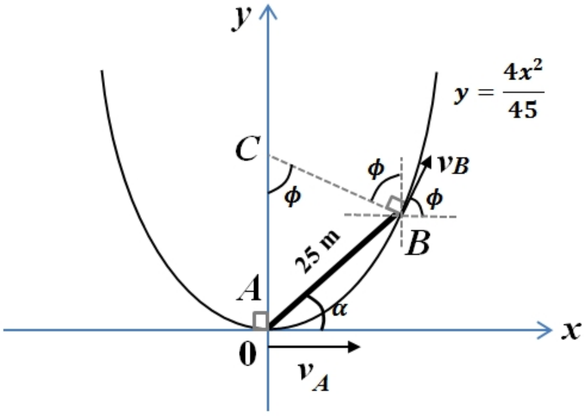
Example 3.3 A rod AB of length 25 m is placed in a parabolic drum (described by $y = \dfrac{4\,x^2}{45}$) as shown in the figure. End A of the rod slides rightwards with 41 m/sec when it is at the lowest point of the drum. Find the velocity of end B and the angular velocity of the rod at the given instant.



Solution

From the figure below, end A is located at $(0,0)$, while the coordinates of end B can be obtained by the following equations.

$$\begin{cases} y = \dfrac{4\,x^2}{45} \\ x^2 + y^2 = 25^2 \end{cases}$$



After solving, the position of B is found as (15, 20). Now, we construct the normal line to the parabola at B, the normal line cuts the y-axis at C. Note that C is the instantaneous center of rotation of the rod because CA is also a normal line to the parabola. Hence, we have

$$\angle ACB = \phi = \tan^{-1}\left(\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=15}\right) = \tan^{-1}\left(\left.\frac{8\,x}{45}\right|_{x=15}\right) = \tan^{-1}\left(\frac{8}{3}\right) = 69.44^\circ$$

On the other hand, $\alpha = \tan^{-1}\frac{20}{15} = 53.13^\circ$ gives $\angle CAB = 90^\circ - \alpha = 36.87^\circ$. Solving the sides of $\triangle CAB$, we have $CA = 25.62\,\mathrm{m}$ and $CB = 16.02\,\mathrm{m}$. Let the angular velocity of the rod about C be ω, then

$$v_A = (CA)\,\omega$$

So $\omega = \dfrac{v_A}{CA} = \dfrac{41\,\mathrm{ms}^{-1}}{25.62\,\mathrm{m}} = 1.6\,\mathrm{rad\,s}^{-1}$.

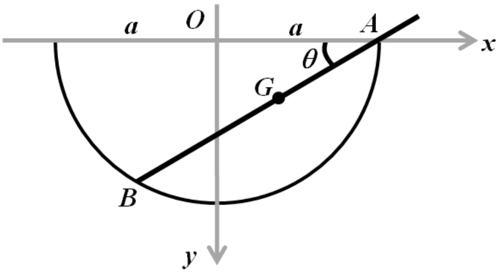
Therefore, $v_B = (CB)\,\omega = (16.02\,\mathrm{m})\,(1.6\,\mathrm{rad\,s}^{-1}) = 25.63\,\mathrm{ms}^{-1}$. ■

Example 3.4 A uniform rod of length $2a$ is placed with its lower end inside a smooth bowl. The bowl is a hemispherical hollow of radius a and it is fixed on a horizontal plane. Find the equilibrium position of the rod.

Solution

Denote G as the center of mass of the rod. The vertical distance of G from the x-axis is y, where

$$\begin{aligned} y &= AG \sin\theta = (AB - GB) \sin\theta \\ &= (2a\cos\theta - a) \sin\theta \end{aligned}$$



When the rod is at equilibrium in the bowl, it occupies the lowest gravitational potential energy. In other words, the vertical distance y obtains the maximum. The derivative of y with respect to θ is

$$\begin{aligned} y' &= 2a\,(\cos^2\theta - \sin^2\theta) - a\cos\theta \\ &= 4a\cos^2\theta - a\cos\theta - 2a \end{aligned}$$

The turning point of y satisfies $y' = 0$ which gives $4a\ \cos^2\theta - a\ \cos\theta - 2a = 0$. The rod reaches its equilibrium when the angle of inclination $\theta_0 = \cos^{-1}(1+\sqrt{33})/8 = 32.5^\circ$. One may check that $y''|_{\theta_0} < 0$, which indicates the maximum value of y at $\theta = \theta_0$. ■

Example 3.5 A light cord of length l has one of its ends connected to a particle of mass m, while the next end of it is fixed at a point O on the ceiling. Initially, the cord is kept horizontally and the particle is at a distance l from O such that the cord is tight, then the particle is released to fall under the gravity. Find the angle that the cord makes with the vertical when the particle obtains its maximum vertical speed.

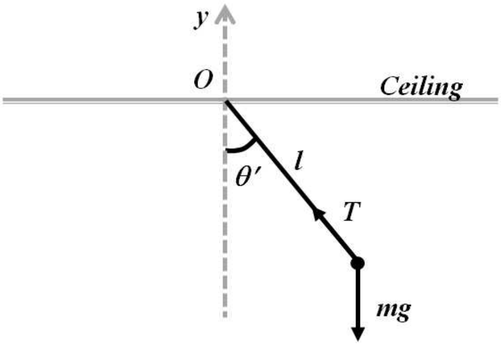
Solution

The particle has zero vertical speed when it is located at the initial position and the lowest position. It means that the particle speed has a turning point when it is descending. Here, the turning point is the maximum speed when the cord has an inclined angle θ' with the vertical. However, the turning point occurs when the net vertical force exerted on the particle is zero, i.e. $F_y = 0$. Thus,

$$T\ \cos\theta' = mg \tag{3.1}$$

The conservation of mechanical energy gives

$$mgl\,\cos\theta' = \frac{1}{2}mv^2 \tag{3.2}$$



The particle performs the circular motion with radius l because the net force along the cord contributes the centripetal force

$$T - mg\,\cos\theta' = \frac{mv^2}{l} \tag{3.3}$$

Eliminating v from equations (3.2) and (3.3), we obtain $T = 3mg\,\cos\theta'$. Using this equation and equation (3.1) to eliminate T, we obtain

$$\cos\theta' = \frac{1}{\sqrt{3}}\,,$$

which gives $\theta' = 54.7°$. ■

Example 3.6 The moment of inertia of a uniform solid sphere about an axis through its center is $2mr^2/5$, where m and r are the mass and radius of the sphere respectively. Deduce the moment of inertia of a uniform spherical hollow of mass M and radius R.

Solution

The moment of inertia of the solid sphere is

$$\begin{aligned} I_{\text{solid}} &= \frac{2}{5}mr^2 = \frac{2}{5}\,(\frac{4}{3}\,\rho\pi r^3)\,r^2 \\ &= \frac{8}{15}\,\rho\pi\,r^5\,, \end{aligned}$$

where ρ is the density of the sphere. Thus, the total differential of I_{solid} becomes

$$\mathrm{d}I_{\text{solid}} = \frac{8}{15}\,\rho\pi\,(5\,r^4)\,\mathrm{d}r = \frac{8}{3}\,\rho\pi\,r^4\,\mathrm{d}r$$

It represents the change in I_{solid} whenever there is a change in radius, i.e. dr.

The change in mass due to the infinitesimal thin hollow is dm, where

$$\mathrm{d}m = 4\,\rho\pi\, r^2\,\mathrm{d}r$$

Thus, we obtain

$$\mathrm{d}I_{\text{solid}} = \frac{8}{3}\,\rho\pi\, r^4\,\mathrm{d}r = \frac{2}{3}\,(4\,\rho\pi\, r^2\,\mathrm{d}r)\, r^2 = \frac{2}{3}\,(\mathrm{d}m)\, r^2$$

Hence, the moment of inertia of a uniform spherical hollow is $I_{\text{hollow}} = \mathrm{d}I_{\text{solid}} = \dfrac{2}{3}\,MR^2$.

■

Example 3.7 A particle moves along the path described by $x = 2t$ and $y = 4t^2$, where t is the time. Determine the magnitudes of the following quantities: velocity, tangential acceleration, centripetal (normal) acceleration of the particle at time t. Find also the radius of curvature when the particle reaches such point at time t.

Solution

Consider the following derivatives with respect to t.

$$\begin{cases} \dot{x} = 2 \\ \ddot{x} = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{y} = 8t \\ \ddot{y} = 8 \end{cases}$$

The velocity (magnitude) of the particle: $v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{4 + 64t^2}$.

The tangential acceleration: $a_T = \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}\,(\sqrt{4 + 64t^2}) = \dfrac{64t}{\sqrt{4 + 64t^2}}$.

Since the acceleration of the particle is $a = \sqrt{\ddot{x}^2 + \ddot{y}^2} = 8$, the centripetal acceleration $a_N = \sqrt{8^2 - \left(\dfrac{64t}{\sqrt{4 + 64t^2}}\right)^2} = \dfrac{16}{\sqrt{4 + 64t^2}}$.

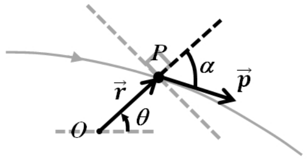
Notice that $a_N = \dfrac{v^2}{\rho}$ gives $\rho = \dfrac{v^2}{a_N}$, where ρ is the radius of curvature. Thus,

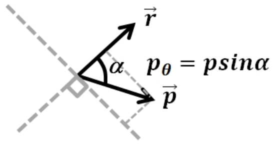
$$\rho = \frac{(4 + 64t^2)^{3/2}}{16}$$

■

Example 3.8 A particle P of mass m is governed to move on a plane by a central force which is originated at a point O. Show that the area swept by a line joining OP is a constant in equal time.

Solution





The angular momentum of the particle $\vec{L}$ about O is a constant as there is no external torque on it, where

$$L = |\vec{L}| = |\vec{r} \times \vec{p}| = |\vec{r}|\,|\vec{p}|\,\sin\alpha = rp\sin\alpha = rp_\theta = r\,mv_\theta = r\,mr\dot{\theta} = m\,r^2\dot{\theta}$$

In the above relations, $\vec{r}$ and $\vec{p}$ are the position vector and the linear momentum of the particle respectively. p_θ and v_θ are the linear momentum and the velocity of the particle normal to the radial direction. Then we have $\dot{\theta} = \dfrac{L}{mr^2}$. Assume $\mathrm{d}\theta$ be very small, the area swept by the radius r in time $\mathrm{d}t$ is $\mathrm{d}A = \dfrac{1}{2}r^2\,\mathrm{d}\theta$. Thus,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2}r^2\,\dot{\theta}$$

Therefore, we obtain $\dfrac{\mathrm{d}A}{\mathrm{d}t} = \dfrac{L}{2m}$ = constant, which states that the areal velocity is a constant. This is also known as the **Kepler's second law** of planetary motion. ■

3.2 L' Hôpital's Rule

Rule 1: $\dfrac{0}{0}$ form

Suppose the functions $f(x)$ and $g(x)$ are differentiable near $x = a$ and $f(a) = g(a) = 0$. Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Rule 2: $\dfrac{\infty}{\infty}$ form

Suppose the functions $f(x)$ and $g(x)$ are differentiable near $x = a$ and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = \infty$. Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Example 3.9 Find $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$.

Solution

The expression $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$ appears as the $\left[\dfrac{0}{0}\right]$ form. Apply the L' Hôpital's rule, we have

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1.$$

■

Example 3.10 Find $\lim\limits_{x\to 1}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)$.

Solution

$$\begin{aligned}
\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) &= \lim_{x\to 1}\left(\frac{x\,\ln x-x+1}{(x-1)\,\ln x}\right) && (\left[\frac{0}{0}\right]\text{ form})\\
&= \lim_{x\to 1}\left(\frac{\ln x}{\frac{x-1}{x}+\ln x}\right) && (\text{L}'\text{ Hopital}'\text{s Rule})\\
&= \lim_{x\to 1}\left(\frac{x\,\ln x}{x-1+x\,\ln x}\right) && (\left[\frac{0}{0}\right]\text{ form})\\
&= \lim_{x\to 1}\left(\frac{1+\ln x}{2+\ln x}\right) && (\text{L}'\text{ Hopital}'\text{s Rule})\\
&= \frac{1}{2}
\end{aligned}$$

■

Example 3.11 Evaluate $\lim\limits_{x\to 0}(1+\sin 2\,x)^{\cot x}$.

Solution

Let $y=(1+\sin 2\,x)^{\cot x}$, then $\ln y=(\cot x\,)\ln(1+\sin 2\,x)=\dfrac{\ln(1+\sin 2\,x)}{\tan x}$. So

$$\begin{aligned}
\lim_{x\to 0}\ln y &= \lim_{x\to 0}\frac{\ln(1+\sin 2\,x)}{\tan x} && (\left[\frac{0}{0}\right]\text{ form})\\
&= \lim_{x\to 0}\frac{\left(\frac{2\,\cos 2\,x}{1+\sin 2\,x}\right)}{\sec^2 x} && (\text{L}'\text{ Hopital}'\text{s Rule})\\
&= \lim_{x\to 0}\frac{2\,\cos 2\,x\,\cos^2 x}{(1+\sin 2\,x)}\\
&= 2
\end{aligned}$$

Therefore $\lim\limits_{x\to 0}(1+\sin 2\,x)^{\cot x}=e^2$. ■

Example 3.12 Evaluate $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\ln(1+x^2)}}$.

Solution

Let $y = (\cos x)^{\frac{1}{\ln(1+x^2)}}$, then we can write $\ln y = \dfrac{\ln \cos x}{\ln(1+x^2)}$. So

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln \cos x}{\ln(1+x^2)} \qquad (\left[\frac{0}{0}\right] \text{ form})$$

Hence, we have

$$\begin{aligned}\lim_{x\to 0} \ln y &= \lim_{x\to 0} \frac{\left(\dfrac{1}{\cos x}\right)(-\sin x)}{\left(\dfrac{1}{1+x^2}\right)(2x)} \qquad (\text{L}' \text{ Hopital}'\text{s Rule})\\ &= -\lim_{x\to 0}\left(\frac{1+x^2}{2\,\cos x}\right) \lim_{x\to 0} \frac{\sin x}{x}\\ &= \left(-\frac{1}{2}\right)(1)\\ &= -\frac{1}{2}\end{aligned}$$

Therefore, $\lim\limits_{x\to 0} (\cos x)^{\frac{1}{\ln(1+x^2)}} = e^{-1/2}$. ■

Example 3.13 A particle of mass m is thrown with speed v_0 and the elevated angle is ϕ. Consider the case that the air resistance (i.e. the force drag $\vec{F} = -k\vec{v}$) is not ignorable, where k is a positive constant and $\vec{v}$ is the velocity of the particle during the flight. Suppose that your classmate John obtains an expression for the required time t when the particle reaches its maximum height.

$$t = \frac{m}{k} \ln\left(1 + \frac{kv_0}{gm} \sin\phi\right)$$

Please check whether this expression agrees with the ideal case when the air resistance is ignorable.

Solution

For the ideal case that the air resistance is ignorable, the force drag constant k becomes zero. The required time $\lim\limits_{k\to 0} t$ is a $\left[\dfrac{0}{0}\right]$ form.

$$\lim_{k\to 0} t = \frac{\lim\limits_{k\to 0} m \ln\left(1 + \dfrac{kv_0}{gm} \sin\phi\right)}{\lim\limits_{k\to 0}\, k}$$

Using L' Hôpital's rule, we have

$$\lim_{k\to 0} t = \frac{\displaystyle\lim_{k\to 0} \frac{\mathrm{d}}{\mathrm{d}k} \left[m \ln\left(1 + \frac{k v_0}{g m} \sin\phi\right)\right]}{\displaystyle\lim_{k\to 0} \frac{\mathrm{d}}{\mathrm{d}k}(k)} = \lim_{k\to 0} m \left(\frac{g m}{g m + k v_0 \sin\phi}\right) \frac{v_0 \sin\phi}{g m} = \frac{v_0 \sin\phi}{g}$$

The last expression is exactly the same as that for the case when air resistance is ignored.

■

Exercise 3.3 Evaluate $\displaystyle\lim_{x\to 1}\left(\frac{x^2}{1-x} + \frac{1}{\ln x}\right)$.

Exercise 3.4 Evaluate $\displaystyle\lim_{x\to\infty}\left(\frac{x+\sin x}{x-\cos x}\right)$.

3.3 Taylor's Series

Assume $f(x)$ is infinitely differentiable at a, then

$$f(x) = f(a) + f'(a)\,(x-a) + \frac{1}{2!}\, f''(a)\,(x-a)^2 + \frac{1}{3!}\, f^{(3)}(a)\,(x-a)^3 + \cdots + \frac{1}{n!}\, f^{(n)}(a)\,(x-a)^n \cdots ,$$

where $f^{(n)}$ is the n-th derivative of $f(x)$. The above expression is called the Taylor's series of $f(x)$. There are many useful results obtained by the Taylor's series and they are widely used in physics.

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots
\end{aligned}$$

Binomial expansion

$$\begin{aligned}
(a+x)^p &= a^p + p\, a^{p-1}\, x + \frac{p\,(p-1)}{2!}\, a^{p-2}\, x^2 + \frac{p\,(p-1)\,(p-2)}{3!}\, a^{p-3}\, x^3 + \cdots \\
&= a^p + \binom{p}{1} a^{p-1}\, x + \binom{p}{2} a^{p-2}\, x^2 + \binom{p}{3} a^{p-3}\, x^3 + \cdots ,
\end{aligned}$$

where p is a real number and $\binom{p}{r} = \dfrac{p\,(p-1)\,(p-2)\cdots(p-r+1)}{r!}$. In particular, when $a = 1$, we have

$$\begin{aligned}(1+x)^p &= 1 + p\,x + \frac{p\,(p-1)}{2!}\,x^2 + \frac{p\,(p-1)\,(p-2)}{3!}\,x^3 + \cdots \\ &= 1 + \binom{p}{1}\,x + \binom{p}{2}\,x^2 + \binom{p}{3}\,x^3 + \cdots .\end{aligned}$$

It is an important result to do approximation if x is small, e.g. $(1+x)^{1/2} \approx 1 + x/2$.

Remark: If p and r are non-negative integers with $0 \le r \le p$, then the combinations formula is $\binom{p}{r} = \dfrac{p!}{r!\,(p-r)!}$. An alternative notation of $\binom{p}{r}$ is C^p_r. Obviously, $\binom{p}{r} = \binom{p}{p-r}$. We should note that $0! = 1$.

Example 3.14 Approximate $(1+x)^n$ by an exponential function if x is small.

Solution

We know from Taylor's series that $(1+x)^n \approx 1 + n\,x$ and $e^{nx} \approx 1 + nx$ when x is small, then we have $(1+x)^n \approx e^{nx}$. ■

Example 3.15 Write down $f(x+h)$ and $f(x-h)$ in Taylor's series. Hence, express $\cos(x+h)$ and $\sin(x-h)$ as power series of h.

Solution

Knowing that Taylor's series has the form

$$f(x) = f(a) + f'(a)\,(x-a) + \frac{1}{2!}\,f''(a)\,(x-a)^2 + \frac{1}{3!}\,f^{(3)}(a)\,(x-a)^3 + \cdots +$$

so we have

$$\begin{cases} f(x+h) = f(x) + h\,f'(x) + \dfrac{h^2}{2!}\,f''(x) + \dfrac{h^3}{3!}\,f^{(3)}(x) + \cdots \\[2ex] f(x-h) = f(x) - h\,f'(x) + \dfrac{h^2}{2!}\,f''(x) - \dfrac{h^3}{3!}\,f^{(3)}(x) + \cdots \end{cases}$$

Then

$$\begin{aligned}\cos(x+h) &= \cos x - h\,\sin x - \frac{h^2}{2!}\,\cos x + \frac{h^3}{3!}\,\sin x + \cdots \\ \sin(x-h) &= \sin x - h\,\cos x - \frac{h^2}{2!}\,\sin x + \frac{h^3}{3!}\,\cos x + \cdots\end{aligned}$$

■

Exercise 3.5 Show that $\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$

Example 3.16 Show that $\tan\left(\frac{\pi}{4} + x\right) = 1 + 2\,x + 2\,x^2 + \frac{8}{3}\,x^3 + \cdots$

Solution

Recall that $f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}\,f''(x) + \frac{h^3}{3!}\,f^{(3)}(x) + \cdots$. Putting $x = \frac{\pi}{4}$, then

$$f(\frac{\pi}{4}+h) = f(\frac{\pi}{4}) + h\,f'(\frac{\pi}{4}) + \frac{h^2}{2!}\,f''(\frac{\pi}{4}) + \frac{h^3}{3!}\,f^{(3)}(\frac{\pi}{4}) + \cdots$$

If we replace h by x, then we have

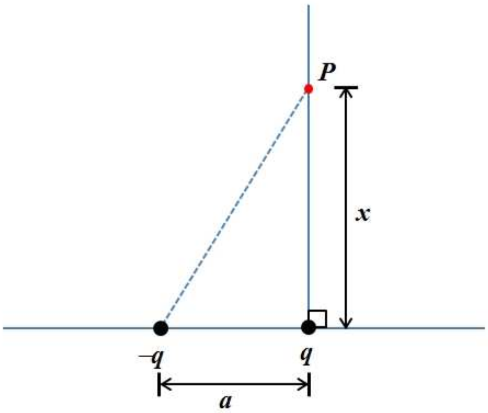
$$f(\frac{\pi}{4}+x) = f(\frac{\pi}{4}) + x\,f'(\frac{\pi}{4}) + \frac{x^2}{2!}\,f''(\frac{\pi}{4}) + \frac{x^3}{3!}\,f^{(3)}(\frac{\pi}{4}) + \cdots \tag{3.4}$$

Let $f(u) = \tan u$, where $\frac{\mathrm{d}}{\mathrm{d}u}\tan u = \sec^2 u$, $\frac{\mathrm{d}^2}{\mathrm{d}u^2}\tan u = 2\,\sec^2 u\,\tan u$, and $\frac{\mathrm{d}^3}{\mathrm{d}u^3}\tan u = 2\,\sec^4 u + 4\,\sec^2 u\,\tan^2 u$. Then $f(\frac{\pi}{4}) = 1$, $f'(\frac{\pi}{4}) = 2$, $f''(\frac{\pi}{4}) = 4$, and $f'''(\frac{\pi}{4}) = 16$. Using equation (3.4), we have

$$\tan\left(\frac{\pi}{4} + x\right) = 1 + 2\,x + 2\,x^2 + \frac{8}{3}\,x^3 + \cdots$$

■

Example 3.17 Two charges q and $-q$ are separated by a distance a. Estimate the electric potential at a point P which has a normal distance x from the line joining the charges ($x \gg a$), as shown in the figure.



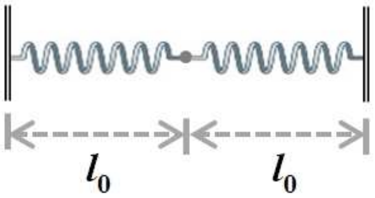
Solution

The total electric potential at P is

$$\begin{aligned} V &= \frac{1}{4\,\pi\epsilon_0}\,\frac{q}{x} - \frac{1}{4\,\pi\epsilon_0}\,\frac{q}{\sqrt{x^2+a^2}} \\ &= \frac{q}{4\,\pi\epsilon_0}\left(\frac{1}{x} - \frac{1}{\sqrt{x^2+a^2}}\right) \\ &= \frac{q}{4\,\pi\epsilon_0\,x}\left[1 - \left(1+\frac{a^2}{x^2}\right)^{-1/2}\right] \\ &\approx \frac{q}{4\,\pi\epsilon_0\,x}\left[1 - \left(1-\frac{a^2}{2\,x^2}\right)\right] \\ &\approx \frac{q\,a^2}{8\,\pi\epsilon_0\,x^3} \end{aligned}$$

■

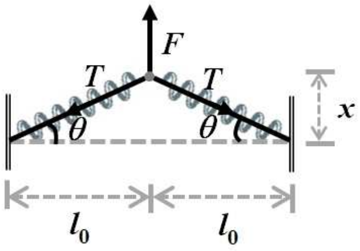
Example 3.18 Two massless springs, each with force constant k and unstretched length l_0 are connected in a straight line as shown in figure.



Find an expression for the work done of a force which moves the point of attachment, i.e. the knot, between the two springs a perpendicular distance x from the equilibrium point. Hence, show that the work done for such movement is given by $\dfrac{kx^4}{4\,l_0^2}$ when $x << l_0$.

Solution

When the vertical displacement of the knot is x, the extension of the spring is $e = \sqrt{l_0^2+x^2} - l_0$. The tension in each spring is $T = ke$. The force exerted by an external agent to displace the knot by x from its equilibrium is $F = 2T\sin\theta = 2ke\,\sin\theta = 2k\,(\sqrt{l_0^2+x^2} - l_0)\left(\dfrac{x}{\sqrt{l_0^2+x^2}}\right) = 2k\,x - \dfrac{2kl_0\,x}{\sqrt{l_0^2+x^2}}$.



Hence, the work done by the force for the displacement is

$$\begin{aligned} W &= \int_0^x F\,\mathrm{d}x \\ &= \int_0^x \left(2k\,x - \frac{2kl_0\,x}{\sqrt{l_0^2+x^2}}\right)\mathrm{d}x \\ &= kx^2 - 2kl_0\sqrt{l_0^2+x^2} + 2kl_0^2 \end{aligned}$$

The binominal expansion of $\sqrt{l_0^2+x^2} = l_0\left(1+(\frac{x}{l_0})^2\right)^{1/2}$ gives

$$\sqrt{l_0^2+x^2} = l_0\left(1+\frac{1}{2}\frac{x^2}{l_0^2}-\frac{1}{8}\frac{x^4}{l_0^4}+\cdots\right).$$

When $x \ll l_0$ we neglect the higher order terms after x^4. Therefore, the work done by the external force is

$$\begin{aligned} W &= kx^2 - 2kl_0\left\{l_0\left(1+\frac{1}{2}\frac{x^2}{l_0^2}-\frac{1}{8}\frac{x^4}{l_0^4}\right)\right\} + 2kl_0^2 \\ &= \frac{kx^4}{4\,l_0^2} \end{aligned}$$

■

Example 3.19 An electric dipole consists of two equal and opposite charges ($\pm q$) separated by a distance s. Show that the approximate potential at a point P far away is given by

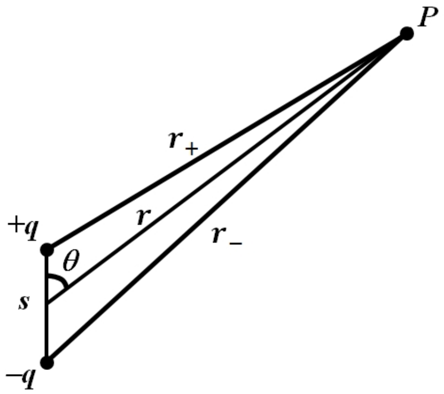
$$\frac{1}{4\pi\epsilon_0}\frac{qs\,\cos\theta}{r^2},$$

where r is the distance measured from P to the mid-point of dipole and θ is the angle between the dipole and the line joining P and the mid-point of dipole.

Solution

The potential due to the dipole is $V(P) = \dfrac{1}{4\pi\epsilon_0}\left(\dfrac{q}{r_+}-\dfrac{q}{r_-}\right)$, where

$$\begin{cases} r_+^2 = r^2 + \left(\dfrac{s}{2}\right)^2 - rs\cos\theta = r^2\left(1-\dfrac{s}{r}\,\cos\theta+\dfrac{s^2}{4\,r^2}\right) \\[2ex] r_-^2 = r^2 + \left(\dfrac{s}{2}\right)^2 + rs\cos\theta = r^2\left(1+\dfrac{s}{r}\,\cos\theta+\dfrac{s^2}{4\,r^2}\right) \end{cases}$$



When P is far away from the dipole, we have $r \gg s$. The higher order terms in the above expressions are negligible. Thus

$$\begin{cases} r_+^2 \approx r^2 \left(1 - \frac{s}{r}\cos\theta\right) \\ r_-^2 \approx r^2 \left(1 + \frac{s}{r}\cos\theta\right) \end{cases}$$

The binominal expansion of them are

$$\begin{cases} \dfrac{1}{r_+} \approx \dfrac{1}{r}\left(1 - \dfrac{s}{r}\cos\theta\right)^{-1/2} = \dfrac{1}{r}\left(1 + \dfrac{s}{2r}\cos\theta\right) \\ \dfrac{1}{r_-} \approx \dfrac{1}{r}\left(1 + \dfrac{s}{r}\cos\theta\right)^{-1/2} = \dfrac{1}{r}\left(1 - \dfrac{s}{2r}\cos\theta\right) \end{cases}$$

Therefore, we have

$$\frac{1}{r_+} - \frac{1}{r_-} \approx \frac{s}{r^2}\cos\theta$$

and hence $V(P) \approx \dfrac{1}{4\pi\epsilon_0}\dfrac{qs\cos\theta}{r^2}$. ■

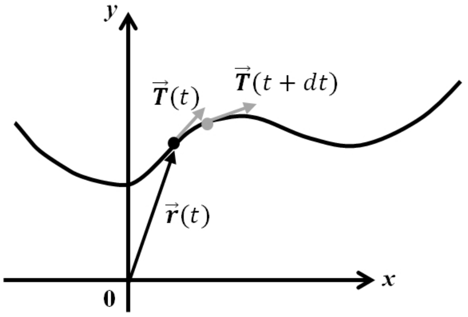
Exercise 3.6 The electric field at a distance z above the center of a flat circular disk of radius R is

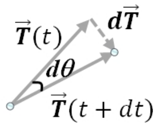
$$\vec{E} = \frac{\sigma z}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{R^2 + z^2}}\right]\hat{k},$$

where σ is the uniform surface charge on the disk (sigma represents the charge per unit area). Please use the binomial expansion to predict the electric field when $z \gg R$.

3.4 Differentiation of Vectors

When an object is moving in a curved path, we can measure its position by setting up a coordinate system with a fixed origin O. The position of the particle is measured by a vector $\vec{r}$ which points from O. It is the position vector. For the velocity and acceleration of the particle, they are represented by $\vec{v}$ and $\vec{a}$ respectively, where $\vec{v} = \dfrac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ and $\vec{a} = \dfrac{\mathrm{d}\vec{v}}{\mathrm{d}t}$. Now, we let $\vec{v} = v\vec{T}$, where v is the magnitude of $\vec{v}$ and $\vec{T}$ is the unit vector along the movement, ie. $|\vec{T}| = 1$.





Notice that the vector $\vec{T} = \dfrac{\mathrm{d}\vec{r}}{\mathrm{d}s}$, where s is the distance travelled by the particle along the path. The acceleration of the particle

$$\begin{aligned} \vec{a} &= \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}(v\vec{T})}{\mathrm{d}t} \\ &= \vec{T}\,\frac{\mathrm{d}v}{\mathrm{d}t} + v\,\frac{\mathrm{d}\vec{T}}{\mathrm{d}t} \end{aligned}$$

Notice that $\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}t}$ is perpendicular to $\vec{T}$ and the proof is simple. Consider the unit vector $\vec{T}$, we have $\vec{T}\cdot\vec{T} = 1$ and $\dfrac{\mathrm{d}}{\mathrm{d}t}(\vec{T}\cdot\vec{T}) = 0$. But, $\dfrac{\mathrm{d}}{\mathrm{d}t}(\vec{T}\cdot\vec{T}) = 2\,\vec{T}\cdot\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}t} = 0$ gives $\vec{T}\cdot\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}t} = 0$. That is to say, $\vec{T}$ and $\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}t}$ are normal to each other.

Now, $\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}t} = \dfrac{\mathrm{d}\vec{T}}{\mathrm{d}s}\dfrac{\mathrm{d}s}{\mathrm{d}t} = \dfrac{\mathrm{d}\vec{T}}{\mathrm{d}s}\,v = \left|\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right|\,v\vec{N} = \left|\dfrac{\mathrm{d}\theta}{\mathrm{d}s}\right|\,v\vec{N} = \kappa v\,\vec{N} = \dfrac{v}{\rho}\,\vec{N}$.

The quantity $\left|\dfrac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right|$ is labelled as $\kappa \geq 0$, i.e. the curvature of the path. Greater this value means a sharper curve along the path. As one can explore more from its alternative definition $\left|\dfrac{\mathrm{d}\theta}{\mathrm{d}s}\right|$. When κ is large it implies that a small change in s causes a large angular change in the direction of motion. It is an abrupt change. The reciprocal of κ is ρ, the radius of curvature of the path. The unit vector $\vec{N}$ is normal to $\vec{T}$.

Hence, we obtain $\vec{a} = a_T\,\vec{T} + a_N\,\vec{N}$, where a_T and a_N are the tangential and normal components of the acceleration vector respectivley.

$$\vec{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\,\vec{T} + \frac{v^2}{\rho}\,\vec{N}$$

An abrupt change along the curved path means that the region has a small radius of curvature.

Remarks:

- The vectorial representations of the tangential and normal accelerations are listed below:

  $$a_T = \frac{\vec{a}\cdot\vec{v}}{v} \qquad \text{and} \qquad a_N = \frac{|\vec{a}\times\vec{v}|}{v}\,,$$

  where a_N is positive but a_T can take positive or negative values.
- The magnitude of velocity $v = \left|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right| = \frac{\mathrm{d}s}{\mathrm{d}t}$ but $v \neq \frac{\mathrm{d}r}{\mathrm{d}t}$.
- The curvature of a path is defined as κ, where $\kappa = \left|\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right|$. Recall that $\frac{\mathrm{d}\vec{T}}{\mathrm{d}s} = \kappa\vec{N}$, where $\vec{N}\perp\vec{T}$ and $\kappa \geq 0$.
- The normal acceleration $a_N = \frac{|\vec{a}\times\vec{v}|}{v} = \kappa v^2$ gives $\kappa = \frac{|\dot{\vec{r}}\times\ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$, where $\dot{\vec{r}}$ and $\ddot{\vec{r}}$ are derivatives with respect to time.

Example 3.20 A particle moves along the path $y = x^2$. Find the radius of curvature at $x = 1$.

Solution

Consider a curved path $y = f(x)$ on a Cartesian plane. The derivative of y gives the elevated angle θ of the tangent line, $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta$. In the following discussion, we denote $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$ and $y'' = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Hence,

$$\begin{aligned} y'' &= (\sec^2\theta)\,\frac{\mathrm{d}\theta}{\mathrm{d}x} \\ &= (1+\tan^2\theta)\,\frac{\mathrm{d}\theta}{\mathrm{d}x} \\ &= (1+(y')^2)\,\frac{\mathrm{d}\theta}{\mathrm{d}x} \end{aligned}$$

Therefore, we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{y''}{1+(y')^2} \tag{3.5}$$

On the other hand, $(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2$ gives $\dfrac{\mathrm{d}s}{\mathrm{d}x} = \left[1+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{1/2}$. Or simply, we write

$$\frac{\mathrm{d}s}{\mathrm{d}x} = [1+(y')^2]^{1/2} \tag{3.6}$$

Using equations (3.5) and (3.6), we obtain

$$\kappa = \left|\frac{\mathrm{d}\theta}{\mathrm{d}s}\right| = \frac{\left|\dfrac{\mathrm{d}\theta}{\mathrm{d}x}\right|}{\left|\dfrac{\mathrm{d}s}{\mathrm{d}x}\right|} = \frac{|y''|}{[1+(y')^2]^{3/2}} \tag{3.7}$$

The radius of curvature

$$\rho = \frac{1}{\kappa} = \frac{[1+(y')^2]^{3/2}}{|y''|} \tag{3.8}$$

Now go back to this example, as $y = x^2$ gives $y' = 2x$ and $y'' = 2$.

When $x = 1$, $y' = 2$ and thus we obtain $\rho = \dfrac{[1+(2)^2]^{3/2}}{2} = \dfrac{\sqrt{125}}{2} = 5.59.$ ■

Exercise 3.7 Show that the curvature can be rewritten as

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}},$$

where $\dot{x}$, $\dot{y}$, $\ddot{x}$ and $\ddot{y}$ are derivatives with respect to time t.

Exercise 3.8 A particle is thrown with speed v_0 and an elevated angle θ. Obtain the radius of curvature of the projectile when the particle reaches the highest point.

Example 3.21 A particle moves along the path $y = \dfrac{1}{3}x^2$ with a constant speed of $8\,\mathrm{ms}^{-1}$. What is the magnitude of the acceleration of the particle when $x = 3$?

Solution

Knowing that $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x}{3}$, so we have $y'\Big|_{x=3} = \dfrac{2\,(3)}{3} = 2\,.$

Again, we have $y'' = \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{2}{3}$, so $y''\Big|_{x=3} = \dfrac{2}{3}\,.$

From equation (3.8), we can write down the radius of curvature of the path at $x = 3$ as

$$\rho = \left.\frac{[1+(y')^2]^{3/2}}{|y''|}\right|_{x=3} = \frac{[1+(2)^2]^{3/2}}{\left|\frac{2}{3}\right|} = 16.77\,\mathrm{m}$$

At $x = 3$, the centripetal acceleration exerted on the particle is

$$a_n = \frac{v^2}{\rho} = \frac{(8\,\mathrm{ms}^{-1})^2}{16.77\,\mathrm{m}} = 3.82\,\mathrm{ms}^{-2}$$

■

Example 3.22 A disk of radius R rolls without slipping inside the parabola $y = a\,x^2$, where $a > 0$. Find the radius of curvature of the vertex of the parabola. Express the condition that allows the disk to roll so that it contacts the parabola at one and only one point, independent of its position.

Solution

We can write $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\,ax$.

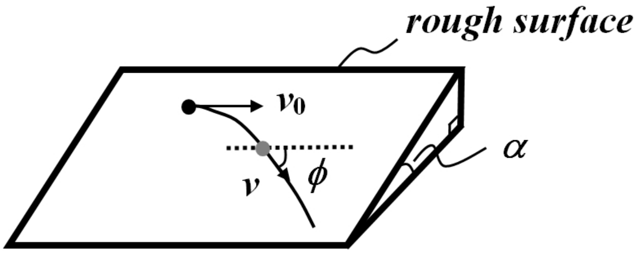
Again, we have $y'' = \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\,a$.

From equation (3.8), the radius of curvature of the parabola at an arbitrary point on it is

$$\rho = \frac{[1+(y')^2]^{3/2}}{|y''|} = \frac{[1+(2\,ax)^2]^{3/2}}{|2\,a|}$$

It implies that $\rho \geq \dfrac{1}{2\,a}$ because the parabola occupies the minimum radius of curvature when $x = 0$. The radius of curvature of the parabola at the vertex ($x = 0$) is $\rho = \dfrac{1}{2\,a}$. So, a disk of radius $R < \dfrac{1}{2\,a}$ can roll on the parabola with only one contact point. ■

Example 3.23 A particle is projected horizontally with speed v_0 on a rough and inclined board, as shown in the figure. The inclination angle of the board is α and the coefficient of friction is $\tan\alpha$.

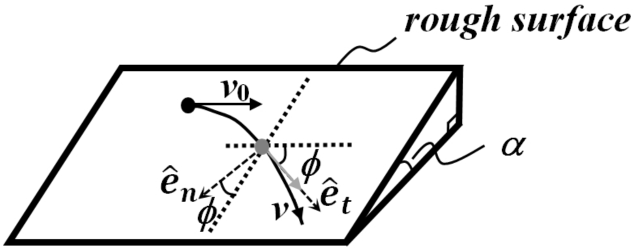


Show that

$$v = \frac{v_0}{1+\sin\phi}\ ,$$

where v is the speed of the particle and ϕ is an acute angle that the velocity vector of the particle makes with a horizontal line on the board.
[Hints: You may consider the equations of motion of a particle along and normal to its path, then use the fact that $\vec{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$, where ρ is the radius of curvature of the path, $\hat{e}_t$ and $\hat{e}_n$ are the unit vectors along and normal to the path respectively. Note that $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ and $\rho = \frac{\mathrm{d}s}{\mathrm{d}\phi}$, where d$s$ is an infinitesimal length of the path.]

Solution



Along the path:

$$mg\sin\alpha\sin\phi - mg\cos\alpha\tan\alpha = m\,\frac{\mathrm{d}v}{\mathrm{d}t} \tag{3.9}$$

where the second term in the right hand side of the equation represents the frictional force and $\tan\alpha$ is the frictional coefficient.
Normal to the path:

$$mg\sin\alpha\cos\phi = \frac{mv^2}{\rho} \tag{3.10}$$

where ρ is the radius of curvature. From equation (3.9), we have

$$g\sin\alpha\,(\sin\phi - 1) = \frac{\mathrm{d}v}{\mathrm{d}t} \tag{3.11}$$

From equation (3.10), we have

$$g\sin\alpha\cos\phi = \frac{v^2}{\rho} \tag{3.12}$$

Since, $v = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} = \rho\,\frac{\mathrm{d}\phi}{\mathrm{d}t}$, where d$s$ is the infinitesimal arc length of the path. That means

$$\frac{1}{\rho} = \frac{1}{v}\,\frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{3.13}$$

Use equations (3.12) and (3.13), we obtain

$$g\sin\alpha\cos\phi = v\,\frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{3.14}$$

Divide equations(3.11) by (3.14), we obtain

$$\frac{\sin\phi - 1}{\cos\phi} = \frac{1}{v}\,\frac{\mathrm{d}v}{\mathrm{d}\phi}$$

Separating the variables, we have

$$(\tan\phi - \sec\phi)\,\mathrm{d}\phi = \frac{1}{v}\,\mathrm{d}v$$

Integrating both sides of the equation

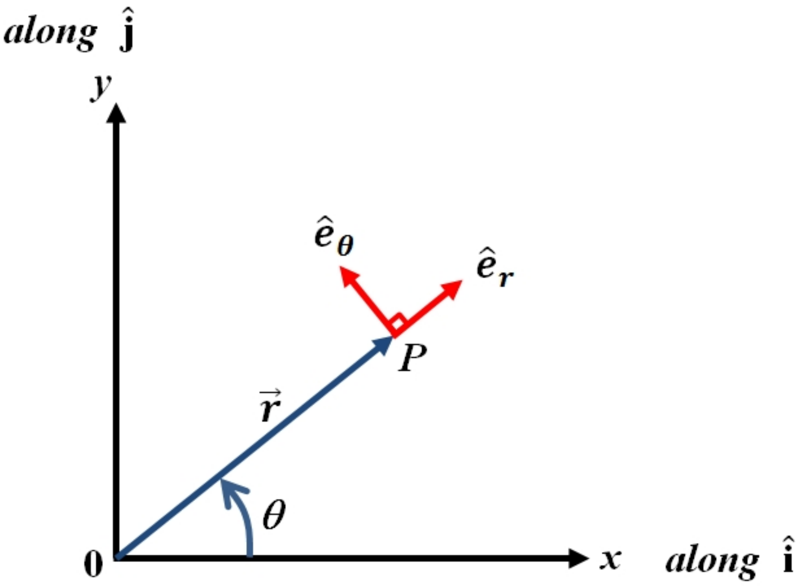
$$\begin{aligned}
\int_0^{\phi} (\tan\phi - \sec\phi)\,\mathrm{d}\phi &= \int_{v_0}^{v} \frac{\mathrm{d}v}{v} \\
[\ln\sec\phi - \ln(\sec\phi + \tan\phi)]\Big|_0^{\phi} &= \ln v\Big|_{v_0}^{v} \\
\ln\left(\frac{1}{1+\sin\phi}\right)\Big|_0^{\phi} &= \ln\left(\frac{v}{v_0}\right) \\
\frac{1}{1+\sin\phi} &= \frac{v}{v_0} \\
v &= \frac{v_0}{1+\sin\phi}
\end{aligned}$$

■

3.5 Polar Coordinates

The location of a point P on a plane is recorded by the Cartesian coordinates (x, y), where x and y are measured from an origin O. The location of P can be re-expressed in the polar coordinates as (r, θ), where r is always positive and it is the length of PO, and θ is the polar angle measured in the counterclockwise direction from the positive x-axis.

In the polar coordinate system, each coordinate is associated with a unit vector, $\hat{e}_r$ for r and $\hat{e}_\theta$ for θ, where $\hat{e}_r \perp \hat{e}_\theta$. This is similar to what we learn from the Cartesian coordinate system, where the unit vectors $\hat{\imath}$ for x and $\hat{\jmath}$ for y, where $\hat{\imath} \perp \hat{\jmath}$.



Please notice that the position vector of P has the following representations:

$$\text{Cartesian coordinates : } \quad \vec{r} = x\hat{\imath} + y\hat{\jmath} \qquad \text{or} \qquad \text{Polar coordinates : } \quad \vec{r} = r\hat{e}_r$$

Let's investigate the unit vectors of the polar coordinate system.

$$\begin{cases} \hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \\ \hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} \end{cases} \tag{3.15}$$

Differentiating both sides with respect to θ, we obtain

$$\begin{cases} \dfrac{\mathrm{d}\hat{e}_r}{\mathrm{d}\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} = \hat{e}_\theta \\ \\ \dfrac{\mathrm{d}\hat{e}_\theta}{\mathrm{d}\theta} = -\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath} = -\hat{e}_r \end{cases} \tag{3.16}$$

The position vector $\vec{r}$ in terms of polar coordinates is given by $\vec{r} = r\hat{e}_r$. (Note that $\hat{e}_r$ and $\hat{e}_\theta$ are functions of θ, i.e. $\hat{e}_r = \hat{e}_r(\theta)$ and $\hat{e}_\theta = \hat{e}_\theta(\theta)$. They are unit vectors, but not constant vectors because their directions are varying.)
Thus the velocity $\vec{v}$ is

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}(r\hat{e}_r) = \frac{\mathrm{d}r}{\mathrm{d}t}\,\hat{e}_r + r\frac{\mathrm{d}\,\hat{e}_r}{\mathrm{d}t}$$

Since $\dfrac{\mathrm{d}\hat{e}_r}{\mathrm{d}t} = \dfrac{\mathrm{d}\hat{e}_r}{\mathrm{d}\theta}\cdot\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}\,\hat{e}_\theta$, we have

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \tag{3.17}$$

The above expression can be interpret as $\vec{v} = v_r\,\hat{e}_r + v_\theta\,\hat{e}_\theta$, where v_r is the radial velocity component of $\vec{v}$ along $\hat{e}_r$ and v_θ is the angular component of $\vec{v}$ along $\hat{e}_\theta$.
The acceleration of the system is given by

$$\begin{aligned}
\vec{a} &= \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta) \\
&= \frac{\mathrm{d}\dot{r}}{\mathrm{d}t}\,\hat{e}_r + \dot{r}\,\frac{\mathrm{d}\hat{e}_r}{\mathrm{d}\theta}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}r}{\mathrm{d}t}\dot{\theta}\,\hat{e}_\theta + r\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t}\,\hat{e}_\theta + r\dot{\theta}\,\frac{\mathrm{d}\hat{e}_\theta}{\mathrm{d}\theta}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} \\
&= \ddot{r}\,\hat{e}_r + \dot{r}(\hat{e}_\theta)\,\dot{\theta} + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\,(-\hat{e}_r)\,\dot{\theta}
\end{aligned}$$

Hence, we have

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta \tag{3.18}$$

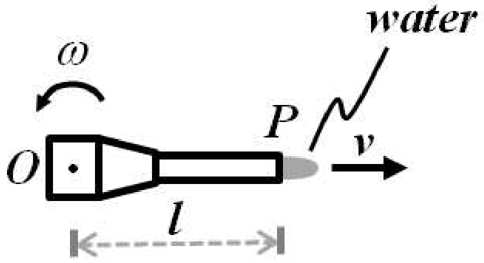
where $(\ddot{r} - r\dot{\theta}^2) = a_r$ is radial acceleration and $(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = a_\theta$ is the angular component of the acceleration.

Example 3.24 Given that c is a constant, please determine whether the position vector $\vec{r} = c\,\hat{e}_r$ of a point P is a constant vector.

Solution

The position vector $\vec{r} = c\,\hat{e}_r$ has a constant magnitude c. It means that point P has a constant distance from the origin, but its' position is not yet fixed. Because $\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ is a function of θ. Hence, $\vec{r}$ is not a constant vector unless θ is fixed. Notice that $\hat{e}_r$ is a unit vector but not a constant vector. ■

Example 3.25 The water nozzle rotates with a constant angular speed ω about an axis through O. A bird's view of the nozzle is shown in the figure. The rotation is on a horizontal plane and the axis of rotation is fixed and vertical (normal to this page). If the water speed relative to the nozzle mouth at P is a constant v and the length of nozzle is l, obtain the magnitude of acceleration of water when it passes the nozzle mouth.



Solution

Refer to equation (3.18), $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$, where $r = l$, $\dot{r} = v$, $\ddot{r} = 0$, $\dot{\theta} = \omega$, and $\ddot{\theta} = 0$. Thus, $\vec{a} = -l\omega^2\,\hat{e}_r + 2\,v\omega\,\hat{e}_\theta$. Hence,

$$a = \sqrt{(-l\omega^2)^2 + (2\,v\omega)^2} = \omega\,\sqrt{l^2\omega^2 + 4\,v^2}$$

■

Exercise 3.9 A constant vector $\vec{c}$ is represented by $\vec{c} = c_1\,\hat{i} + c_2\,\hat{j}$ in the Cartesian coordinate system, where c_1 and c_2 are constants. Express $\vec{c}$ again in the polar coordinate system.

Example 3.26 A particle of mass m is traveling along the curve $r\,(1 + \cos\theta) = 2a$, in a horizontal plane ($a > 0$), so that the position vector of the particle is rotating at constant angular speed ω about the origin O. Given that (r, θ) are the polar coordinates of the path and a is a constant.

(a) Show that the speed of the particle at any instant is $\omega\sqrt{\dfrac{r^3}{a}}$.

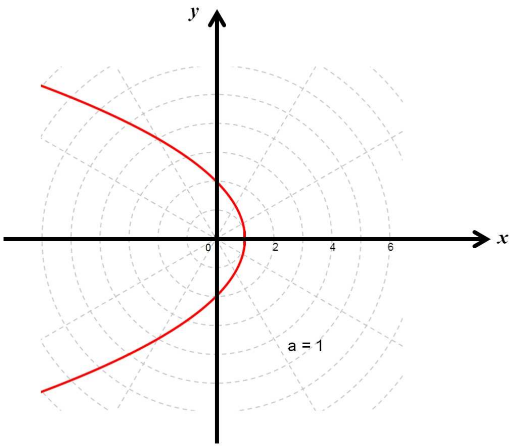
(b) Find also the acceleration of the particle along the path of motion when $\theta = \pi/2$.

(c) Find the magnitude of the x-component of the force acting on the particle when $\theta = \pi/2$.

(d) Find the magnitude of the y-component of the force acting on the particle when $\theta = \pi/2$, find also total force acting on the particle at the same instant.

Solution

(a) The figure below shows the path of the particle if $a = 1$.



Knowing that $\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$ and $\dot{\theta} = \omega$, we have $v^2 = \dot{r}^2 + r^2\,\omega^2$.
Since $r\,(1+\cos\theta) = 2\,a$, we have $r = \dfrac{2\,a}{1+\cos\theta}$ and $\dot{r} = \dfrac{2\,a\,\dot{\theta}\,\sin\theta}{(1+\cos\theta)^2}$. So

$$\dot{r} = \frac{2\,a\,\omega\,\sin\theta}{(1+\cos\theta)^2} \tag{3.19}$$

Then

$$\begin{aligned}
v^2 &= \left[\frac{2\,a\,\omega\,\sin\theta}{(1+\cos\theta)^2}\right]^2 + \left[\frac{2\,a}{1+\cos\theta}\right]^2\omega^2 \\
&= \frac{4\,a^2\omega^2}{(1+\cos\theta)^2}\left[\frac{\sin^2\theta}{(1+\cos\theta)^2} + 1\right] \\
&= \frac{4\,a^2\omega^2}{(1+\cos\theta)^4}\left[\sin^2\theta + (1+\cos\theta)^2\right] \\
&= \frac{8\,a^2\omega^2}{(1+\cos\theta)^3} \\
&= \frac{r^3\,\omega^2}{a}
\end{aligned}$$

Hence, we obtain $v = \omega\sqrt{\dfrac{r^3}{a}}$.

(b) From the result of (a), we have $v = \dfrac{\omega}{\sqrt{a}}\,r^{3/2}$, so the tangential acceleration of the particle is

$$a_T = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{3\,\omega}{2\,\sqrt{a}}\,\sqrt{r}\,\dot{r}$$

When $\theta = \dfrac{\pi}{2}$, $r = 2\,a$, $\dot{r} = 2\,a\omega$. Hence, we have $a_T = 3\,\sqrt{2}\,a\,\omega^2$.

(c) Consider the acceleration of the particle: $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$. When $\theta = \dfrac{\pi}{2}$, the x-component of the acceleration is given by $\vec{a}_x = -(r\,\ddot{\theta} + 2\,\dot{r}\dot{\theta})\,\hat{\imath}$, where $r = 2\,a$, $\dot{r} = 2\,a\,\omega$, $\dot{\theta} = \omega$, and $\ddot{\theta} = 0$, so $|a_x| = 4\,a\,\omega^2$. Hence, we obtain the magnitude of the x-component of the force: $|F_x| = m\,|a_x| = 4\,ma\,\omega^2$.

(d) Differentiating both sides of equation (3.19) with respect to t, we have

$$\begin{aligned}
\ddot{r} &= 2\,a\,\omega\,\dot{\theta}\left[\frac{(1+\cos\theta)^2\,\cos\theta + 2\,\sin^2\theta(1+\cos\theta)}{(1+\cos\theta)^4}\right] \\
&= 2\,a\,\omega^2\left[\frac{(1+\cos\theta)\,\cos\theta + 2\,\sin^2\theta}{(1+\cos\theta)^3}\right] \\
&= 2\,a\,\omega^2\left[\frac{2+\cos\theta-\cos^2\theta}{(1+\cos\theta)^3}\right]
\end{aligned}$$

Then, we have

$$\ddot{r} = \frac{2\,a\,\omega^2\,(2-\cos\theta)}{(1+\cos\theta)^2}$$

Consider the acceleration of the particle again: $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$. When $\theta = \dfrac{\pi}{2}$, the y-component of the acceleration is given by $\vec{a}_y = (\ddot{r} - r\dot{\theta}^2)\,\hat{\jmath}$, where $r = 2\,a$, $\ddot{r} = 4\,a\,\omega^2$, and $\dot{\theta} = \omega$, so $|a_y| = 2\,a\,\omega^2$. Hence, we obtain the magnitude of the y-component of the force: $|F_y| = m\,|a_y| = 2\,ma\,\omega^2$.

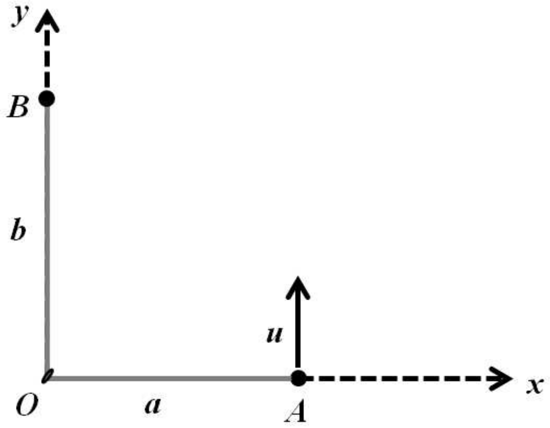
The total force exerted on the particle at the instant is

$$F = \sqrt{F_x^2 + F_y^2} = m\,a\,\omega^2\,\sqrt{4^2+2^2} = 2\sqrt{5}\,m\,a\,\omega^2$$

■

Exercise 3.10 Find the acceleration of the particle normal to the path of motion when $\theta = \pi/2$.

Example 3.27 Two particles A and B of equal masses are attached to the ends of a light and inextensible string which passes through a small and fixed smooth ring O on a smooth horizontal table such that the portions OA and OB of the string are straight and perpendicular with $OA = a$ and $OB = b$. Initially, the particle A is given a speed u in the direction of OB, as shown in the figure. Let (r, θ) be the polar coordinates of particle A at time t in the subsequent motion, where $a \le r \le a + b$. The origin of the coordinate system is at the ring O and OA lies on the x-axis when $t = 0$.



(a) Show that $2\ddot{r} = \dfrac{a^2u^2}{r^3}$.

(b) Show also that $\dot{r}^2 = \dfrac{u^2}{2}\left(1 - \dfrac{a^2}{r^2}\right)$.

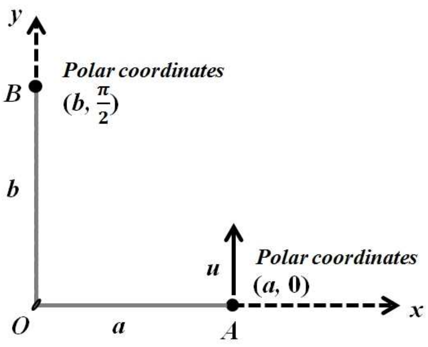
(c) Hence, show that particle B has speed $\sqrt{\dfrac{b\,(2a+b)\,u^2}{2\,(a+b)^2}}$ when it reaches the ring O.

(d) Show further that $\theta = \sqrt{2}\,\cos^{-1}\left(\dfrac{a}{r}\right)$.

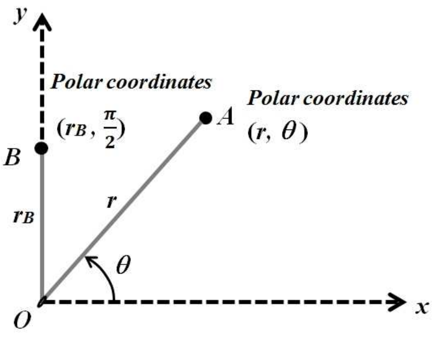
Solution

(a) Notice that the total length of the string is fixed. We have $r + r_B = a + b$, where r_B is the instantaneous length of OB at time t. Thus

$$\ddot{r} + \ddot{r}_B = 0 \tag{3.20}$$



At time $t = 0$



At time t

Consider the equations of motion of particle A along and normal to the string respectively.

Along the radial direction:

$$m(\ddot{r} - r\dot{\theta}^2) = -T \tag{3.21}$$

Along the tangential direction:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{3.22}$$

Eqn. (3.22) can be rewritten as

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0 \tag{3.23}$$

And thus, we obtain $r^2\dot{\theta} = c$, where c is a constant.

At time $t = 0$, $r = a$ and $\dot{\theta} = \dfrac{u}{a}$. We have $a^2\left(\dfrac{u}{a}\right) = c$ which gives $c = au$. Therefore, we can write

$$\dot{\theta} = \frac{au}{r^2} \tag{3.24}$$

Consider the equation of motion of particle B along the string. We have $m\ddot{r}_B = -T$. Using eqn.(3.20) we obtain

$$m\ddot{r} = T \tag{3.25}$$

Eqns. (3.21) and (3.25) imply $m(\ddot{r} - r\dot{\theta}^2) = -m\ddot{r}$ which gives $2\ddot{r} = r\dot{\theta}^2$. Use the result in Eqn. (3.24), hence we obtain $2\ddot{r} = \dfrac{a^2u^2}{r^3}$.

(b) From the result in (a), we have

$$2\ddot{r} = \frac{a^2u^2}{r^3}$$

Rewrite $\ddot{r}$ in the L.H.S. of the above equation as $\dot{r}\,\dfrac{\mathrm{d}\dot{r}}{\mathrm{d}r}$, then we have

$$2\dot{r}\,\frac{\mathrm{d}\dot{r}}{\mathrm{d}r} = \frac{a^2u^2}{r^3}$$

Rearrange the above equation and use integration, we obtain

$$2\int_0^{\dot{r}} \dot{r}\,\mathrm{d}\dot{r} = a^2u^2\int_a^r \frac{\mathrm{d}r}{r^3}$$

Therefore, we have

$$\dot{r}^2 = \frac{u^2}{2}\left(1 - \frac{a^2}{r^2}\right) \tag{3.26}$$

(c) Particle B reaches O when $r = a + b$. Notice that $\dot{r}_B = -\dot{r}$ and we have $\dot{r}_B^2 = \dot{r}^2$. Use the result in (b), we have $\dot{r}_B^2 = \dot{r}^2 = \dfrac{u^2}{2}\left(1 - \dfrac{a^2}{(a+b)^2}\right)$. Hence, we obtain

$$|\dot{r}_B| = \sqrt{\frac{b\,(2a+b)\,u^2}{2(a+b)^2}}$$

(d) From equations (3.24) and (3.26), we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{au}{r^2} \qquad \text{and} \qquad \frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\frac{u^2}{2}\left(1 - \frac{a^2}{r^2}\right)}$$

Using the chain rule, we have $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = \dfrac{\mathrm{d}r}{\mathrm{d}t}\cdot\dfrac{\mathrm{d}t}{\mathrm{d}\theta}$, thus

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{r}{\sqrt{2}\,a}\sqrt{r^2 - a^2}$$

which gives

$$\int_a^r \frac{\mathrm{d}r}{r\sqrt{r^2-a^2}} = \frac{1}{\sqrt{2}a}\int_0^\theta \mathrm{d}\theta \tag{3.27}$$

Let $r = a\,\sec\alpha$, we have $\mathrm{d}r = a\,\sec\alpha\,\tan\alpha\,\mathrm{d}\alpha$, equation (3.27) becomes

$$\int_0^{\cos^{-1}\left(\frac{a}{r}\right)} \mathrm{d}\alpha = \frac{1}{\sqrt{2}}\,\theta$$

Therefore, we have

$$\theta = \sqrt{2}\,\cos^{-1}\left(\frac{a}{r}\right)$$

■

Exercise 3.11 Investigate example 3.27 again, show that

$$t = \frac{\sqrt{2}a}{u}\tan\left(\frac{\theta}{\sqrt{2}}\right) \qquad \text{and} \qquad t = \frac{\sqrt{2\left(r^2-a^2\right)}}{u}$$

Chapter 4

Integration

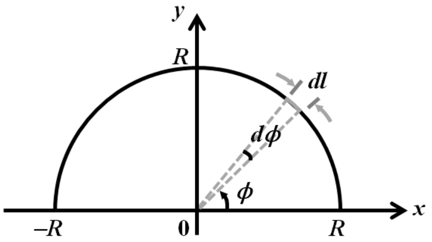
4.1 Center of Mass

The center of mass of an object is the point at which the entire mass of the object may be considered concentrated. For a 2-D object lying on the xy-plane, this point is expressed by integrals as shown below.

$$\bar{x} = \frac{\displaystyle\int x\,\mathrm{d}m}{\displaystyle\int \mathrm{d}m} \qquad \text{and} \qquad \bar{y} = \frac{\displaystyle\int y\,\mathrm{d}m}{\displaystyle\int \mathrm{d}m}$$

Example 4.1 A uniform wire of radius R is bent into a semi-circle. Locate the center of mass of the wire.

Solution



Due to symmetry, the x-coordinate of the center of mass always lies on the y-axis (i.e. $\bar{x} = 0$). Consider an infinitesimal element of length $\mathrm{d}l$ on the wire, where $\mathrm{d}l = R\,\mathrm{d}\phi$. The mass of the element is $\mathrm{d}m = \lambda\,\mathrm{d}l = \lambda\,R\,\mathrm{d}\phi$. By the definition of centre of mass

$$\begin{aligned}
\bar{y} = \frac{\displaystyle\int_{\text{wire}} y\,\mathrm{d}m}{\displaystyle\int_{\text{wire}} \mathrm{d}m} &= \frac{\displaystyle\int_0^{\pi} (R\,\sin\phi)\;(\lambda\,R\,\mathrm{d}\phi)}{\displaystyle\int_0^{\pi} \lambda\,R\,\mathrm{d}\phi} \\
&= \frac{R^2\,\lambda \displaystyle\int_0^{\pi} \sin\phi\,\mathrm{d}\phi}{R\lambda \displaystyle\int_0^{\pi} \mathrm{d}\phi} \\
&= -\frac{R}{\pi}\,\cos\phi\,\Big|_{\phi=0}^{\pi} = -\frac{R}{\pi}\,(-1-1) = \frac{2R}{\pi}
\end{aligned}$$

■

4.2 Moment of Inertia

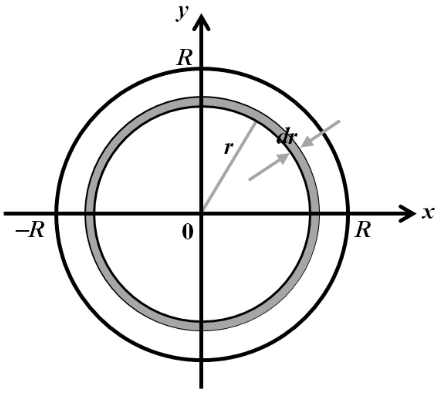
The moment of inertia of an object is the measure of an object's resistance to change its rotational motion. It is defined as

$$I = \int r^2\,\mathrm{d}m\,,$$

where r is the distance between the infinitesimal mass element $\mathrm{d}m$ and the axis of rotation. A simplest example is for a uniform ring. The moment of inertia of a uniform ring of mass m and radius r through an axis passing through the ring's center and normal to the ring's plane is mr^2 because $I = \displaystyle\int r^2\,\mathrm{d}m = r^2 \int \mathrm{d}m = m\,r^2$.

Example 4.2 Find the moment of inertia of a uniform disk about an axis through its center and normal to the disk. The disk has mass M and radius R.

Solution



A disk can be regarded as a collection of numerous rings which has infinitesimal thickness $\mathrm{d}r$. The moment of inertia of a ring having a radius r and thickness $\mathrm{d}r$ about an axis passing through the center of ring is $\mathrm{d}I = (\mathrm{d}m)\, r^2$, where $\mathrm{d}m = 2\pi\sigma\, r\, \mathrm{d}r$ and σ is the surface density of ring.

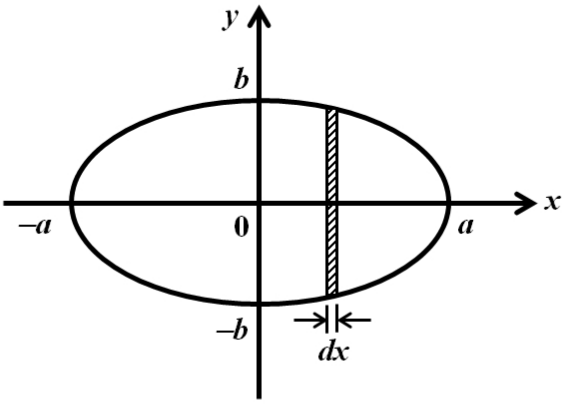
Therefore, the moment of inertia of a disk is

$$\begin{aligned} I &= \int_{\text{disk}} \mathrm{d}I = \int_{\text{disk}} r^2\, \mathrm{d}m \\ &= \int_0^R r^2\, (2\pi\sigma\, r\, \mathrm{d}r) \\ &= 2\pi\sigma \int_0^R r^3\, \mathrm{d}r \\ &= 2\,\pi\sigma\, (\frac{R^4}{4}) \\ &= \frac{MR^2}{2}, \end{aligned}$$

where $M = \sigma\pi R^2$. ■

Example 4.3 An elliptic laminar $\mathcal{E}$ is described by the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. It is uniform and has mass m. Find the moment of inertia of $\mathcal{E}$ about the y-axis.

Solution



Let the density of the laminar be σ, so $\sigma = \dfrac{m}{\pi ab}$. Divide the laminar into numerous vertical strips each having mass $\mathrm{d}m$ and width $\mathrm{d}x$, where $\mathrm{d}m = \sigma\,(2y)\,\mathrm{d}x = \dfrac{2\,b\,\sigma}{a}\,(a^2 - x^2)^{1/2}\,\mathrm{d}x$.

Note: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ gives $y = \dfrac{b}{a}\,(a^2 - x^2)^{1/2}$. Hence, we have

$$\begin{aligned} I_y = \int_{\mathcal{E}} x^2\, \mathrm{d}m &= \frac{2\,b\,\sigma}{a} \int_{-a}^{a} x^2\,(a^2 - x^2)^{1/2}\,\mathrm{d}x \\ &= \frac{4\,b\,\sigma}{a} \int_0^a x^2\,(a^2 - x^2)^{1/2}\,\mathrm{d}x \end{aligned}$$

Using the substitution $x = a\sin\theta$, then we have $\mathrm{d}x = a\cos\theta\,\mathrm{d}\theta$, So

$$\begin{aligned} I_y &= 4\,\sigma\,a^3 b \int_0^{\frac{\pi}{2}} \sin^2\theta\,\cos^2\theta\,\mathrm{d}\theta \\ &= \sigma\,a^3 b \int_0^{\frac{\pi}{2}} \sin^2 2\,\theta\,\mathrm{d}\theta && (\sin 2A = 2\,\sin A\,\cos A) \\ &= \frac{\sigma\,a^3 b}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 4\,\theta)\,\mathrm{d}\theta && (\sin^2 A = \frac{1}{2}\,(1 - \cos 2A)) \end{aligned}$$

Therefore

$$\begin{aligned} I_y &= \frac{\sigma\,a^3 b}{2} \left(\theta - \frac{1}{4}\,\sin 4\,\theta\right) \Bigg|_0^{\frac{\pi}{2}} \\ &= \frac{\sigma\,a^3 b\,\pi}{4} \\ &= \frac{m}{\pi ab} \left(\frac{a^3 b\,\pi}{4}\right) \\ &= \frac{m\,a^2}{4} \end{aligned}$$

■

Exercise 4.1 Locate the center of mass of a uniform semi-circular disk having mass M and radius R.

Exercise 4.2 Obtain the moment of inertia of a uniform sphere about an axis passing through the center. The mass and radius of sphere are M and R respectively.

4.3 Gravitational Force

We know clearly that the gravitational force between two particles of mass m_1 and m_2 respectively is an attractive force and the magnitude of it obeys the inverse square law, i.e.

$$F = \frac{G\,m_1 m_2}{d^2}$$

where d is the distance between the particle and G is the universal gravitational constant. Now we proceed to work out the gravitational force between a particle and a uniform spherical hollow. Two shell theorems follows.

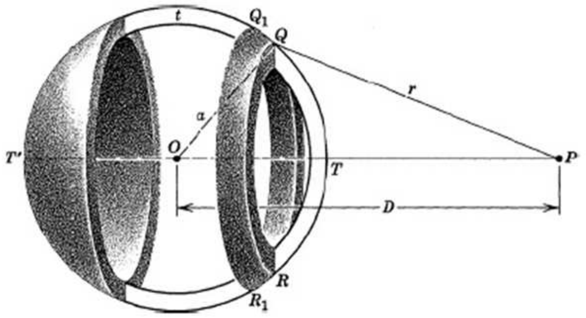
Shell theorem 1:

A uniform spherical shell attracts an external particle as if all the mass of the shell were concentrated at the center of the shell.

Shell theorem 2:

A uniform spherical shell exerts no force on a particle located inside the shell.

Let's derive the two theorems one by one. Consider a uniform and thin spherical hollow of radius a, mass M and very small thickness t. A particle located a distance D from the center of the hollow experiences attactive force from every mass element of the hollow, e.g an element U_1UTT_1. However, the net force due to the ring zone is along the line PO. This is a result of symmetry.



For example, the pull on a particle of mass m at P due to the mass elements QQ_1 and RR_1 will offset each other except the components along the line PO. If the mass element QQ_1 has volume ΔV, then the component of force along PO is

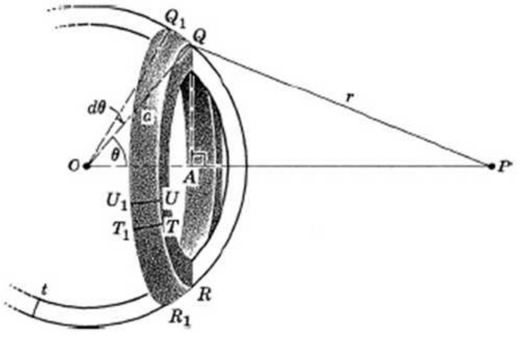
$$\frac{G\,\rho\,\Delta V\,m}{r^2}\cos\angle OPQ\,,$$

where ρ is the density of the hollow. The sum of these components resulted from the ring zone is

$$f=\frac{G\,\rho\,V\,m}{r^2}\cos\angle OPQ\,,$$

where V is the volume of the ring (see the figure below) and

$$V=2\,\pi\cdot QA\cdot(a\,\mathrm{d}\theta)\cdot t=2\,\pi\cdot(a\,\sin\theta)\cdot(a\,\mathrm{d}\theta)\cdot t$$



Then

$$f = \frac{G\,\rho\, mt\, 2\,\pi\, a^2 \sin\theta\, \mathrm{d}\theta}{r^2} \cos\angle OPQ \tag{4.1}$$

Now, we try to find an expression for $\cos\angle OPQ$.

$$\cos\angle OPQ = \frac{AP}{PQ} = \frac{OP - OA}{PQ} = \frac{D - a\,\cos\theta}{r}$$

Thus

$$f = \frac{G\,\rho\, mt\, 2\,\pi\, a^2 \sin\theta\, \mathrm{d}\theta}{r^2} \cdot \frac{D - a\,\cos\theta}{r} \tag{4.2}$$

However, cosine rule for $\triangle OPQ$ gives

$$r^2 = a^2 + D^2 - 2\,aD\,\cos\theta \tag{4.3}$$

Differentiating both sides, we have $r\,\mathrm{d}r = aD\,\sin\theta\,\mathrm{d}\theta$, so

$$a\,\sin\theta\,\mathrm{d}\theta = \frac{r\,\mathrm{d}r}{D}$$

then

$$f = \frac{G\,\rho\, mt\, 2\,\pi\, a\, \mathrm{d}r}{D\, r^2} \cdot (D - a\,\cos\theta) \tag{4.4}$$

From equation (4.3) again, we have

$$r^2 + D^2 - a^2 = 2\,D^2 - 2\,aD\,\cos\theta$$

which gives

$$\frac{r^2 + D^2 - a^2}{2\,D} = D - a\,\cos\theta$$

So equation (4.4) becomes

$$f = \frac{G\,\rho\,\pi\, mat}{D^2} \cdot \frac{r^2 + D^2 - a^2}{r^2}\,\mathrm{d}r \tag{4.5}$$

The total force exerted on the particle is $F = \int \mathrm{d}f$, so

$$\begin{aligned}
F &= \frac{G\,\rho\,\pi\, mat}{D^2} \int_{D-a}^{D+a} \left(1 + \frac{D^2 - a^2}{r^2}\right) \mathrm{d}r \\
&= \frac{G\,\rho\,\pi\, mat}{D^2} \left(\int_{D-a}^{D+a} \mathrm{d}r + (D^2 - a^2) \int_{D-a}^{D+a} \frac{1}{r^2}\mathrm{d}r\right) \\
&= \frac{G\,\rho\,\pi\, mat}{D^2} \left((D+a) - (D-a) + (D^2 - a^2)\left[\frac{-1}{D+a} + \frac{1}{D-a}\right]\right) \\
&= \frac{G\,\rho\,\pi\, mat}{D^2} \left((D+a) - (D-a) - (D-a) + (D+a)\right)
\end{aligned}$$

Therefore

$$F = \frac{G\,\rho\,m\,4\,\pi\,a^2 t}{D^2} = \frac{G\,M m}{D^2}\,,$$

where $M = 4\,\pi\,a^2\,\rho\,t$. This completes the proof of shell theorem 1.

Now, we proceed to derive theorem 2 for which the particle is inside the shell. Suppose that the ring is on the left side of the particle as shown in figure (a). Then

$$\cos\angle OPQ = \frac{AP}{r} = \frac{OP + OA}{r} = \frac{D - a\,\cos\theta}{r}$$

If the ring is on the right side of the particle as shown in figure (b). Then

$$\cos\angle OPQ = -\cos\angle QPA = -\frac{PA}{PQ} = -\frac{OA - OP}{r} = -\frac{a\,\cos\theta - D}{r} = \frac{D - a\,\cos\theta}{r}$$

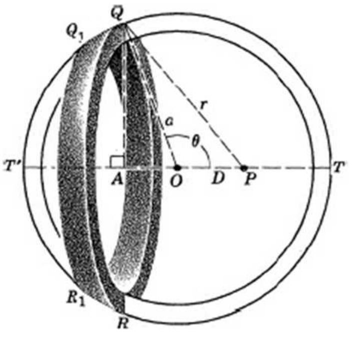


Figure (a)

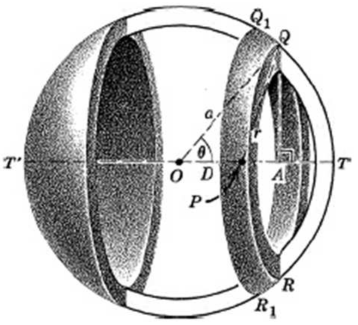


Figure (b)

That means the expression of $\cos\angle PQ$ are identical no matter the particle is outside or inside the hollow. So, the force expression of f is the same as equation (4.5) and thus the total force is given by

$$\begin{aligned}
F &= \frac{G\,\rho\,\pi\,mat}{D^2}\int_{a-D}^{a+D}\left(1 + \frac{D^2 - a^2}{r^2}\right)\mathrm{d}r \\
&= \frac{G\,\rho\,\pi\,mat}{D^2}\left(\int_{a-D}^{a+D}\mathrm{d}r + (D^2 - a^2)\int_{a-D}^{a+D}\frac{1}{r^2}\mathrm{d}r\right) \\
&= \frac{G\,\rho\,\pi\,mat}{D^2}\left((a+D) - (a-D) + (D^2 - a^2)\left[\frac{-1}{a+D} + \frac{1}{a-D}\right]\right) \\
&= \frac{G\,\rho\,\pi\,mat}{D^2}\left((a+D) - (a-D) + (a-D) - (a+D)\right) \\
&= 0
\end{aligned}$$

This complete the proof of theorem 2.

4.4 Force Field and Potential Energy

A force field is called *conservative* if the force vector $\vec{F}$ of the field depends only on the position $\vec{r}$ of the particle and the work integral $\int_A^B \vec{F}\cdot \mathrm{d}\vec{r}$ is independent of the path of integration, depending only on the initial point A and the final point B, of the path. Such force can be introduced by a scalar function, i.e. the potential energy $U(r)$, where

$$U(r) = -\int_P^r \vec{F}\cdot \mathrm{d}\vec{r}$$

In the above equation, P is an arbitrary point that we choose as a reference point in the force field, in particular, we define the value of U at this point as zero, i.e. $U(P) = 0$. For one dimensional case, we have

$$F(x) = -\frac{\mathrm{d}U(x)}{\mathrm{d}x}$$

Example 4.4 The gravitational force is a conservative force, find the gravitational potential energy of an object of mass m due to the Earth which has mass M.

Solution

For simplicity, we consider the motion that is along a straight line and it passes through the center of Earth. The initial and final positions of the object are located at A and B respectively. The work done by the gravitational force is given by

$$\begin{aligned} W &= \int_{r_A}^{r_B} \vec{F}\cdot \mathrm{d}\vec{r} \\ &= \int_{r_A}^{r_B} \left(-\frac{GMm}{r^2}\hat{r}\right)\cdot \mathrm{d}\vec{r} \\ &= \int_{r_A}^{r_B} \left(-\frac{GMm}{r^2}\hat{r}\right)\cdot \mathrm{d}r\,\hat{r} \\ &= \int_{r_A}^{r_B} \left(-\frac{GMm}{r^2}\right) \mathrm{d}r \\ &= -\left\{\left(-\frac{GMm}{r_B}\right) - \left(-\frac{GMm}{r_A}\right)\right\} \end{aligned}$$

Denote $U(r_A) = -\frac{GMm}{r_A}$ and $U(r_B) = -\frac{GMm}{r_B}$, we have $W = -\{U(r_B) - U(r_A)\}$. Take $r_A = r$ and choose r_B as the reference point at infinity and define $U(r_B) = 0$. We obtain

$$W_{r\infty} = U(r) = -\frac{GMm}{r}$$

Example 4.5 An object has mass m and is attracted by the Earth of mass M. Find the gravitational force exerted on the object if the gravitational potential energy of the object is $-GMm/r$.

Solution

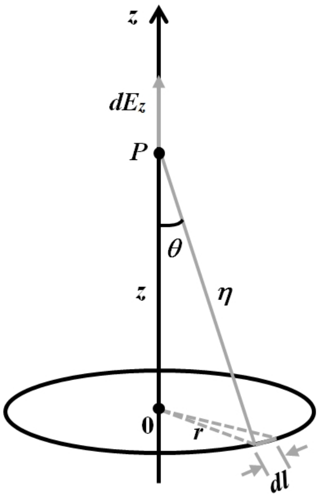
$$F(r) = -\frac{\mathrm{d}U(x)}{\mathrm{d}x} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{GMm}{r}\right) = -\frac{GMm}{r^2}$$

■

Example 4.6 Find the electric field a distance z above the center of a circular loop of radius r, which carries a uniform line charge density λ (i.e. the charge per unit length).

Solution

Let P be the point having a distance z above the center of loop. Consider a small arc of length dl on the loop. The horizontal component of the E-field due to this small charge element cancels because of the symmetry of the system.



The vertical component of the E-field due to this small charge element is dE_z, where

$$\mathrm{d}E_z = \frac{1}{4\pi\epsilon_0}\frac{\lambda\,\mathrm{d}l}{\eta^2}\cos\theta\,,$$

where $\eta^2 = z^2 + r^2$ and $\cos\theta = z/\eta$. Hence, the E-field at P due to the entire loop is

$$\begin{aligned}
\int_{\text{loop}} \mathrm{d}E_z &= \int_{\text{loop}} \frac{1}{4\pi\epsilon_0} \frac{\lambda\,\mathrm{d}l}{\eta^2} \cos\theta \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda\,\cos\theta}{\eta^2} \int_{\text{loop}} \mathrm{d}l \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda\,\cos\theta}{\eta^2} (2\pi r) \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda\,(2\pi r)\,z}{(r^2+z^2)^{3/2}}
\end{aligned}$$

Therefore, the electric field at a point z above the center the circular loop is

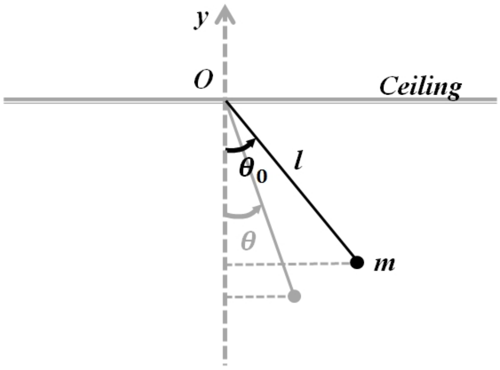
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\lambda\,(2\pi r)\,z}{(r^2+z^2)^{3/2}}\,\hat{k}$$

■

Exercise 4.3 Find the electric field a distance z above the center of a circular disk of radius R, which carries a uniform surface charge σ (representing the charge per unit area).

4.5 The Period of a Plane Pendulum

A plane pendulum has a light string of length l and a metal bob of mass m. One end of the string is attached to the bob and the next end is fixed to the ceiling. The string is kept tight when the bob is released from a point that the string makes an initial angle θ_0 with the vertical.



Suppose that the mechanical energy of this system is conserved, the initial angle θ_0 becomes the amplitude of oscillation and the following points are noted.

(a) The period of a plane pendulum is approximately a constant when the amplitude of oscillation is very small.

(b) The plane pendulum is not isochronous when the amplitude of oscillation is not small. That is to say, the period depends on the amplitude of the oscillation.

Set the point on the ceiling as zero gravitational potential, the conservation of mechanical energy gives

$$\begin{aligned}\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta &= -mgl\cos\theta_0\\ \frac{1}{2}ml^2\dot{\theta}^2 &= mgl\left(\cos\theta - \cos\theta_0\right)\\ \frac{1}{2}ml^2\dot{\theta}^2 &= 2mgl\left[\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]\end{aligned}$$

Hence,

$$\dot{\theta} = 2\sqrt{\frac{g}{l}}\left[\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]^{1/2}$$

Since $\dot{\theta} = \mathrm{d}\theta/\mathrm{d}t$, then we have

$$\mathrm{d}t = \frac{1}{2}\sqrt{\frac{l}{g}}\left[\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]^{-1/2}\mathrm{d}\theta$$

The period of the oscillation is given by

$$\tau = 4\int_0^{\theta_0}\frac{1}{2}\sqrt{\frac{l}{g}}\left[\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]^{-1/2}\mathrm{d}\theta$$

This is an elliptic integral that we can proceed our work with the following substitutions.

$$z = \frac{\sin(\theta/2)}{\sin(\theta_0/2)} \qquad \text{and} \qquad k = \sin(\theta_0/2)$$

Then

$$\mathrm{d}z = \frac{\cos(\theta/2)}{2\,\sin(\theta_0/2)}\,\mathrm{d}\theta = \frac{\sqrt{1-k^2z^2}}{2k}\,\mathrm{d}\theta$$

Now, we obtain

$$\tau = 4\sqrt{\frac{l}{g}}\int_0^1\left[\left(1-z^2\right)\left(1-k^2z^2\right)\right]^{-1/2}\mathrm{d}z$$

The power series expansion for $(1-k^2z^2)^{-1/2}$ gives

$$(1-k^2z^2)^{-1/2} = 1 + \frac{k^2z^2}{2} + \frac{3k^4z^4}{8} + \dots$$

The expression for the period can be rewritten as

$$\begin{aligned}
\tau &= 4\sqrt{\frac{l}{g}} \int_0^1 \frac{\mathrm{d}z}{(1-z^2)^{1/2}} \left[1 + \frac{k^2z^2}{2} + \frac{3k^4z^4}{8} + \dots\right] \\
&= 4\sqrt{\frac{l}{g}} \left\{\int_0^1 \frac{\mathrm{d}z}{(1-z^2)^{1/2}} + \frac{k^2}{2}\int_0^1 \frac{z^2\,\mathrm{d}z}{(1-z^2)^{1/2}} + \frac{3k^4}{8}\int_0^1 \frac{z^4\,\mathrm{d}z}{(1-z^2)^{1/2}} + \dots\right\} \\
&= 4\sqrt{\frac{l}{g}} \left\{\frac{\pi}{2} + \frac{k^2}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{3k^4}{8}\cdot\frac{3}{8}\cdot\frac{\pi}{2} + \dots\right\} \\
&= 2\pi\sqrt{\frac{l}{g}} \left\{1 + \frac{k^2}{4} + \frac{9k^4}{64} + \dots\right\}
\end{aligned}$$

When θ_0 is very small, $k = \sin\left(\frac{\theta_0}{2}\right) \approx \frac{\theta_0}{2}$. If we neglect the higher order terms of k, we have

$$\tau \approx 2\pi\sqrt{\frac{l}{g}}$$

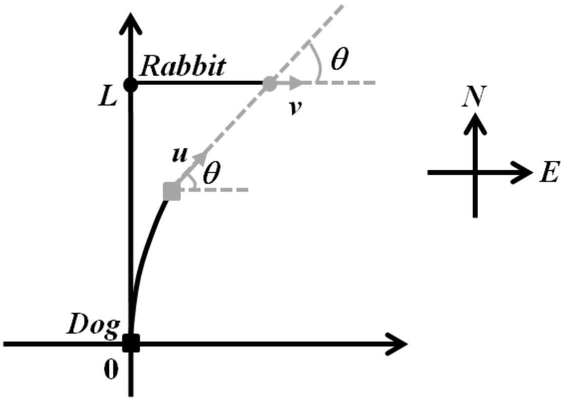
For more accurate result of τ, we write $k = \sin\left(\frac{\theta_0}{2}\right) \approx \frac{\theta_0}{2} - \frac{\theta_0^3}{48}$, thus

$$\tau \approx 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4\right]$$

It indicates that the plane pendulum is not isochronous.

4.6 The Dog-And-Rabbit Chase Problem

The classic dog-and-rabbit chase problem is an interesting topic in calculus. A dog is at a distance L due south of a rabbit, it observes the rabbit running in a vast field at time $t = 0$. The positions of them at time $t = 0$ are shown in the figure. When the dog sees the rabbit, it starts to pursue the rabbit and its motion always points to the rabbit. Given that the rabbit keeps running due east with a constant speed v and the dog' speed is a constant u, where $v < u$. Find the time elapsed when the dog catches the rabbit.



Let x be the horizontal displacement of the rabbit relative to the dog and τ be the time elapsed when the dog catches the rabbit. Then, at arbitrary time t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v - u\cos\theta \tag{4.6}$$

Integrating both sides of the equation (4.6) from $t = 0$ to $t = \tau$, we have

$$\begin{aligned} \int_{x=0}^{0} \mathrm{d}x &= \int_0^{\tau} (v - u\cos\theta)\,\mathrm{d}t \\ 0 &= v\tau - u\int_0^{\tau} \cos\theta\,\mathrm{d}t \end{aligned}$$

That is to say,

$$\int_0^{\tau} \cos\theta\,\mathrm{d}t = \frac{v\tau}{u} \tag{4.7}$$

Let r be the displacement of the rabbit relative to the dog. At any instant of time,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = v\,\cos\theta - u \tag{4.8}$$

Integrating both sides of equation (4.8) from $t = 0$ to $t = \tau$, we have

$$\begin{aligned} \int_{r=L}^{0} \mathrm{d}r &= \int_0^{\tau} (v\,\cos\theta - u)\,\mathrm{d}t \\ -L &= v\int_0^{\tau} \cos\theta\,\mathrm{d}t - u\tau \end{aligned}$$

Hence, we obtain

$$L = u\tau - v\int_0^{\tau} \cos\theta\,\mathrm{d}t \tag{4.9}$$

Substituting equation (4.7) into (4.9), we have

$$L = u\tau - v\left(\frac{v\tau}{u}\right)$$

Therefore, the required time τ is $\dfrac{Lu}{u^2 - v^2}$.

4.7 Impulse

The impulse I on an object is defined as the change of momentum of an object during the impact.

$$I = \Delta p$$

The impulse I can be expressed as an integral of force with respect to time

$$I = \int F \, \mathrm{d}t,$$

where F is the force exerted on the object and t is the time.

Example 4.7 A particle of mass m is thrown vertically upward with speed v_0 and it returns to the initial point with speed v_1, where v_0 and v_1 are positive quantities. Suppose the retarding force F due to the air resistance is linearly proportional to the instantaneous velocity of particle, e.g. $F = -kv$, where $k > 0$ and v is the velocity of the particle. By considering the total impulse acting on the particle during its motion, show that the time elapsed is given by

$$t = \frac{1}{g}(v_0 + v_1) \, .$$

Solution

The net force on the particle F_{net} equals to the rate of change of momentum of the particle.

$$F_{\text{net}} = \frac{\mathrm{d}p}{\mathrm{d}t}$$

Integrate on both sides with respect to t, we have

$$\begin{aligned} \int F_{\text{net}} \, \mathrm{d}t &= \int \mathrm{d}p \\ \int F_{\text{net}} \, \mathrm{d}t &= \Delta p \end{aligned}$$

Taking the upward motion as positive, the net force $F_{\text{net}} = -mg - kv$, hence,

$$\begin{aligned} \int_0^t (-mg - kv) \, \mathrm{d}t &= m\,(-v_1 - v_0) \\ -mgt - k\int_0^t v \, \mathrm{d}t &= m\,(-v_1 - v_0) \end{aligned}$$

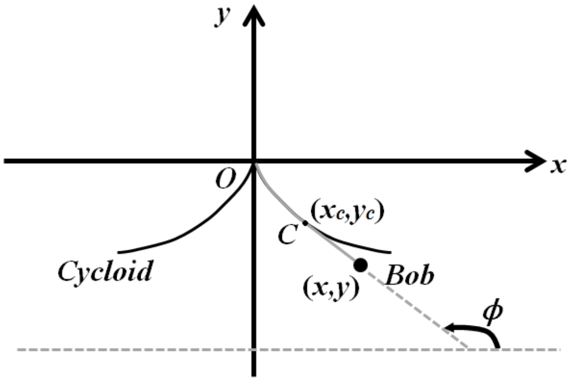
The integral $\int_0^t v \, \mathrm{d}t$ in the left hand side of the above equation gives the total displacement of the particle and its value is zero. We then have

$$mgt = m\,(v_1 + v_0)$$

Thus, the required time is given by $\frac{1}{g}\,(v_0 + v_1)$. ■

4.8 Isochronous Pendulum

The idea of a cycloidal pendulum was introduced by a Dutch physicist Christiaan Huygens in 1656. A pendulum is suspended from the cusp of a cycloid cut in a rigid support. The path described by the cycloid is given by $x = a\,(\theta - \sin\theta)$ and $y = a\,(\cos\theta - 1)$, where θ is the angle of rotation of the disk of radius a generating the cycloid. The length of the pendulum is $l = 4a$. The oscillations are exactly isochronous with a period $2\pi\sqrt{\dfrac{l}{g}}$, independent of the amplitude. The proof is shown below.



When the bob is oscillating, the contact point on the cycloid by the string is $C(x_c, y_c)$, where

$$\begin{cases} x_c = a\,(\theta - \sin\theta) \\ y_c = a\,(\cos\theta - 1) \end{cases}$$

The length of the cycloid from the origin O to this contact point is

$$\begin{aligned} l_c &= \int \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \\ &= \sqrt{2}a \int_0^\theta \sqrt{1-\cos\theta}\,\mathrm{d}\theta \\ &= 4a\left(1 - \sqrt{\frac{1+\cos\theta}{2}}\right) \end{aligned}$$

The slope of the string is $\tan\phi = \dfrac{\mathrm{d}y_c}{\mathrm{d}x_c} = -\dfrac{\sin\theta}{1-\cos\theta}$. Therefore, the coordinates of the bob is

$$\begin{cases} x = x_c - (l - l_c)\,\cos\phi = a\,(\theta + \sin\theta) \\ y = y_c - (l - l_c)\,\sin\phi = -a\,(3 + \cos\theta) \end{cases}$$

The velocity of the bob $v = \sqrt{\dot{x}^2 + \dot{y}^2} = a\dot{\theta}\,\sqrt{2\,(1 + \cos\theta)}$.
The conservation of mechanical energy gives

$$\frac{1}{2}m(0)^2 + mgy_0 = \frac{1}{2}mv^2 + mgy$$

where $y_0 = -a\,(3 + \cos\theta_0)$ is the y-coordinate of the bob when the bob is stationary. Hence, we obtain

$$a\dot{\theta}^2\,(1 + \cos\theta) - g\cos\theta = -g\,\cos\theta_0$$

and thus

$$\dot{\theta} = \sqrt{\frac{g}{a}\left(\frac{\cos\theta - \cos\theta_0}{1 + \cos\theta}\right)}$$

For the period of the oscillation $T = \displaystyle\int \mathrm{d}t = 4\int_0^{\theta_0} \frac{\mathrm{d}\theta}{\dot{\theta}}$, thus

$$T = 4\sqrt{\frac{a}{g}}\int_0^{\theta_0}\sqrt{\frac{1 + \cos\theta}{\cos\theta - \cos\theta_0}}\,\mathrm{d}\theta$$

Making use the substitution $\sigma = \dfrac{\cos\theta - \cos\theta_0}{1 - \cos\theta_0}$, we obtain

$$T \;=\; 4\sqrt{\frac{a}{g}}\int_0^1 \frac{\mathrm{d}\sigma}{\sqrt{\sigma\,(1 - \sigma)}}$$

Using the substitution $\sigma = \sin^2\beta$, we obtain

$$T = 8\sqrt{\frac{a}{g}}\int_0^{\pi/2}\mathrm{d}\beta$$

Therefore, we have $T = 2\pi\sqrt{\dfrac{l}{g}}$.

4.9 Integration by Parts

When we compute the integrals, a special technique called the 'Integration by Parts' will facilitate your work and it is very useful to obtain the reduction formula for an integral.

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

The following examples illustrate its application.

Example 4.8 Find the following integrals.

(a) $\int x\,e^{5x}\,\mathrm{d}x$ (b) $\int x^2\,\cos x\,\mathrm{d}x$ (c) $\int \frac{\ln x}{x^{2020}}\,\mathrm{d}x$

Solution

(a)

$$\begin{aligned}\int x\,e^{5x}\,\mathrm{d}x &= \frac{1}{5}\int x\,\mathrm{d}\,e^{5x}\\ &= \frac{1}{5}\,x\,e^{5x} - \frac{1}{5}\int e^{5x}\,\mathrm{d}x\\ &= \frac{1}{5}\,x\,e^{5x} - \frac{1}{25}\int e^{5x}\,\mathrm{d}(5\,x)\\ &= \frac{1}{5}\,x\,e^{5x} - \frac{1}{25}\,e^{5x} + C\end{aligned}$$

(b)

$$\begin{aligned}\int x^2\,\cos x\,\mathrm{d}x &= \int x^2\,\mathrm{d}\sin x\\ &= x^2\,\sin x - \int \sin x\,\mathrm{d}x^2\\ &= x^2\,\sin x - 2\int x\,\sin x\,\mathrm{d}x\\ &= x^2\,\sin x + 2\int x\,\mathrm{d}\cos x\\ &= x^2\,\sin x + 2\,x\,\cos x - 2\int \cos x\,\mathrm{d}x\\ &= x^2\,\sin x + 2\,x\,\cos x - 2\,\sin x + C\end{aligned}$$

(c)

$$\begin{aligned}\int \frac{\ln x}{x^{2020}}\,\mathrm{d}x &= -\frac{1}{2019}\int \ln x\,\mathrm{d}(x^{-2019})\\ &= -\frac{1}{2019}\left(x^{-2019}\,\ln x - \int x^{-2019}\,\mathrm{d}(\ln x)\right)\\ &= -\frac{1}{2019}\left(x^{-2019}\,\ln x - \int x^{-2020}\,\mathrm{d}x\right)\\ &= -\frac{1}{2019}\left(x^{-2019}\,\ln x + \frac{1}{2019}\,x^{-2019}\right) + C\\ &= -\frac{1}{2019}\,x^{-2019}\,\ln x - \frac{1}{(2019)^2}\,x^{-2019} + C\end{aligned}$$

■

Example 4.9 Let $I_n = \int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}\,\mathrm{d}x$. Show that $I_n = \frac{2n-1}{2n}\,I_{n-1}$. Hence, find I_5.

Solution

$$\begin{aligned} I_n &= \int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}\,\mathrm{d}x \\ &= -\int_0^1 x^{2n-1}\,\mathrm{d}\sqrt{1-x^2} \\ &= -\left(x^{2n-1}\sqrt{1-x^2}\,\Big|_0^1 - \int_0^1 \sqrt{1-x^2}\,\mathrm{d}x^{2n-1} \right) \\ &= (2n-1)\int_0^1 \frac{x^{2n-2}\,(1-x^2)}{\sqrt{1-x^2}}\,\mathrm{d}x \\ &= (2n-1)\,(I_{n-1} - I_n) \end{aligned}$$

So $I_n = \frac{2n-1}{2n}\,I_{n-1}$. Using the substituting $x = \sin\theta$, we have

$$I_0 = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_0^{\pi/2} \mathrm{d}\theta = \frac{\pi}{2}$$

Hence, $I_5 = \frac{9}{10}\,I_4 = \frac{9}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\,I_0 = \frac{9}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{63\,\pi}{512}$. ■

Example 4.10 Denote $I_n = \int_0^{\pi/2} \sin^n x\,\mathrm{d}x$, where n is a positive integer. Obtain a reduction formula for I_n Hence, find I_5.

Solution

$$\begin{aligned} I_n &= \int_0^{\pi/2} \sin^n x\,\mathrm{d}x \\ &= -\int_0^{\pi/2} \sin^{n-1} x\,\mathrm{d}\cos x \\ &= -\sin^{n-1} x\,\cos x\Big|_0^{\frac{\pi}{2}} + \int_0^{\pi/2} \cos x\,\mathrm{d}\sin^{n-1} x \\ &= (n-1)\int_0^{\pi/2} \sin^{n-2} x\,\cos^2 x\,\mathrm{d}x \\ &= (n-1)\int_0^{\pi/2} \sin^{n-2} x\,(1-\sin^2 x)\,\mathrm{d}x \\ &= (n-1)\,(I_{n-2} - I_n) \end{aligned}$$

Hence, we obtain $I_n = \frac{n-1}{n}\,I_{n-2}$.

Finally, we have $I_5 = \frac{4}{5}\,I_3 = \frac{4}{5}\cdot\frac{2}{3}\,I_1 = \frac{8}{15}$. ■

Exercise 4.4 Denote $I_n = \displaystyle\int_0^{\pi/2} \cos^n x \,\mathrm{d}x$, where n is a positive integer. Obtain a reduction formula for I_n.

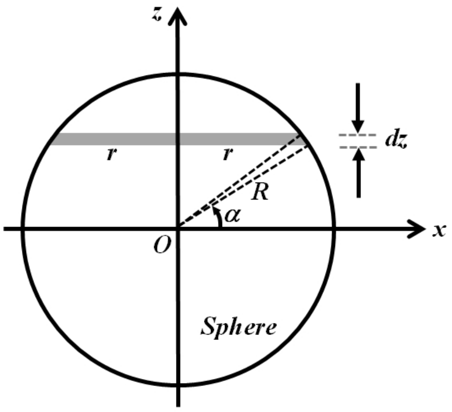
Exercise 4.5 Denote $J_n = \displaystyle\int \sec^n x \,\mathrm{d}x$, where n is a positive integer. Obtain a reduction formula for J_n.

Exercise 4.6 If $I_{m,n} = \displaystyle\int_{-1}^{1} (x+1)^m (x-1)^n \,\mathrm{d}x$, where m and n are positive integers. Show that $I_{m,n} = -\dfrac{m}{n+1} I_{m-1,\,n+1}$. Hence, evaluate $I_{10,5}$.

Exercise 4.7 Let $I_n = \displaystyle\int_0^{\pi} \frac{\sin nx}{\sin x} \,\mathrm{d}x$. Find a reduction formula for I_n. Distinguish the cases for n being odd and even.

Example 4.11 Find the moment of inertia of a uniform sphere about its diameter. The radius and mass of the sphere are R and M respectively.

Solution



The sphere can be regarded as a collection of numerous disks, each having infinitesimal thickness dz. The density is ρ, where $M = \dfrac{4}{3}\pi\rho R^3$. For an arbitrary disk inside the sphere, the radius of it is $r = R\cos\alpha$ and the distance of it from the origin is $z = R\sin\alpha$. The mass of the disk is $\mathrm{d}m = \pi\rho r^2 \,\mathrm{d}z = (\pi\rho R^2\cos^2\alpha)(R\cos\alpha\,\mathrm{d}\alpha) = \pi\rho R^3\cos^3\alpha\,\mathrm{d}\alpha$. From example 4.2, we know that the moment of inertia of such disk about the z-axix is $r^2\,\mathrm{d}m/2$. Now, the total moment of inertia

$$\begin{aligned} I &= \int \mathrm{d}I = \int \frac{r^2}{2}\,\mathrm{d}m \\ &= \int (\frac{R^2\cos^2\alpha}{2})\,(\pi\rho R^3\cos^3\alpha\,\mathrm{d}\alpha) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi\rho R^5}{2}\int_{-\pi/2}^{\pi/2}\cos^5\alpha\,\mathrm{d}\alpha \\
&= \pi\rho R^5\int_{0}^{\pi/2}\cos^5\alpha\,\mathrm{d}\alpha
\end{aligned}
$$

Making use the result of Exercise 4.4, we have

$$
\begin{aligned}
I &= \pi\rho R^5\,(\frac{4}{5}\cdot\frac{2}{3})\int_0^{\pi/2}\cos\alpha\,\mathrm{d}\alpha \\
&= \frac{8}{15}\pi\rho R^5
\end{aligned}
$$

Hence, $I = \frac{2}{5}\,R^2\left(\frac{4}{3}\,\pi\rho R^3\right) = \frac{2}{5}\,MR^2.$ ■

4.10 Sum of an Infinite Series

The sum of an infinite series can be computed by using integration. The summation sign can be converted to a definite integral while taking out the common factor $1/n$ as $\mathrm{d}x$, where $n \to \infty$. To construct the integrand, the fraction k/n is replaced by x. If $k = 1,\ 2,\ 3,\ \cdots,\ n$, then the lower and upper limits of integration are $x = 0$ and $x = 1$ respectively.

Example 4.12 Find $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+2\,n}} + \ \cdots\ + \frac{1}{\sqrt{n^2+n^2}}\right).$

Solution

$$
\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+2\,n}} + \ \cdots\ + \frac{1}{\sqrt{n^2+n^2}}\right) \\
&\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+\frac{1}{n}}} + \frac{1}{\sqrt{1+\frac{2}{n}}} + \ \cdots\ + \frac{1}{\sqrt{1+\frac{n}{n}}}\right) \\
= &\int_0^1\frac{\mathrm{d}x}{\sqrt{1+x}} \\
= &\ 2\sqrt{1+x}\,\Big|_0^1 \\
= &\ 2\,(\sqrt{2}-1)
\end{aligned}
$$

■

Example 4.13 Find $\lim\limits_{n\to\infty}\sum_{k=1}^{3n}\frac{1}{n}\,\sin\left(\pi + \frac{k\,\pi}{2\,n}\right).$

Solution

$$\begin{aligned}
& \lim_{n\to\infty}\sum_{k=1}^{3n}\frac{1}{n}\sin\left(\pi+\frac{k\,\pi}{2\,n}\right)\\
= & \lim_{n\to\infty}\left[\frac{1}{n}\sin\left(\pi+\frac{\pi}{2\,n}\right)+\frac{1}{n}\sin\left(\pi+\frac{2\,\pi}{2\,n}\right)+\ \cdots\ +\frac{1}{n}\sin\left(\pi+\frac{3\,n\,\pi}{2\,n}\right)\right]\\
= & \lim_{n\to\infty}\left[\frac{1}{n}\left\{\sin\left(\pi+\frac{\pi}{2}\cdot\frac{1}{n}\right)+\sin\left(\pi+\frac{\pi}{2}\cdot\frac{2}{n}\right)+\ \cdots\ +\sin\left(\pi+\frac{\pi}{2}\cdot\frac{3\,n}{n}\right)\right\}\right]\\
= & \int_0^3 \sin\left(\pi+\frac{\pi x}{2}\right)\,\mathrm{d}x\\
= & -\int_0^3 \sin\left(\frac{\pi x}{2}\right)\,\mathrm{d}x\\
= & \frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\Bigg|_0^3\\
= & -\frac{2}{\pi}
\end{aligned}$$

■

Exercise 4.8 $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ \cdots\ +\frac{n}{n^2+n^2}\right)$

Exercise 4.9 $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+3}+\frac{1}{n+5}\ \cdots\ +\frac{1}{3\,n+1}\right)$

4.11 Finding the Power Series

If the series for a given function be known, or can be found, then the series for its integral can be obtained by integrating this series term by term.

Example 4.14 By considering the integral $\int_0^x \frac{\mathrm{d}u}{1+u^2}$, find the power series of $\tan^{-1}x$, where $|x|<1$. Hence, show that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ \cdots$.

Solution

Knowing that $\frac{1}{1+u^2}=1-u^2+u^4-u^6+\ \cdots$, where $u^2<1$, then

$$\begin{aligned}
\int_0^x \frac{\mathrm{d}u}{1+u^2} &= \int_0^x (1-u^2+u^4-u^6+\ \cdots)\,\mathrm{d}u\\
&= x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ \cdots
\end{aligned}$$

Using the substitution $u = \tan\theta$, we have $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$, then the LHS of the above equation becomes

$$\int_0^x \frac{\mathrm{d}u}{1+u^2} \;=\; \int_0^{\tan^{-1}x} \mathrm{d}\theta \;=\; \tan^{-1}x$$

Hence, we have $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \;\cdots$.

Putting $x = 1$, we have $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \;\cdots$. ■

Example 4.15 Find the power series of $\ln(1{+}x)$ by integrating a function, where $|x| < 1$.

Solution

We start from the expression $\frac{1}{1+u} = 1 - u + u^2 - u^3 + \;\cdots$, where $|u| < 1$, then

$$\begin{aligned}
\int_0^x \frac{\mathrm{d}u}{1+u} &= \int_0^x (1 - u + u^2 - u^3 + \;\cdots)\,\mathrm{d}u \\
\ln(1+u)\Big|_0^x &= \left(u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \;\cdots\right)\Big|_0^x \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \;\cdots
\end{aligned}$$

■

Exercise 4.10 Find the power series of $\sin^{-1}x$ by integrating a function. Determine the function.

Chapter 5

Ordinary Differential Equations

The ordinary differential equations (ODE) are widely used in physics and engineering. In this chapter, we will focus the discussion on the first order differential equations.

5.1 Separation of Variables

If the right-hand side of the first order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

can be expressed as a function that depends only on x times a function that depends on y, then the differential equation is called separable. Such equation has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)\, p(y)$$

To solve the equation, we multiply both sides by $1/p(y)$ and $\mathrm{d}x$. Hence, we obtain

$$\frac{\mathrm{d}y}{p(y)} = g(x)\, \mathrm{d}x$$

Then integrate both sides of the equation

$$\int \frac{\mathrm{d}y}{p(y)} = \int g(x)\, \mathrm{d}x$$

This technique is known as the separation of variables.

Example 5.1 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x-5}{y^2}$.

Solution

We separate the variables on two sides of the equation as $y^2\,\mathrm{d}y = (x-5)\,\mathrm{d}x$. Integrating, we have

$$\begin{aligned}
\int y^2\,\mathrm{d}y &= \int (x-5)\,\mathrm{d}x \\
\frac{y^3}{3} &= \frac{x^2}{2} - 5x + C \\
y &= \left(\frac{3x^2}{2} - 15x + 3C\right)^{1/3}
\end{aligned}$$

Replace $3C$ by K, we then have

$$y = \left(\frac{3x^2}{2} - 15x + K\right)^{1/3}$$

■

Example 5.2 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cos(x+y)$.

Solution

Let $z = x+y$, then $\dfrac{\mathrm{d}z}{\mathrm{d}x} = 1 + \dfrac{\mathrm{d}y}{\mathrm{d}x}$. So, we have

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}x} - 1 &= \cos z \\
\frac{\mathrm{d}z}{\cos z + 1} &= \mathrm{d}x \\
\int \frac{\mathrm{d}z}{\cos z + 1} &= \int \mathrm{d}x \\
\int \frac{\mathrm{d}z}{2\cos^2\left(\frac{z}{2}\right) - 1 + 1} &= \int \mathrm{d}x \\
\frac{1}{2}\int \sec^2\frac{z}{2}\,\mathrm{d}z &= \int \mathrm{d}x \\
\tan\frac{z}{2} &= x + C
\end{aligned}$$

Hence, $\tan\dfrac{x+y}{2} = x + C$. ■

Example 5.3 Solve $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - (x+y-1)\dfrac{\mathrm{d}y}{\mathrm{d}x} + x\,(y-1) = 0$.

Solution

The original equation can be rewritten as

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x} - x\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - y + 1\right) = 0$$

which gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x \qquad \text{or} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = y - 1$$

then, we have

$$\int \mathrm{d}y = \int x \,\mathrm{d}x \qquad\qquad \int \frac{\mathrm{d}y}{y-1} = \int \mathrm{d}x$$

Finally, we get $y = \dfrac{x^2}{2} + C$ or $\ln(y-1) = x + C'$. The latter can be rewritten as $y = 1 + A\,e^x$ where $A = e^{C'}$. ■

Example 5.4 Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2+y^2}{x^2+xy}$.

Solution
Let $y = vx$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = v + x\,\dfrac{\mathrm{d}v}{\mathrm{d}x}$. The original equation becomes

$$\begin{aligned}
v + x\,\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1+\left(\dfrac{y}{x}\right)^2}{1+\dfrac{y}{x}} \\
v + x\,\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1+v^2}{1+v} \\
x\,\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1-v}{1+v}
\end{aligned}$$

Integrating both sides of the equation, we have

$$\begin{aligned}
\int \frac{1+v}{1-v}\,\mathrm{d}v &= \int \frac{\mathrm{d}x}{x} \\
\int \frac{-1-v}{1-v}\,\mathrm{d}v &= -\int \frac{\mathrm{d}x}{x} \\
\int \frac{1-v-2}{1-v}\,\mathrm{d}v &= -\ln(x) \\
\int \left(1 - \frac{2}{1-v}\right)\mathrm{d}v &= \ln\left(\frac{1}{x}\right) \\
v + 2\,\ln(1-v) + C &= \ln\left(\frac{1}{x}\right) \\
\ln(e^C\,e^v\,(1-v)^2) &= \ln\left(\frac{1}{x}\right) \\
e^C\,e^v\,(1-v)^2 &= \frac{1}{x}
\end{aligned}$$

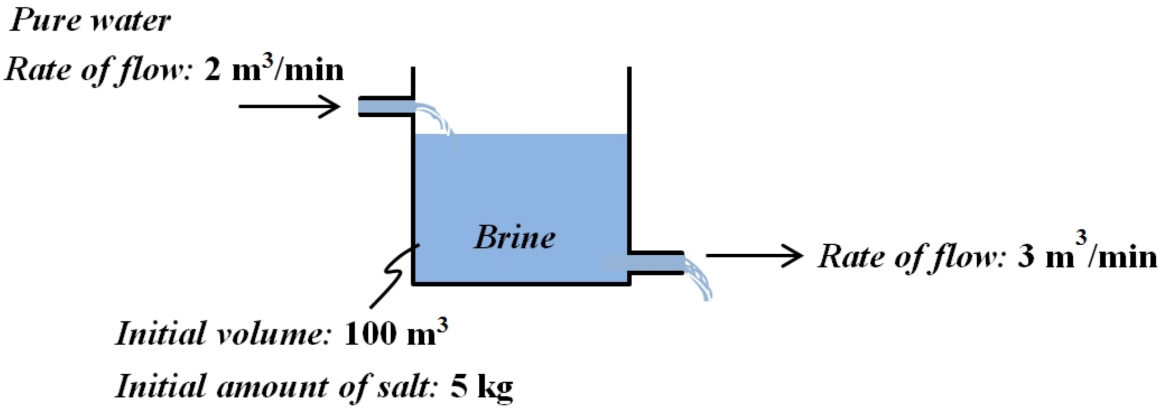
Thus, we have $e^C\,e^{\frac{y}{x}}\left(1-\dfrac{y}{x}\right)^2 = \dfrac{1}{x}$ which gives $e^{\frac{y}{x}}\,(x-y)^2 = A\,x$, where $A = e^{-C}$. ■

Exercise 5.1 Solve the equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sec y \, \tan x$.

Exercise 5.2 Solve the equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 10^x \, 10^y$.

Example 5.5 Initially, a tank of brine contains 5 kg of salt dissolved in 100 m^3 of water. At $t = 0$, pure water begins to flow into the tank at the rate of 2 m^3/min. The stirred mixture runs out of it at the rate of 3 m^3/min. Find the amount of salt in the tank after 50 min.

Solution



Let x be the amount of salt in the tank at time t. The volume of brine at time t is $(100 - t)\,\mathrm{m}^3$ because it decreases by 1 m^3 per minute. Then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{3\,x}{100 - t} \tag{5.1}$$

Separating the variables, we have

$$\frac{\mathrm{d}x}{x} = \frac{-3\,\mathrm{d}t}{100 - t}$$

Integrating both sides, we have

$$\begin{aligned} \int_5^x \frac{\mathrm{d}x}{x} &= -3 \int_0^t \frac{\mathrm{d}t}{100 - t} \\ \ln x \Big|_5^x &= 3\,\ln(100 - t)\Big|_0^t \end{aligned}$$

$$\begin{aligned}
\ln\left(\frac{x}{5}\right) &= \ln\left(\frac{100-t}{100}\right)^3 \\
\frac{x}{5} &= \left(\frac{100-t}{100}\right)^3 \\
x &= 5\left(\frac{100-t}{100}\right)^3
\end{aligned}$$

When $t = 50\,\text{min}$, $x = 5\left(\frac{100-50}{100}\right)^3 = \frac{5}{8}\,\text{kg}$. ■

Example 5.6 A particle of mass m is projected with speed v_0 at an angle ϕ to the horizontal. Find the time when the particle reaches its maximum height if air resistance is not negligible. Given that the air resistance (i.e. the force drag) is linearly proportional to the speed of the particle.

Solution

Let the air resistance a force constant k, where k is positive. The equation of motion of the particle along the vertical is

$$-mg - kv_y = ma_y,$$

where a_y is the acceleration of the particle along the vertical. But, $a_y = \frac{\mathrm{d}v_y}{\mathrm{d}t}$, which gives

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\left(\frac{mg + kv_y}{m}\right)$$

Arrange the equation by separating the variables,

$$\frac{\mathrm{d}v_y}{mg + kv_y} = -\frac{\mathrm{d}t}{m}$$

Integrate both sides of the equation, we obtain

$$\begin{aligned}
\int_{v_0 \sin\phi}^{v_y} \frac{\mathrm{d}v_y}{mg + kv_y} &= -\frac{1}{m}\int_0^t \mathrm{d}t \\
\frac{1}{k}\ln(mg + kv_y)\Big|_{v_y = v_0 \sin\phi}^{v_y} &= -\frac{t}{m} \\
\frac{1}{k}\ln\left(\frac{mg + kv_y}{mg + kv_0\,\sin\phi}\right) &= -\frac{t}{m} \\
\frac{mg + kv_y}{mg + kv_0\,\sin\phi} &= e^{-\frac{kt}{m}}
\end{aligned}$$

Therefore, the vertical velocity of the particle is given by

$$v_y = \frac{1}{k}\left[(mg + kv_0\,\sin\phi)\,e^{-\frac{kt}{m}} - mg\right]$$

At the maximum point, $v_y = 0$, thus

$$\begin{aligned}(mg + kv_0 \sin\phi)\, e^{-\frac{kt}{m}} &= mg \\ e^{\frac{kt}{m}} &= 1 + \frac{kv_0}{mg}\sin\phi \\ t &= \frac{m}{k}\ln\left(1 + \frac{kv_0}{mg}\sin\phi\right)\end{aligned}$$

■

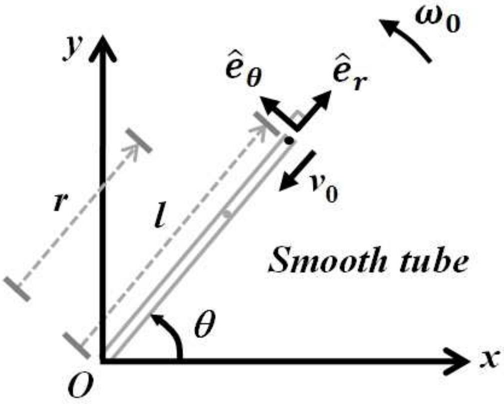
Example 5.7 A straight and smooth tube has length l and is rotating about its end O horizontally with a constant angular speed ω_0. A particle is placed at the next end inside the tube and it is projected with speed $v_0 = l\omega_0$ towards O and along the tube.

(a) Find the speed of the particle along the tube when it is located at r from O.

(b) Find the time when the particle is located at $l/2$ from O.

(c) Determine whether the particle can reach O.

Solution



(a) We consider the motion of the particle in the polar coordinate system, where the origin is located at the axis of rotation. The acceleration vector $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$. As the tube is smooth, there is no force acting on the particle along the tube, i.e. $\ddot{r} - r\,\omega_0^2 = 0$. Hence,

$$\begin{aligned}\dot{r}\,\frac{\mathrm{d}\dot{r}}{\mathrm{d}r} - r\,\omega_0^2 &= 0 \\ \int_{-v_0}^{v} \dot{r}\,\mathrm{d}\dot{r} &= \omega_0^2 \int_{l}^{r} r\,\mathrm{d}r\end{aligned}$$

$$\begin{aligned}
\left.\frac{\dot{r}^2}{2}\right|_{-v_0}^{v} &= \left.\frac{\omega_0^2\, r^2}{2}\right|_{l}^{r} \\
v^2 - v_0^2 &= \omega_0^2\,(r^2 - l^2) \\
v^2 &= \omega_0^2\,(r^2 - l^2) + v_0^2
\end{aligned}$$

Substituting the initial speed $v_0 = l\,\omega_0$, we obtain

$$\begin{aligned}
v^2 &= \omega_0^2\,(r^2 - l^2) + l^2\,\omega_0^2 \\
v^2 &= \omega_0^2\, r^2
\end{aligned}$$

As the motion of the particle points O, the velocity of the particle should be negative. That is, the velocity $v = -\omega_0\, r$, and the speed is $\omega_0\, r$.

(b) The velocity $v = \dfrac{\mathrm{d}r}{\mathrm{d}t} = -\omega_0\, r$. Separating the variables, we obtain

$$\begin{aligned}
\int_l^r \frac{\mathrm{d}r}{r} &= -\omega_0 \int_0^t \mathrm{d}t \\
\ln r\Big|_l^r &= -\omega_0\, t \\
\ln\left(\frac{r}{l}\right) &= -\omega_0\, t \\
\frac{r}{l} &= e^{-\omega_0 t} \\
r &= l\, e^{-\omega_0 t}
\end{aligned}$$

Let $r = l/2$, we have $\dfrac{1}{2} = e^{-\omega_0 t}$, which gives $t = \dfrac{1}{\omega_0}\ln 2$.

(c) Consider again the relation, $r = l\, e^{-\omega_0 t}$. When the particle reaches the next end of the tube, $r \to 0$, it gives $t \to \infty$. Thus, the particle can never reach the point O.

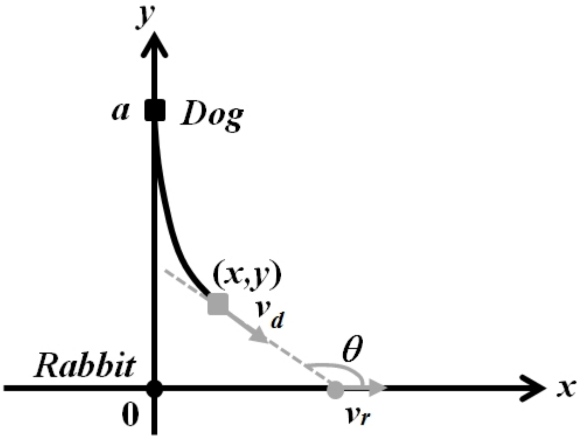
■

Example 5.8 Using the Cartesian coordinate system, the initial positions of a dog and a rabbit are recorded as $(0, a)$ and $(0, 0)$ respectively in a vast field. When the dog sees the rabbit, it starts to pursue the rabbit and its motion always directs to the rabbit. Given that the rabbit keeps running along the positive x-axis with a constant speed v_r and the dog's speed is a constant v_d. Show that the locus of the dog satisfies

$$y\frac{\mathrm{d}^2x}{\mathrm{d}y^2} = k\sqrt{1 + (\frac{\mathrm{d}x}{\mathrm{d}y})^2}\ ,$$

where $k = v_r/v_d$.

Solution



From the figure,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x - v_r t} \tag{5.2}$$

$$-\int_a^y \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y = v_d t \tag{5.3}$$

From equation (5.2), we obtain $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{x - v_r t}{y}$, which gives

$$x - y\,\frac{\mathrm{d}x}{\mathrm{d}y} = v_r t \tag{5.4}$$

From equations (5.3) and (5.4), we have

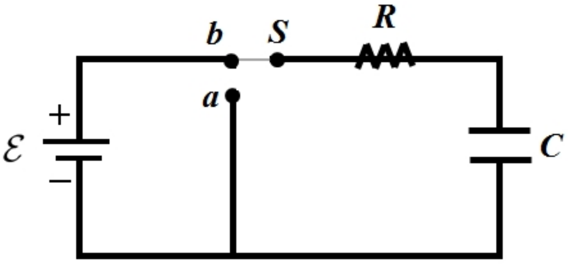
$$-v_r \int_a^y \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y = v_d \left(x - y\,\frac{\mathrm{d}x}{\mathrm{d}y}\right) \tag{5.5}$$

Differentiating both sides of the equation with respect to y

$$\begin{aligned}
-v_r \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} &= v_d \left(\frac{\mathrm{d}x}{\mathrm{d}y} - y\,\frac{\mathrm{d}^2x}{\mathrm{d}y^2} - \frac{\mathrm{d}x}{\mathrm{d}y}\right) \\
\frac{v_r}{v_d} \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} &= y\,\frac{\mathrm{d}^2x}{\mathrm{d}y^2} \\
k \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} &= y\,\frac{\mathrm{d}^2x}{\mathrm{d}y^2} \qquad \text{where } k = v_r/v_d.
\end{aligned}$$

■

Example 5.9 At time $t = 0$, the switch S in the circuit is connected to b such that the capacitor C is charged from empty. The circuit has a resistor R and a battery $\mathcal{E}$ in series with the capacitor. Find the amount of charge Q stored in the capacitor at time t. Find also the current I at time t.



Solution

The Kirchhoff's loop rule gives

$$\mathcal{E} - IR - \frac{Q}{C} = 0 \tag{5.6}$$

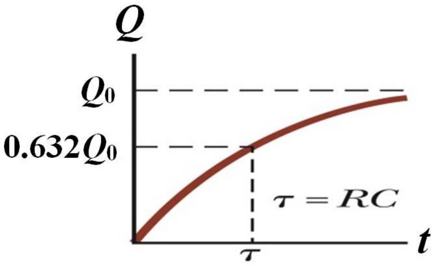
Knowing that $I = \mathrm{d}Q/\mathrm{d}t$

$$\begin{aligned} \mathcal{E} - R\frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{Q}{C} &= 0 \\ \frac{\mathrm{d}Q}{\mathrm{d}t} &= \frac{\mathcal{E}C - Q}{RC} \end{aligned}$$

Separating the variables, we have

$$\begin{aligned} \frac{\mathrm{d}Q}{\mathcal{E}C - Q} &= \frac{\mathrm{d}t}{RC} \\ \int_0^Q \frac{\mathrm{d}Q}{\mathcal{E}C - Q} &= \int_0^t \frac{\mathrm{d}t}{RC} \\ -\ln(\mathcal{E}C - Q)\Big|_0^Q &= \frac{t}{RC} \\ \ln\left(\frac{\mathcal{E}C - Q}{\mathcal{E}C}\right) &= -\frac{t}{RC} \\ \frac{\mathcal{E}C - Q}{\mathcal{E}C} &= e^{-\frac{t}{RC}} \end{aligned}$$

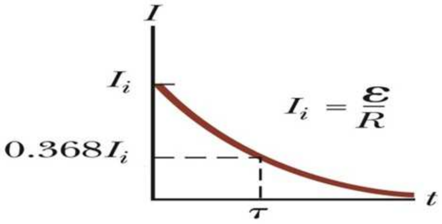
Hence, we have $Q = \mathcal{E}C\,(1 - e^{-\frac{t}{RC}}) = Q_0\,(1 - e^{-\frac{t}{RC}})$, where $Q_0 = \mathcal{E}C$ is the final charge in the capacitor when it is fully charged.



The current in the circuit at time t is $I = \mathrm{d}Q/\mathrm{d}t$. Thus,

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} = \mathcal{E}C\,\frac{\mathrm{d}}{\mathrm{d}t}\,(1 - e^{-\frac{t}{RC}}) = \frac{\mathcal{E}}{R}\,e^{-\frac{t}{RC}} = I_0\,e^{-\frac{t}{RC}}$$

where $I_0 = \mathcal{E}/R$.



Alternatively, we can differentiate both sides of equation (5.6) with respect to t,

$$\begin{aligned} -R\,\frac{\mathrm{d}I}{\mathrm{d}t} - \frac{1}{C}\,\frac{\mathrm{d}Q}{\mathrm{d}t} &= 0 \\ R\,\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{I}{C} &= 0 \end{aligned}$$

Rearranging the equation, we have

$$\begin{aligned} \frac{\mathrm{d}I}{I} &= -\frac{1}{RC}\,\mathrm{d}t \\ \int_{I_0}^{I} \frac{\mathrm{d}I}{I} &= -\frac{1}{RC}\int_0^t \mathrm{d}t \\ \ln\left(\frac{I}{I_0}\right) &= -\frac{t}{RC} \\ \frac{I}{I_0} &= e^{-\frac{t}{RC}} \end{aligned}$$

Hence, $I = I_0\,e^{-\frac{t}{RC}}$, where $I_0 = \mathcal{E}/R$. ■

Exercise 5.3 The switch in example 5.9 is removed from b after the capacitor is fully charged. Then the switch is connected to a for discharging. Set the instant as $t = 0$, what is the total charge in the capacitor at time t.

5.2 First-order Differential Equations in Standard Form

If the first order differential equation has the standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\,y = Q(x)\,,$$

we should first calculate the integrating factor $\mu(x) = e^{\int P(x)\,\mathrm{d}x}$. Multiply both sides of the equation by $\mu(x)$ and, we have

$$\mu(x)\,\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\,\mu(x)y = \mu(x)\,Q(x)\,,$$

Notice that the left-hand side is just $\frac{\mathrm{d}}{\mathrm{d}x}(\mu(x)\,y)$. Let check it!

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu(x)\,y) &= \mu(x)\,\frac{\mathrm{d}y}{\mathrm{d}x} + y\,\frac{\mathrm{d}}{\mathrm{d}x}\,(\mu(x)) \\
&= \mu(x)\,\frac{\mathrm{d}y}{\mathrm{d}x} + y\,\frac{\mathrm{d}}{\mathrm{d}x}\,\left(e^{\int P(x)\,\mathrm{d}x}\right) \\
&= \mu(x)\,\frac{\mathrm{d}y}{\mathrm{d}x} + y\,\left(e^{\int P(x)\,\mathrm{d}x}\right)\,\frac{\mathrm{d}}{\mathrm{d}x}\,\left(\int P(x)\,\mathrm{d}x\right) \\
&= \mu(x)\,\frac{\mathrm{d}y}{\mathrm{d}x} + y\,\mu(x)\,P(x)
\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu(x)\,y) = \mu(x)\,Q(x)$$

Integrate the last equation with respect to x and solve for y.

$$y = \frac{1}{\mu(x)}\int \mu(x)\,Q(x)\,\mathrm{d}x$$

Example 5.10 Find the general solutions of

(a) $x\,\frac{\mathrm{d}y}{\mathrm{d}x} - ky = x^2$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} + y\,\tan x = \sec x$

Solution

(a) The ODE $x\,\frac{\mathrm{d}y}{\mathrm{d}x} - ky = x^2$ becomes $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{-k}{x}\right)\,y = x$. The integrating facor is $\exp\left(\int -\frac{k\,\mathrm{d}x}{x}\right) = x^{-k}$. Multiply the factor to both sides of the equation. The ODE can be reduced to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(x^{-k}y) &= x^{1-k} \\
x^{-k}y &= \int x^{1-k}\,\mathrm{d}x \\
x^{-k}y &= \frac{x^{2-k}}{2-k} + C \\
y &= \frac{x^2}{2-k} + Cx^k
\end{aligned}$$

(b) The integrating factor is $\exp(\int \tan x \, \mathrm{d}x) = \exp(\ln \sec x) = \sec x$. Multiply the factor to both sides of the ODE $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y \tan x = \sec x$, the equation can be reduced to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(y \sec x) &= \sec^2 x \\ y \sec x &= \int \sec^2 x \, \mathrm{d}x \\ y \sec x &= \tan x + C \\ y &= \sin x + C \cos x \end{aligned}$$

■

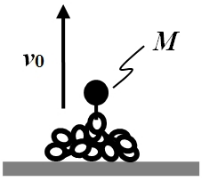
Exercise 5.4 Solve the equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x+y}{2x}$ by the following methods.

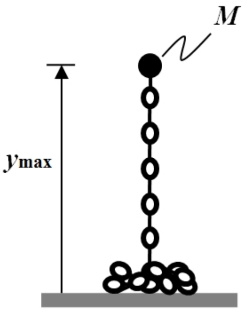
(a) Use the substitution $z = y/x$ and then the separation of variables.

(b) Use the integrating factor

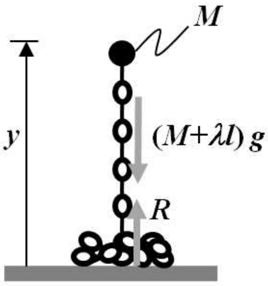
Exercise 5.5 Solve the equation $x \dfrac{\mathrm{d}y}{\mathrm{d}x} - 2y = x^3 \cos 4x$.

Example 5.11 A pile of chain of mass density λ is at rest on a smooth table. One of its end is attached to a particle of mass M. The particle is projected up with a vertical speed v_0. Find the maximum height that the particle can raise above the table.





Solution



Let l be the total length of the chain and take upward direction as positive. Consider the net force acting on the entire chain when the particle is at a height y above the table.

$$F_{\text{net}} = -(M + \lambda l)\, g + R\,, \qquad \text{where } R \text{ is the reaction force on the chain}$$

As the length of the moving part is y, the reaction force R equals to the weight of the resting part of the chain, i.e. $R = (l - y)\,\lambda g$ when $t > 0$. Thus, $F_{\text{net}} = -(M + \lambda y)\, g$. From Newton's second law of motion, we know that $F_{\text{net}} = \dfrac{\mathrm{d}p}{\mathrm{d}t}$. Hence, we can write

$$-(M + \lambda y)\, g = \frac{\mathrm{d}p}{\mathrm{d}t}$$

Note that the resting part has zero momentum, the total momentum of the chain equal to the momentum of the moving part of the chain and is equal to $(M + \lambda y)\, v$.

$$-(M + \lambda y)\, g = \frac{\mathrm{d}}{\mathrm{d}t}[(M + \lambda y)\, v] \tag{5.7}$$

Use the fact that $\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\mathrm{d}v}{\mathrm{d}y} \cdot \dfrac{\mathrm{d}y}{\mathrm{d}t} = v\,\dfrac{\mathrm{d}v}{\mathrm{d}y}$, we have

$$\begin{aligned}
-(M + \lambda y)\, g &= (M + \lambda y)\, v\,\frac{\mathrm{d}v}{\mathrm{d}y} + \lambda\, v^2 \\
-(M + \lambda y)\, g &= \left(\frac{M + \lambda y}{2}\right) \frac{\mathrm{d}v^2}{\mathrm{d}y} + \lambda\, v^2
\end{aligned}$$

$$\frac{\mathrm{d}v^2}{\mathrm{d}y} + \left(\frac{2\lambda}{M + \lambda y}\right) v^2 = -2g \tag{5.8}$$

This is the standard form of the first order ODE and the integrating factor is

$$\exp\left(\int \frac{2\lambda}{M + \lambda y}\,\mathrm{d}y\right) = \exp\left[2\,\ln(M + \lambda y)\right] = \exp\left[\ln(M + \lambda y)^2\right] = (M + \lambda y)^2$$

Equation (5.8) can be reduced to

$$\frac{\mathrm{d}}{\mathrm{d}y}[v^2\,(M + \lambda y)^2] = -2g\,(M + \lambda y)^2$$

Integrating both sides with respect to y

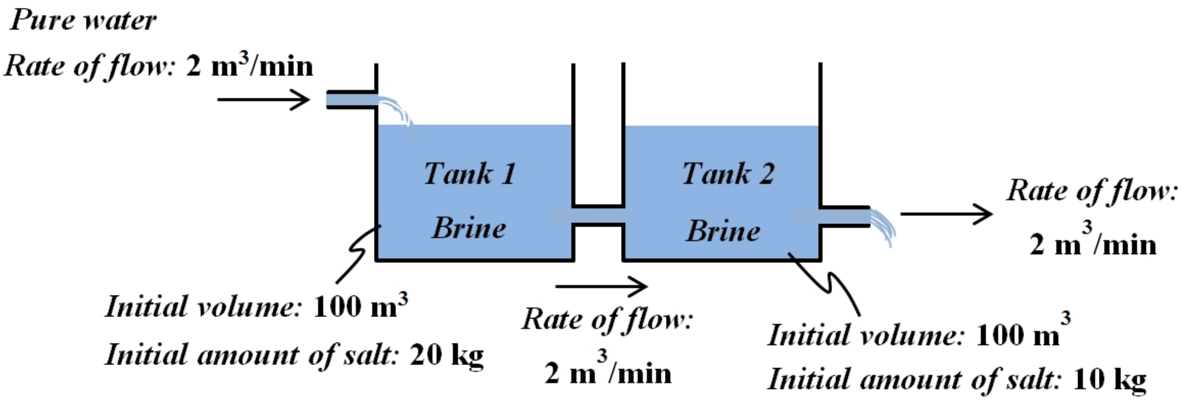
$$\begin{aligned}
v^2\,(M + \lambda y)^2 \Big|_{v=v_0,\ y=0}^{v=0,\ y=y_{\max}} &= -2g \int_0^{y_{\max}} (M + \lambda y)^2\,\mathrm{d}y \\
-v_0^2 M^2 &= -\frac{2g}{\lambda}\,\frac{(M + \lambda y)^3}{3}\Big|_0^{y_{\max}}
\end{aligned}$$

$$
\begin{aligned}
-v_0^2 M^2 &= -\frac{2g}{3\lambda}\left[(M+\lambda y_{\max})^3 - M^3\right] \\
y_{\max} &= \frac{1}{\lambda}\left[\left(\frac{3\lambda v_0^2 M^2}{2g} + M^3\right)^{1/3} - M\right] \\
y_{\max} &= \frac{M}{\lambda}\left[\left(\frac{3\lambda v_0^2}{2Mg} + 1\right)^{1/3} - 1\right]
\end{aligned}
$$

■

Example 5.12 Two tanks each initially contains 100 m^3 of brine solution, with 20 kg of salt in the first tank and 10 kg of salt in the second. At $t = 0$, pure water begins to flow into the first tank at the rate of 2 m^3/min. The stirred mixture runs into the second tank at the rate of 2 m^3/min and the stirred mixture in the second tank runs out at the same rate. Find the amount of salt in each tank (a) after 20 min, (b) after a long time.

Solution



Let x_1 and x_2 be the amount of salt (in kg) in the first and second tank at time t respectively. The volume of solution in each tank is unchanged,

$$
\begin{cases}
\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = -\dfrac{2}{100}x_1 \\[2ex]
\dfrac{\mathrm{d}x_2}{\mathrm{d}t} = \dfrac{2}{100}x_1 - \dfrac{2}{100}x_2
\end{cases}
\tag{5.9}
$$

From the first equation of equation set (5.9),

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{x_1} &= -\frac{2}{100}\,\mathrm{d}t \\
\int_{20}^{x_1}\frac{\mathrm{d}x_1}{x_1} &= -\frac{1}{50}\int_0^t \mathrm{d}t
\end{aligned}
$$

$$\begin{aligned}
\ln x_1 \Big|_{20}^{x_1} &= -\frac{1}{50}\,t \\
\ln\left(\frac{x_1}{20}\right) &= -\frac{1}{50}\,t \\
x_1 &= 20\,e^{-\frac{t}{50}}
\end{aligned}$$

Substituting the above result into the second equation of equation set (5.9), we have

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{2}{5}\,e^{-\frac{t}{50}} - \frac{1}{50}\,x_2$$

It becomes the standard form of the first order differential equation, i.e.

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} + \frac{1}{50}\,x_2 = \frac{2}{5}\,e^{-\frac{t}{50}} \tag{5.10}$$

where $P(t) = \dfrac{1}{50}$ and $Q(t) = \dfrac{2}{5}\,e^{-\frac{t}{50}}$. The integrating factor is $e^{\int \frac{\mathrm{d}t}{50}} = e^{\frac{t}{50}}$. Thus, equation (5.10) becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(x_2\,e^{\frac{t}{50}}\right) &= \frac{2}{5}\,e^{-\frac{t}{50}}\,e^{\frac{t}{50}} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(x_2\,e^{\frac{t}{50}}\right) &= \frac{2}{5} \\
\int_{t=0,\ x_2=10}^{t,\ x_2} \mathrm{d}\left(x_2\,e^{\frac{t}{50}}\right) &= \frac{2}{5}\int_0^t \mathrm{d}t \\
x_2\,e^{\frac{t}{50}} \Big|_{t=0,\ x_2=10}^{t,\ x_2} &= \frac{2}{5}\,t \\
x_2\,e^{\frac{t}{50}} - 10 &= \frac{2}{5}\,t \\
x_2 &= \left(\frac{2}{5}\,t + 10\right)\,e^{-\frac{t}{50}}
\end{aligned}$$

(a) When $t = 20\,\mathrm{min}$, $x_1 = 20\,e^{-\frac{2}{5}}$ and $x_2 = 18\,e^{-\frac{2}{5}}$.

(b) If $t \to \infty$, then $x_1 \to 0$ and $x_2 \to 0$. ■

Exercise 5.6 The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x\cos y + \sin 2y}$$

is regarded as nonlinear in $y(x)$. Rewrite the equation such that it becomes a linear differential equation in $x(y)$ and has the form $\dfrac{\mathrm{d}x}{\mathrm{d}y} + P(y)x = Q(y)$. Find P and Q, and hence solve the equation.

5.3 Second-order Homogeneous Differential Equations with Constant Coefficients

The second order homogeneous and linear differential equation has the form

$$a\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\,\frac{\mathrm{d}y}{\mathrm{d}x}+c\,y=0 \tag{5.11}$$

where a, b, and c are constants. If $e^{\lambda x}$ satisfies the equation, then

$$(a\,\lambda^2+b\,\lambda+c)\,e^{\lambda x}=0$$

If equation (5.11) has non-trivial solution, then $e^{\lambda x}\neq 0$ for all x. So

$$a\,\lambda^2+b\,\lambda+c=0 \tag{5.12}$$

We call equation (5.12) as the characteristic equation or the auxiliary equation of the differential equation. Let λ_1 and λ_2 be the roots of it, then $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are the solutions of equation (5.11). We should note that there are two independent solutions for a second order homogeneous differential equation. If λ_1, λ_2 are real and $\lambda_1\neq\lambda_2$, the general solution of equation (5.11) is

$$y=A\,e^{\lambda_1 x}+B\,e^{\lambda_2 x}$$

If $\lambda_1=\lambda_2=\lambda$, then

$$y=(A+B\,x)\,e^{\lambda x}$$

If the roots are complex, we can set $\lambda_1=p+q\,i$ and $\lambda_2=p-q\,i$. The general solution of equation (5.11) becomes

$$y=e^{px}\,(A\,e^{qix}+B\,e^{-qix})$$

which can be rewritten as

$$y=e^{px}\,(c_1\,\cos qx+c_2\,\sin qx)$$

The last expression comes from the Euler's formula: $e^{i\theta}=\cos\theta+i\,\sin\theta$.

Example 5.13 Solve the following second order homongeneous differential equations.

(a) $y''+4\,y'=0$ (b) $y''-7\,y'+6\,y=0$ (c) $y''+9\,y=0$

(d) $y''-4\,y'+5\,y=0$ (e) $y''-6\,y'+9\,y=0$

Solution

(a) The characteristic equation of $y''+4\,y'=0$ is $\lambda\,(\lambda+4)=0$, then $\lambda=0$ or -4. Thus, e^{0x} and e^{-4x} are solutions of y. Hence,

$$y = A\,e^{0x} + B\,e^{-4x} = A + B\,e^{-4x}$$

where A and B are constants.
(b) The characteristic equation of $y'' - 7\,y' + 6\,y = 0$ is $\lambda^2 - 7\,\lambda + 6 = 0$, so $(\lambda - 6)\,(\lambda - 1) = 0$ gives $\lambda = 6$ or 1. Hence,

$$y = A\,e^{6x} + B\,e^{x}$$

where A and B are constants.
(c) The characteristic equation of $y'' + 9\,y = 0$ is $\lambda^2 + 9 = 0$, so $\lambda = \pm 3\,i$. Hence, we can write down the solutions of the differential equation:

$$y = A\,e^{i3x} + B\,e^{-i3x}, \qquad \text{where } A \text{ and } B \text{ are constants.}$$

Alternatively, we can write

$$y = C\cos 3x + D\sin 3x, \qquad \text{where } C \text{ and } D \text{ are constants.}$$

(d) The characteristic equation of $y'' - 4\,y' + 5\,y = 0$ is $\lambda^2 - 4\,\lambda + 5 = 0$, so $\lambda = 2 \pm i$. Hence, we can write down the solutions of the differential equation:

$$y = e^{2x}\,(A\,e^{ix} + B\,e^{-ix}), \qquad \text{where } A \text{ and } B \text{ are constants.}$$

Alternatively, we can write

$$y = e^{2x}\,(C\cos x + D\sin x), \qquad \text{where } C \text{ and } D \text{ are constants.}$$

(e) The characteristic equation of $y'' - 6\,y' + 9\,y = 0$ is $(\lambda - 3)^2 = 0$, so $\lambda = 3$ (equal roots). The multiplicity is 2. Thus, other than e^{3x} is a solution of y, there is another independent solution, i.e. $x\,e^{3x}$. To conclude,

$$y = A\,e^{3x} + B\,x\,e^{3x}, \qquad \text{where } A \text{ and } B \text{ are constants.}$$

One can verify that $x\,e^{3x}$ is a solution of y. ■

Example 5.14 Solve the system of equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x + y \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = 4\,x + y \end{cases} \tag{5.13}$$

Solution

Differentiating the first equation of equation set (5.13) with respect to t, we have

$$\begin{aligned}
\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t} \\
\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \frac{\mathrm{d}x}{\mathrm{d}t} + (4\,x + y) \\
\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \frac{\mathrm{d}x}{\mathrm{d}t} + \left(4\,x + \frac{\mathrm{d}x}{\mathrm{d}t} - x\right)
\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\,\frac{\mathrm{d}x}{\mathrm{d}t} - 3\,x = 0$$

The auxiliary equation is $\lambda^2 - 2\,\lambda - 3 = 0$ which gives $\lambda = 3,\ -1$. So e^{3t} and e^{-t} are the solutions of the differential equation. Hence, we obtain the general solution

$$x = A\,e^{3t} + B\,e^{-t}$$

From the above equation, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3\,A\,e^{3t} - B\,e^{-t}$$

where A and B are arbitrary constants to be determined by boundary conditions. From the second equation of equation set (5.13), we have

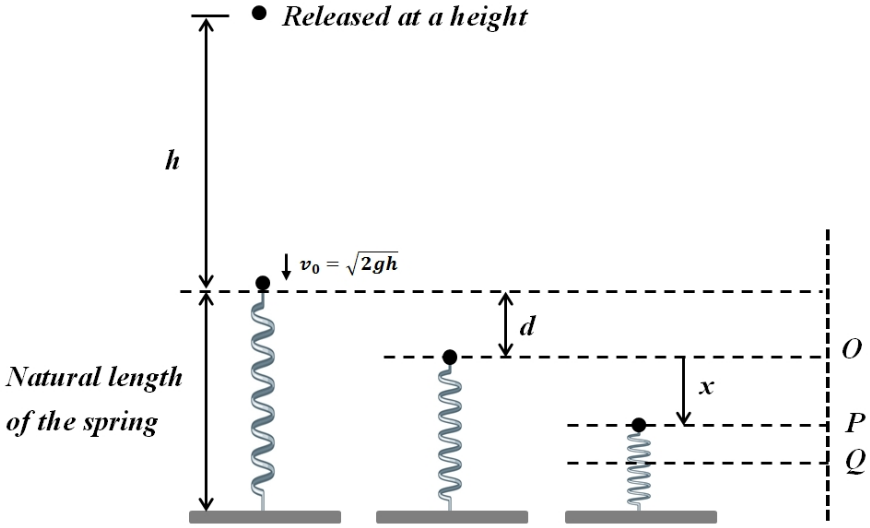
$$\begin{aligned}
y &= \frac{\mathrm{d}x}{\mathrm{d}t} - x \\
&= 3\,A\,e^{3t} - B\,e^{-t} - \left(A\,e^{3t} + B\,e^{-t}\right) \\
&= 2\,A\,e^{3t} - 2\,B\,e^{-t}
\end{aligned}$$

■

Example 5.15 A light spring is fixed at its lower end with its axis vertical. A particle of mass m is dropped on the spring from a height h. If the particle is placed at rest on the upper end of the spring, the spring would compress by a distance d. Show that the particle will be shot off on the rebound after the remaining on the spring for a time

$$2\sqrt{\frac{d}{g}}\left(\pi - \tan^{-1}\sqrt{\frac{2h}{d}}\right)$$

Solution



The velocity of the particle is $v_0 = \sqrt{2gh}$ when it first meets the spring. Let k be the force constant of the spring, then $mg = kd$.

Next, we let O be the equilibrium position of the particle and we take downward as positive in the following calculations. Suppose that the particle is located at point P at time t, where P has a displacement x from O. The lowest point that the particle can reach is Q. Newton's second law of motion gives

$$m\ddot{x} = mg - k\,(d + x)$$

It becomes

$$\ddot{x} = -\omega^2\, x \qquad\qquad (\text{where } \omega^2 = \frac{k}{m})$$

After solving, we have $x = A\cos\omega t + B\sin\omega t$, where A and B are arbitrary constants to be determined. Knowing that at $t = 0$, $x = -d$ and $\dot{x} = v_0 = \sqrt{2gh}$, then we have $A = -d$ and $B = \dfrac{\sqrt{2gh}}{\omega} = \sqrt{2hd}$.

Hence, we can write

$$x = -d\,\cos\omega t + \sqrt{2hd}\,\sin\omega t$$

The particle will be shot off when $x = -d$, so $-d = -d\,\cos\omega t + \sqrt{2hd}\,\sin\omega t$. Then

$$\begin{aligned} -(1 - \cos\omega t) &= \sqrt{\frac{2h}{d}}\,\sin\omega t \\ -2\,\sin^2\frac{\omega t}{2} &= 2\sqrt{\frac{2h}{d}}\,\sin\frac{\omega t}{2}\,\cos\frac{\omega t}{2} \\ \sin\frac{\omega t}{2}\left(\sin\frac{\omega t}{2} + \sqrt{\frac{2h}{d}}\,\cos\frac{\omega t}{2}\right) &= 0 \end{aligned}$$

If $\sin\dfrac{\omega t}{2} = 0$, then we have $t = 0$, and $t = \dfrac{2\,n\pi}{\omega}$, where n is a positive integer. However, $t = 0$ is not the required answer and $t = \dfrac{2\,\pi}{\omega}$ equals to the period of oscillation and it is longer than the required time for such rebound. So, the answer comes from

$$\begin{aligned}
\sin\frac{\omega t}{2} + \sqrt{\frac{2h}{d}}\cos\frac{\omega t}{2} &= 0 \\
\tan\frac{\omega t}{2} &= -\sqrt{\frac{2h}{d}}
\end{aligned}$$

$$\begin{aligned}
\tan\left(\pi - \frac{\omega t}{2}\right) &= \sqrt{\frac{2h}{d}} \\
\frac{\omega t}{2} &= \pi - \tan^{-1}\sqrt{\frac{2h}{d}} \\
t &= 2\sqrt{\frac{d}{g}}\left(\pi - \tan^{-1}\sqrt{\frac{2h}{d}}\right)
\end{aligned}$$

■

Exercise 5.7 Show that the substitution $u = \cos x$ simplifies the differential equation

$$\sin x\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \cos x\,\frac{\mathrm{d}y}{\mathrm{d}x} - y\,\sin^3 x = 0 \qquad (*)$$

to $\dfrac{\mathrm{d}^2y}{\mathrm{d}u^2} - y = 0$. Hence, solve (*)

5.4 Method of Undetermined Coefficients

In this section, we proceed to study the solutions of a nonhomogeneous differential equation which has the form

$$y'' + a\,y' + b\,y = r(x) \tag{5.14}$$

where a and b are constants. The general solution is $y = y_c + y_p$, where y_c is the complementary solution of equation (5.14) and it satisfies the homogeneous equation $y'' + a\,y' + b\,y = 0$. The particular solution y_p satisfies equation (5.14) specifically. The following table shows the choice for y_p.

Term in $r(x)$	**Choice for** y_p
$k\,e^{\gamma x}$	$c\,e^{\gamma x}$
$k\,x^n \quad (n = 0,\ 1,\ 2,\ \cdots)$	$c_n\,x^n + c_{n-1}\,x^{n-1} + \cdots + c_1\,x + c_0$
$k\,x^n e^{\gamma x} \quad (n = 0,\ 1,\ 2,\ \cdots)$	$(c_n\,x^n + c_{n-1}\,x^{n-1} + \cdots + c_1\,x + c_0)\,e^{\gamma x}$
$k\,\cos\omega x$	$p\,\cos\omega x + q\,\sin\omega x$
$k\,\sin\omega x$	$p\,\cos\omega x + q\,\sin\omega x$
$k\,x^n\,\cos\omega x \quad (n = 0,\ 1,\ 2,\ \cdots)$	$(p_n\,x^n + p_{n-1}\,x^{n-1} + \cdots + p_1\,x + p_0)\,\cos\omega x +$ $(q_n\,x^n + q_{n-1}\,x^{n-1} + \cdots + q_1\,x + q_0)\,\sin\omega x$
$k\,x^n\,\sin\omega x \quad (n = 0,\ 1,\ 2,\ \cdots)$	$(p_n\,x^n + p_{n-1}\,x^{n-1} + \cdots + p_1\,x + p_0)\,\cos\omega x +$ $(q_n\,x^n + q_{n-1}\,x^{n-1} + \cdots + q_1\,x + q_0)\,\sin\omega x$
$k\,e^{\alpha x}\,\cos\omega x$	$e^{\alpha x}\,(p\,\cos\omega x + q\,\sin\omega x)$
$k\,e^{\alpha x}\,\sin\omega x$	$e^{\alpha x}\,(p\,\cos\omega x + q\,\sin\omega x)$

Modification rule: If a term in your choice for y_p is a solution of the homogeneous equation, then multiply y_p by x^k, where k is the smallest positive integer such that no term of $x^k\,y_p$ is a solution of the homogeneous equation.

Example 5.16 Solve the equation $y'' + y' - 2\,y = x^2$.

Solution

Consider the differential equation $y'' + y' - 2\,y = 0$ first. The characteristic equation of it is $\lambda^2 + \lambda - 2 = 0$, then $\lambda = 1, -2$. So the complementary solution of the given equation is

$$y_c = A\,e^{x} + B\,e^{-2\,x}$$

For the particular solution, we let $y_p = C\,x^2 + D\,x + E$, then $y_p' = 2\,Cx + D$ and $y_p'' = 2\,C$. Thus,

$$\begin{aligned} y_p'' + y_p' - 2\,y_p &= x^2 \\ 2\,C + 2\,Cx + D - 2\,(C\,x^2 + D\,x + E) &= x^2 \\ -2\,C\,x^2 + 2\,(C - D)\,x + (2\,C + D - 2\,E) &= x^2 \end{aligned}$$

Comparing the coefficients of the polynomial on both sides, we have

$$\left\{ \begin{aligned} -2\,C &= 1 \\ C - D &= 0 \\ 2\,C + D - 2\,E &= 0 \end{aligned} \right.$$

Then $C = -\dfrac{1}{2}$, $D = -\dfrac{1}{2}$, $E = -\dfrac{3}{4}$ and the particular solution is $y_p = -\dfrac{1}{2}\,x^2 - \dfrac{1}{2}\,x - \dfrac{3}{4}$. Hence, the general solution is

$$y = y_c + y_p = A\,e^{x} + B\,e^{-2\,x} - \frac{1}{2}\,x^2 - \frac{1}{2}\,x - \frac{3}{4}$$

Example 5.17 Solve the equation $y'' - y' = 2 + e^{x}$.

Solution

Consider the differential equation $y'' - y' = 0$ first. The characteristic equation of it is $\lambda\,(\lambda - 1) = 0$, then $\lambda = 0,\ 1$. So the complementary solution of the given equation is

$$y_c = A + B\,e^{x}$$

From the nonhomogenous terms, i.e. 2 and e^{x}, one should realized that the trial particular solution repeats the complementary solution and the multiplcity is $k = 1$. Then we let the particular solution $y_p = C\,x + D\,x\,e^{x}$, then $y_p' = C + D\,e^{x} + D\,x\,e^{x}$ and $y_p'' = 2\,D\,e^{x} + D\,x\,e^{x}$. Thus,

$$\begin{aligned} y_p'' - y_p' &= 2 + e^{x} \\ 2\,D\,e^{x} + D\,x\,e^{x} - (C + D\,e^{x} + D\,x\,e^{x}) &= 2 + e^{x} \\ -C + D\,e^{x} &= 2 + e^{x} \end{aligned}$$

Comparing the coefficients of the polynomial on both sides, we have $C = -2$, $D = 1$, and the particular solution is $y_p = -2\,x + x\,e^{x}$. Hence, the general solution is

$$y = y_c + y_p = A - 2\,x + B\,e^{x} + x\,e^{x}$$

Exercise 5.8 Solve the equation $y'' + y = x$.

Exercise 5.9 Solve $y'' - 2\,y' - 3\,y = 2\,e^{x} - 10\,\sin x$, where $y(0) = 2$ and $y'(0) = 4$.

Exercise 5.10 Solve the equation $y'' + y = x\,\sin x$.

Chapter 6

Partial Differentiation

6.1 Partial Derivatives

Consider a function f of three variables x, y and z,

$$f(x, y, z)$$

If y and z are held constant and only x is allowed to vary, the partial derivative with respect to x is denoted by $\dfrac{\partial f}{\partial x}$ or f_x and is defined as the limit

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

The total differential of f is given by

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z\,, \tag{6.1}$$

where $\mathrm{d}f$ represents the change in f due to the infinitesimal changes in x, y and z respectively.

The proof of Equation (6.1) is shown below.

Consider the difference of the functional values at two adjacent points $P(x, y, z)$ and $Q(x + \Delta x, y + \Delta y, z + \Delta z)$,

$$\begin{aligned}
\Delta f &= f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z) \\
&= [f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y + \Delta y, z + \Delta z)] \\
&\qquad +[f(x, y + \Delta y, z + \Delta z) - f(x, y, z + \Delta z)] \\
&\qquad\quad +[f(x, y, z + \Delta z) - f(x, y, z)]
\end{aligned}$$

$$
\begin{aligned}
&= \left[\left(\frac{f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x, y+\Delta y, z+\Delta z)}{\Delta x}\right)\Delta x\right] \\
&\quad + \left[\left(\frac{f(x, y+\Delta y, z+\Delta z) - f(x, y, z+\Delta z)}{\Delta y}\right)\Delta y\right] \\
&\quad\quad + \left[\left(\frac{f(x, y, z+\Delta z) - f(x, y, z)}{\Delta z}\right)\Delta z\right]
\end{aligned}
$$

In the limiting case as $\Delta x \to 0$, $\Delta y \to 0$, and $\Delta z \to 0$, we have $\Delta x \cong \mathrm{d}x$, $\Delta y \cong \mathrm{d}y$, and $\Delta z \cong \mathrm{d}z$, and $\Delta f \cong \mathrm{d}f$. The above equation can be reduced into

$$
\begin{aligned}
\mathrm{d}f &= \left[\lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x, y+\Delta y, z+\Delta z)}{\Delta x}\right)\Delta x\right] \\
&\quad + \left[\lim_{\Delta y \to 0}\left(\frac{f(x, y+\Delta y, z+\Delta z) - f(x, y, z+\Delta z)}{\Delta y}\right)\Delta y\right] \\
&\quad\quad + \left[\lim_{\Delta z \to 0}\left(\frac{f(x, y, z+\Delta z) - f(x, y, z)}{\Delta z}\right)\Delta z\right]
\end{aligned}
$$

Therefore, we have $\mathrm{d}f = \dfrac{\partial f}{\partial x}\,\mathrm{d}x + \dfrac{\partial f}{\partial y}\,\mathrm{d}y + \dfrac{\partial f}{\partial z}\,\mathrm{d}z$.

Remarks:

1. If f is a three-variable function $f(x, y, z)$, where x, y and z are functions of t, i.e. $x(t)$, $y(t)$ and $z(t)$, then the derivative of f with respect to t is the total differentiation of f and is given by the chain rule

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

2. Suppose that y and z are functions of x, then f becomes a function of x. So, we have

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x}$$

   Example 6.3 gives an illustration of this.

3. Suppose that x and y are independent but that z is a function of x and y, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ and f becomes a function of x and y. So, we have

$$\left(\frac{\partial f}{\partial x}\right)_y = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}$$

   Note that f in the LHS is purely a function of x and y because z has been substituted into it. The subscript indicates y being held constant. In the RHS, f corresponds to a function of x, y and z. Similarly, we have

$$\left(\frac{\partial f}{\partial y}\right)_x = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}$$

if y is allowed to vary but x is fixed. As a reminder, the subscript labels the quantity being fixed. Read examples 6.4.

Example 6.1 If $f(x,y) = e^{xy}$, find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$, and $\dfrac{\partial^2 f}{\partial x\,\partial y}$.

Solution

Consider the function $f(x,y) = e^{xy}$, we obtain

$$\frac{\partial f}{\partial x} = y\,e^{xy} \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2} = y^2\,e^{xy}$$

$$\frac{\partial f}{\partial y} = x\,e^{xy} \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2} = x^2\,e^{xy}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x\,e^{xy}) = xy\,e^{xy} + e^{xy} = e^{xy}\,(xy+1)$$

Remark

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(y\,e^{xy}) = xy\,e^{xy} + e^{xy} = e^{xy}\,(xy+1)$$

■

Example 6.2 Find $\dfrac{\partial}{\partial x}\displaystyle\int_{x+y}^{xy} \frac{\sin t}{t}\,\mathrm{d}t$.

Solution

$$\begin{aligned}
\frac{\partial}{\partial x}\int_{x+y}^{xy} \frac{\sin t}{t}\,\mathrm{d}t &= \frac{\partial}{\partial x}\int_{x+y}^{c} \frac{\sin t}{t}\,\mathrm{d}t + \frac{\partial}{\partial x}\int_{c}^{xy} \frac{\sin t}{t}\,\mathrm{d}t \\
&= -\frac{\partial}{\partial x}\int_{c}^{x+y} \frac{\sin t}{t}\,\mathrm{d}t + \frac{\partial}{\partial x}\int_{c}^{xy} \frac{\sin t}{t}\,\mathrm{d}t \\
&= -\left(\frac{\partial}{\partial(x+y)}\int_{c}^{x+y} \frac{\sin t}{t}\,\mathrm{d}t\right)\frac{\partial(x+y)}{\partial x} + \left(\frac{\partial}{\partial(xy)}\int_{c}^{xy} \frac{\sin t}{t}\,\mathrm{d}t\right)\frac{\partial(xy)}{\partial x} \\
&= -\left(\frac{\sin(x+y)}{x+y}\right)(1) + \left(\frac{\sin xy}{xy}\right)(y) \\
&= -\frac{\sin(x+y)}{x+y} + \frac{\sin xy}{x}
\end{aligned}$$

■

Example 6.3 Given that $z = e^x \sin y$, find $\dfrac{\partial z}{\partial x}$. If $y = e^x$, find $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

Solution

Consider $z = e^x \sin y$, we have $\dfrac{\partial z}{\partial x} = e^x \sin y$.

Now, we put $y = e^x$, the z can be rewritten as a single variable function, where $z = e^x \sin e^x$, and

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (\sin e^x)\, e^x + e^x\,(\cos e^x)\, e^x = e^x \sin e^x + e^{2x} \cos e^x$$

Alternatively, we apply the chain rule

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= e^x \sin y + (e^x \cos y)\, e^x \\ &= e^x \sin y + e^{2x} \cos y \\ &= e^x \sin e^x + e^{2x} \cos e^x \end{aligned}$$

■

Exercise 6.1 Given a two-variable function $f(x, y) = \dfrac{x^2}{x+y}$, find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$.

Example 6.4 Let $f(x, y, z) = x^2 + xy^2 + z$, where $z(x, y) = 3x - y$. Obtain $\dfrac{\partial f}{\partial x}$ and $\left(\dfrac{\partial f}{\partial x}\right)_y$.

Solution

Obviously, $\dfrac{\partial f}{\partial x} = 2x + y^2$, where $\dfrac{\partial f}{\partial x}$ is the differentiation of $f(x, y, z)$ with respect to x while taking y and z as constants.

Let's illustrate the meaning of $\left(\dfrac{\partial f}{\partial x}\right)_y$:

The function $f(x, y, z)$ can be rewritten as $f(x, y, 3x - y) = x^2 + xy^2 + 3x - y$ after substituting $z = 3x - y$. Then f is no longer a three-variable function but instead it becomes a two-variable function (i.e. x and y only). Hence, $\left(\dfrac{\partial f}{\partial x}\right)_y = 2x + y^2 + 3$, where y is being fixed in the derivative.

One can obtain $\left(\dfrac{\partial f}{\partial x}\right)_y$ by using chain rule. Knowing that $\dfrac{\partial f}{\partial x} = 2x + y^2$, $\dfrac{\partial f}{\partial z} = 1$ and $\dfrac{\partial z}{\partial x} = 3$, we have

$$\begin{aligned} \left(\frac{\partial f}{\partial x}\right)_y &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \\ &= 2x + y^2 + (1)(3) \\ &= 2x + y^2 + 3 \end{aligned}$$

■

Example 6.5 Given that $z(x, y) = x^2 + y^2$ and $u = xy$. Find the following quantities. (a) $\dfrac{\partial z}{\partial x}$, (b) $\left(\dfrac{\partial z}{\partial x}\right)_u$, (c) $\left(\dfrac{\partial z}{\partial u}\right)_x$, and (d) $\left(\dfrac{\partial u}{\partial z}\right)_x$.

Solution

(a) It is important to note that $\dfrac{\partial z}{\partial x} = \left(\dfrac{\partial z}{\partial x}\right)_y$ without saying because z is a function of x and y. Differentiate both sides of $z(x, y) = x^2 + y^2$ with respect to x while keeping y unchanged, we have

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{\partial z}{\partial x} = 2\,x + 0 = 2\,x$$

(b) Refer to $z(x, y) = x^2 + y^2$ again, if we differentiate both sides with respect to x while u is fixed, we have

$$\left(\frac{\partial z}{\partial x}\right)_u = 2\,x\,(1) + 2y\left(\frac{\partial y}{\partial x}\right)_u$$

Knowing that $u = xy$, if we differentiate both sides with respect to x while u is fixed, we have $0 = x\left(\dfrac{\partial y}{\partial x}\right)_u + y\,(1)$ which gives $\left(\dfrac{\partial y}{\partial x}\right)_u = -\dfrac{y}{x}$. Hence, we have

$$\left(\frac{\partial z}{\partial x}\right)_u = 2\,x + 2\,y\left(-\frac{y}{x}\right) = \frac{2\left(y^2 - x^2\right)}{x}$$

(c) Differentiating both sides of $z(x, y) = x^2 + y^2$ with respect to u while keeping x unchanged, we have

$$\left(\frac{\partial z}{\partial u}\right)_x = 0 + 2y\left(\frac{\partial y}{\partial u}\right)_x$$

Knowing that $u = xy$, if we differentiate both sides with respect to u while x is fixed, then we have $1 = x\left(\dfrac{\partial y}{\partial u}\right)_x + y\,(0)$ which gives $\left(\dfrac{\partial y}{\partial u}\right)_x = \dfrac{1}{x}$. Hence, we have

$$\left(\frac{\partial z}{\partial u}\right)_x = 2\,y\left(\frac{1}{x}\right) = \frac{2\,y}{x}$$

(d) Now, we proceed to differentiate both sides of $u = xy$ with respect to z while x is fixed, so we have $\left(\dfrac{\partial u}{\partial z}\right)_x = x\left(\dfrac{\partial y}{\partial z}\right)_x + y\,(0)$ which gives $\left(\dfrac{\partial u}{\partial z}\right)_x = x\left(\dfrac{\partial y}{\partial z}\right)_x$.

On the other hand, if we differentiate both sides of $z(x, y) = x^2 + y^2$ with respect to z while x is fixed, then we obtain $1 = 0 + 2y\left(\dfrac{\partial y}{\partial z}\right)_x$, which gives $\left(\dfrac{\partial y}{\partial z}\right)_x = \dfrac{1}{2\,y}$

Hence, we have

$$\left(\frac{\partial u}{\partial z}\right)_x = x\left(\frac{1}{2\,y}\right) = \frac{x}{2\,y}$$

Remark: The results of (c) and (d) are reciporal to each other. The reason is straight forwarded as when x is fixed, u is simply a single valued function of z (i.e. z is a function of y) or z is a single valued function of u (i.e. z is a function of y). Obviously, the reciporal rule of differentiation for single valued function is valid. ■

Exercise 6.2 Given $x^2 + y^2 + z^2 = 6$, and $w^3 + z^3 = 5\,xy + 12$, find the following partial derivatives at the point $(x, y, z, w) = (1, -2, 1, 1)$.
(a) $\left(\dfrac{\partial z}{\partial x}\right)_y$, (b) $\left(\dfrac{\partial z}{\partial x}\right)_w$, (c) $\left(\dfrac{\partial w}{\partial x}\right)_z$, and (d) $\left(\dfrac{\partial x}{\partial w}\right)_z$.

Example 6.6 The Cartesian coordinates relate the polar coordinates in the following forms: $x = r\,\cos\theta$ and $y = r\,\sin\theta$, find

(a) $\dfrac{\partial x}{\partial r}, \dfrac{\partial x}{\partial \theta}$ in terms of r and θ, hence find $\dfrac{\partial^2 x}{\partial r^2}, \dfrac{\partial^2 x}{\partial \theta^2}$.

(b) $\dfrac{\partial r}{\partial x}$ and $\dfrac{\partial \theta}{\partial x}$ in terms of r and θ, hence $\dfrac{\partial^2 r}{\partial x^2}, \dfrac{\partial^2 \theta}{\partial x^2}$.

Solution

(a) $\dfrac{\partial x}{\partial r} = \cos\theta$ and $\dfrac{\partial x}{\partial \theta} = -r\sin\theta$.

Hence, $\dfrac{\partial^2 x}{\partial r^2} = 0$ and $\dfrac{\partial^2 x}{\partial \theta^2} = -r\cos\theta$.

(b) Consider the expression $x = r\,\cos\theta$ and differentiate both sides with respect to x, we obtain

$$1 = r\,\frac{\partial}{\partial x}\,(\cos\theta) + \cos\theta\,\frac{\partial r}{\partial x}$$

which implies

$$1 = -r\,\sin\theta\,\frac{\partial \theta}{\partial x} + \cos\theta\,\frac{\partial r}{\partial x} \tag{6.2}$$

Consider the expression $y = r\,\sin\theta$ and differentiate both sides with respect to x, we obtain

$$0 = r\,\frac{\partial}{\partial x}\,(\sin\theta) + \sin\theta\,\frac{\partial r}{\partial x}$$

which implies

$$0 = r\,\cos\theta\,\frac{\partial \theta}{\partial x} + \sin\theta\,\frac{\partial r}{\partial x} \tag{6.3}$$

Solve equations 6.2 and 6.3, we obtain $\dfrac{\partial r}{\partial x} = \cos\theta$ and $\dfrac{\partial \theta}{\partial x} = -\dfrac{\sin\theta}{r}$.

Alternatively, one may consider the expression $r^2 = x^2 + y^2$. Differentiating both sides with respect to x, we obtain

$$\begin{aligned} 2r\,\frac{\partial r}{\partial x} &= 2x \\ \frac{\partial r}{\partial x} &= \frac{x}{r} = \frac{r\,\cos\theta}{r} = \cos\theta \end{aligned}$$

Next, we know that $\tan\theta = \dfrac{y}{x}$. If we differentiate both sides with respect to x, we obtain $\sec^2\theta\left(\dfrac{\partial\theta}{\partial x}\right) = \dfrac{x\left(\frac{\partial y}{\partial x}\right) - y\left(\frac{\partial x}{\partial x}\right)}{x^2} = \dfrac{x\,(0) - y\,(1)}{x^2} = -\dfrac{y}{x^2}$. So, we obtain $\dfrac{\partial\theta}{\partial x} = -\cos^2\theta\left(\dfrac{y}{x^2}\right) = -\dfrac{\sin\theta}{r}$.

Hence, we have

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}(\cos\theta) = -\sin\theta\,\frac{\partial\theta}{\partial x} = -\sin\theta\left(-\frac{\sin\theta}{r}\right) = \frac{\sin^2\theta}{r}$$

Repeat similar process, we have $\dfrac{\partial^2\theta}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial\theta}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(-\dfrac{\sin\theta}{r}\right)$, then

$$\begin{aligned} \frac{\partial^2\theta}{\partial x^2} &= \frac{-\left\{r\,\dfrac{\partial}{\partial x}(\sin\theta) - \sin\theta\,\dfrac{\partial r}{\partial x}\right\}}{r^2} \\ &= \frac{-\left\{r\,\cos\theta\left(-\dfrac{\sin\theta}{r}\right) - \sin\theta\,\cos\theta\right\}}{r^2} \\ &= \frac{2\sin\theta\cos\theta}{r^2} \\ &= \frac{\sin 2\theta}{r^2} \end{aligned}$$

Remark: The reciprocal rule resulted from the differentiation of single-variable function may not apply to partial differentiation. In this example, we observe that $\dfrac{\partial x}{\partial r}\,\dfrac{\partial r}{\partial x} \neq 1$. In fact, $\dfrac{\partial x}{\partial r} = \left(\dfrac{\partial x}{\partial r}\right)_\theta$ and $\dfrac{\partial r}{\partial x} = \left(\dfrac{\partial r}{\partial x}\right)_y$, where the parameters being fixed are not the same in each derivative. Similarly, $\dfrac{\partial x}{\partial\theta}\,\dfrac{\partial\theta}{\partial x} \neq 1$, where $\dfrac{\partial x}{\partial\theta} = \left(\dfrac{\partial x}{\partial\theta}\right)_r$ and $\dfrac{\partial\theta}{\partial x} = \left(\dfrac{\partial\theta}{\partial x}\right)_y$. The reciprocal rule is valid for multivariable functions if the parameter(s) being fixed is(are) the same. ■

Example 6.7 If $f = f(x, y)$ and $x = r\,\cos\theta$, $y = r\,\sin\theta$, then prove that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

Solution

Given $f = f(x, y)$, $x = r\,\cos\theta$, $y = r\,\sin\theta$, we have

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \tag{6.4}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \tag{6.5}$$

where $\dfrac{\partial x}{\partial r} = \cos\theta$, $\dfrac{\partial x}{\partial \theta} = -r\,\sin\theta$, $\dfrac{\partial y}{\partial r} = \sin\theta$, and $\dfrac{\partial y}{\partial \theta} = r\,\cos\theta$. So

$$\frac{\partial f}{\partial r} = \cos\theta\,\frac{\partial f}{\partial x} + \sin\theta\,\frac{\partial f}{\partial y} \tag{6.6}$$

$$\frac{\partial f}{\partial \theta} = -r\,\sin\theta\,\frac{\partial f}{\partial x} + r\,\cos\theta\,\frac{\partial f}{\partial y} \tag{6.7}$$

Then

$$\left(\frac{\partial f}{\partial r}\right)^2 = \cos^2\theta\left(\frac{\partial f}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial f}{\partial y}\right)^2 + 2\,\sin\theta\,\cos\theta\,\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \tag{6.8}$$

$$\left(\frac{\partial f}{\partial \theta}\right)^2 = r^2\left\{\sin^2\theta\left(\frac{\partial f}{\partial x}\right)^2 + \cos^2\theta\left(\frac{\partial f}{\partial y}\right)^2 - 2\,\sin\theta\,\cos\theta\,\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right\} \tag{6.9}$$

Therefore,

$$\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

■

Exercise 6.3 Given $x^2y - y^2v = 1$, and $x + y = uv$, find (a) $\left(\dfrac{\partial x}{\partial u}\right)_v$ and (b) $\left(\dfrac{\partial x}{\partial u}\right)_y$.

Example 6.8 Given $f(x, y, z) = 0$ and $g(x, y, z) = 0$, find a formula for $\mathrm{d}y/\mathrm{d}x$.

Solution

Knowing that the total differentials $\mathrm{d}f = \mathrm{d}g = 0$, we can write

$$\begin{cases} \dfrac{\partial f}{\partial x}\,\mathrm{d}x + \dfrac{\partial f}{\partial y}\,\mathrm{d}y + \dfrac{\partial f}{\partial z}\,\mathrm{d}z &= 0 \\ \dfrac{\partial g}{\partial x}\,\mathrm{d}x + \dfrac{\partial g}{\partial y}\,\mathrm{d}y + \dfrac{\partial g}{\partial z}\,\mathrm{d}z &= 0 \end{cases}$$

Eliminating dz from the above equations, we have

$$\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}-\frac{\partial g}{\partial x}\frac{\partial f}{\partial z}\right)\mathrm{d}x+\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z}-\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\right)\mathrm{d}y=0$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(\frac{\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial z}-\dfrac{\partial g}{\partial x}\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial z}-\dfrac{\partial g}{\partial y}\dfrac{\partial f}{\partial z}}\right)=-\frac{f_xg_z-g_xf_z}{f_yg_z-g_yf_z}$$

■

Example 6.9 Evaluate $\{(4.1)^2+(1.9)^3+2.7\}^{1/3}$ to the first order approximation without using calculator.

Solution

Let $w=(x^2+y^3+z)^{1/3}$. The total differential of w is

$$\mathrm{d}w=\frac{\partial w}{\partial x}\mathrm{d}x+\frac{\partial w}{\partial y}\mathrm{d}y+\frac{\partial w}{\partial z}\mathrm{d}z\,,$$

where $\dfrac{\partial w}{\partial x}=\dfrac{1}{3}(x^2+y^3+z)^{-2/3}(2\,x)$, $\dfrac{\partial w}{\partial y}=\dfrac{1}{3}(x^2+y^3+z)^{-2/3}(3\,y^2)$ and $\dfrac{\partial w}{\partial z}=\dfrac{1}{3}(x^2+y^3+z)^{-2/3}$.

Hence, $\mathrm{d}w=\dfrac{1}{3}(x^2+y^3+z)^{-2/3}(2\,x\,\mathrm{d}x+3\,y^2\,\mathrm{d}y+\mathrm{d}z)$.

Let $x=4$, $y=2$, and $z=3$. Set $\mathrm{d}x=0.1$, $\mathrm{d}y=-0.1$ and $\mathrm{d}z=-0.3$, then we have $w=(4^2+2^3+3)^{1/3}=(27)^{1/3}=3$, and thus

$$\begin{aligned}\mathrm{d}w &= \frac{1}{3}(4^2+2^3+3)^{-2/3}[2\,(4)\,(0.1)+3\,(2^2)\,(-0.1)-0.3]\\ &= \frac{1}{3}\cdot(27)^{-2/3}(0.8-1.2-0.3)\\ &= -\frac{1}{3}\cdot\frac{1}{9}\cdot\frac{7}{10}=-\frac{7}{270}=-0.0259\end{aligned}$$

Therefore $\left\{(4.1)^2+(1.9)^3+2.7\right\}^{1/3}=3-0.0259=2.9741$. ■

Exercise 6.4

(a) Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, where $f(x,y)=y^x$ and $y>0$. [Hint: Consider $y^x=e^{\ln y^x}$.]

(b) Hence, evaluate $(0.97)^{1.05}$ to the first order approximation.

Example 6.10 Functions x and y are described by $\begin{cases} x = e^{u}\cos v \\ y = e^{u}\sin v \end{cases}$.

(a) Write down $\mathrm{d}x$ and $\mathrm{d}y$. Hence, show that

$$\begin{cases} \mathrm{d}u = e^{-u}\cos v\,\mathrm{d}x + e^{-u}\sin v\,\mathrm{d}y \\ \mathrm{d}v = -e^{-u}\sin v\,\mathrm{d}x + e^{-u}\cos v\,\mathrm{d}y \end{cases}$$

(b) If $z = uv$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ by using the results in (a).

Solution

(a) Differentiate x and y with respect to u and v, we have

$$\begin{cases} \dfrac{\partial x}{\partial u} = e^{u}\cos v \\[2ex] \dfrac{\partial x}{\partial v} = -e^{u}\sin v \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial y}{\partial u} = e^{u}\sin v \\[2ex] \dfrac{\partial y}{\partial v} = e^{u}\cos v \end{cases}$$

The total differential of x is $\mathrm{d}x = \dfrac{\partial x}{\partial u}\,\mathrm{d}u + \dfrac{\partial x}{\partial v}\,\mathrm{d}v$. We can write

$$\mathrm{d}x = e^{u}\cos v\,\mathrm{d}u - e^{u}\sin v\,\mathrm{d}v \tag{6.10}$$

The total differential of y is $\mathrm{d}y = \dfrac{\partial y}{\partial u}\,\mathrm{d}u + \dfrac{\partial y}{\partial v}\,\mathrm{d}v$. We can write

$$\mathrm{d}y = e^{u}\sin v\,\mathrm{d}u + e^{u}\cos v\,\mathrm{d}v \tag{6.11}$$

Solving equations (6.10) and (6.11), we obtain

$$\begin{aligned} \mathrm{d}u &= e^{u}\cos v\,\mathrm{d}x + e^{-u}\sin v\,\mathrm{d}y \\ \mathrm{d}v &= -e^{-u}\sin v\,\mathrm{d}x + e^{-u}\cos v\,\mathrm{d}y \end{aligned}$$

(b) Since $z = uv$, we obtain $\mathrm{d}z = u\,\mathrm{d}v + v\,\mathrm{d}u$. Hence,

$$\mathrm{d}z = u\left(-e^{-u}\sin v\,\mathrm{d}x + e^{-u}\cos v\,\mathrm{d}y\right) + v\left(e^{u}\cos v\,\mathrm{d}x + e^{-u}\sin v\,\mathrm{d}y\right)$$

Arrange the equation, we have

$$\mathrm{d}z = \left(-ue^{-u}\sin v + ve^{-u}\cos v\right)\mathrm{d}x + \left(ue^{-u}\cos v + ve^{-u}\sin v\right)\mathrm{d}y$$

But, $\mathrm{d}z = \dfrac{\partial z}{\partial x}\,\mathrm{d}x + \dfrac{\partial z}{\partial y}\,\mathrm{d}y$.

Hence, We can write

$$\begin{cases} \dfrac{\partial z}{\partial x} = -ue^{-u}\sin v + ve^{-u}\cos v \\ \\ \dfrac{\partial z}{\partial x} = ue^{-u}\cos v + ve^{-u}\sin v \end{cases}$$

■

Example 6.11 If $f(x,y,z)=0$ show that $\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial x}\dfrac{\partial x}{\partial z} = -1$

Solution

If $f(x,y,z)=0$, we can write $z(x,y)$ and $\left(\dfrac{\partial f}{\partial y}\right)_x = \dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial y} = 0$. Hence, we obtain

$$\frac{\partial z}{\partial y} = -\frac{\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial z}} \qquad \text{where} \quad \frac{\partial z}{\partial y} = \left(\frac{\partial z}{\partial y}\right)_x$$

Similarly, we can write $y(z,x)$ and $\left(\dfrac{\partial f}{\partial x}\right)_z = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial x} = 0$. Hence, we obtain

$$\frac{\partial y}{\partial x} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} \qquad \text{where} \quad \frac{\partial y}{\partial x} = \left(\frac{\partial y}{\partial x}\right)_z$$

Similarly, we can write $x(y,z)$ and $\left(\dfrac{\partial f}{\partial z}\right)_y = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial z} + \dfrac{\partial f}{\partial z} = 0$. Hence, we obtain

$$\frac{\partial x}{\partial z} = -\frac{\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial x}} \qquad \text{where} \quad \frac{\partial x}{\partial z} = \left(\frac{\partial x}{\partial z}\right)_y$$

Therefore, we obtain the product of the partial derivatives $\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial x}\dfrac{\partial x}{\partial z} = -1$. This formula is widely used in thermodynamics. ■

Exercise 6.5 Given a function $u(x,y,z)$, where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, where r, θ and ϕ are spherical coordinates.

(a) Show that $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\left(x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}\right)$.

(b) State ONE feature of u if we have $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$.

Exercise 6.6 If $f(x,y,z)=0$ show that $\dfrac{\partial z}{\partial x}=\left(\dfrac{\partial x}{\partial z}\right)^{-1}$, where the derivatives are given when the value of y is fixed.

Exercise 6.7 In the Cartesian coordinate system, the infinitesimal change in the position vector $\vec{R}=x\,\hat{\imath}+y\,\hat{\jmath}$ is represented by $\mathrm{d}\vec{R}=\mathrm{d}x\,\hat{\imath}+\mathrm{d}y\,\hat{\jmath}$. Express $\mathrm{d}\vec{R}$ in the polar coordinate system.

Exercise 6.8 The Laplacian of a scalar function $u(x,y)$ is defined by

$$\nabla^2 u=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u\ ,$$

where (x,y) are Cartesian coordinates. Show that the Laplacian of u can be rewritten as

$$\nabla^2 u=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)u\ ,$$

where (r,θ) are plane polar coordinates.

Hints: You can use the following expressions directly.

$$\frac{\partial r}{\partial x}=\cos\theta,\qquad \frac{\partial\theta}{\partial x}=-\frac{\sin\theta}{r},\qquad \frac{\partial r}{\partial y}=\sin\theta,\qquad \text{and}\qquad \frac{\partial\theta}{\partial y}=\frac{\cos\theta}{r}.$$

6.2 Geometrical Meaning of Partial Derivatives

Figure 6.1 shows the intersection between the curved surface $z=f(x,y)$ and the vertical plane $x=x_0$. It is a curve given by $z(x_0,y)$. The slope of this curve at (x_0,y_0) is given by the partial derivative of z with respect to y, i.e. $\left.\dfrac{\partial z}{\partial y}\right|_{(x_0,y_0)}$. While doing the differentiation, only the values of y is allowed to vary but the value of x is always fixed at x_0. The derivative gives the rate of change of z along the positive y direction. Similarly, if we cut the curved surface by another vertical plane $y=y_0$, the intersecting curve is $z(x,y_0)$ and the slope of it at (x_0,y_0) is $\left.\dfrac{\partial z}{\partial x}\right|_{(x_0,y_0)}$. This quantity gives the rate of change of z along the positive x direction.

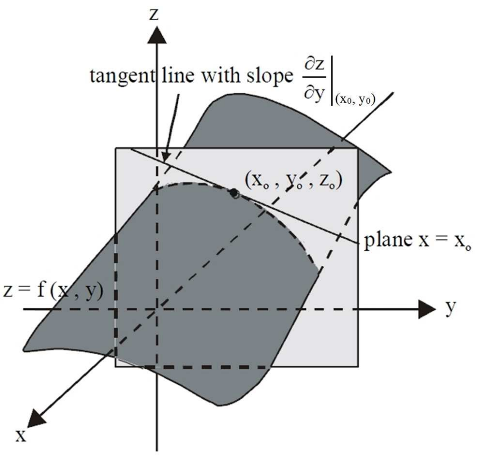


Figure 6.1: The meaning of partial derivative

6.3 Cartesian Coordinates

A Cartesian coordinate system has coordinates (x, y, z) as shown in figure 6.2. The position vector of a point P in the space is represented by

$$\vec{l} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}\,,$$

where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are "constant vectors". The directions and magnitudes of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ never change. The length of each of them is 1.

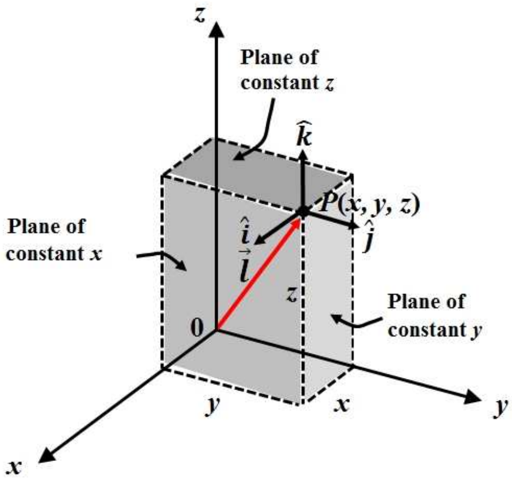


Figure 6.2: The Cartesian coordinate system

Then, the infinitesimal change of the position vector is

$$\mathrm{d}\vec{l} = \mathrm{d}x\,\hat{\imath} + \mathrm{d}y\,\hat{\jmath} + \mathrm{d}z\,\hat{k}$$

The answer is straight forward, but the idea behind it relates to partial differentiation. Let's reveal it! Recall that the total differential of a scalar function f is given by

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z$$

It is also true for vector function, so

$$\mathrm{d}\vec{l} = \frac{\partial \vec{l}}{\partial x}\,\mathrm{d}x + \frac{\partial \vec{l}}{\partial y}\,\mathrm{d}y + \frac{\partial \vec{l}}{\partial z}\,\mathrm{d}z \tag{6.12}$$

On the other hand, the position vector $\vec{l} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ gives

$$\begin{cases} \dfrac{\partial \vec{l}}{\partial x} = \hat{\imath} & \text{along } \hat{\imath} \\[2ex] \dfrac{\partial \vec{l}}{\partial y} = \hat{\jmath} & \text{along } \hat{\jmath} \\[2ex] \dfrac{\partial \vec{l}}{\partial z} = \hat{k} & \text{along } \hat{k} \end{cases}$$

Obviously, the magnitudes of them are $\left|\dfrac{\partial \vec{l}}{\partial x}\right| = 1$, $\left|\dfrac{\partial \vec{l}}{\partial y}\right| = 1$, and $\left|\dfrac{\partial \vec{l}}{\partial z}\right| = 1$, respectively. Equation 6.12 is equivalent to

$$\mathrm{d}\vec{l} = \left|\frac{\partial \vec{l}}{\partial x}\right|\,\mathrm{d}x\,\hat{\imath} + \left|\frac{\partial \vec{l}}{\partial y}\right|\,\mathrm{d}y\,\hat{\jmath} + \left|\frac{\partial \vec{l}}{\partial z}\right|\,\mathrm{d}z\,\hat{k}$$

Therefore, the vectorial representation of an infinitesimal change using the Cartesian coordinates is

$$\mathrm{d}\vec{l} = \mathrm{d}x\,\hat{\imath} + \mathrm{d}y\,\hat{\jmath} + \mathrm{d}z\,\hat{k}$$

The answer looks trivial, but the approach to obtain it is applicable to other coordinate systems.

6.4 Cylindrical Coordinates

A cylindrical coordinate system has coordinates (ρ, ϕ, z). It is an extension of the plane polar coordinates to include the z axis as shown in figure 6.3. We should note that the plane polar coordinates are written as (ρ, ϕ) instead of (r, θ). This is to avoid confusion with the spherical coordinates to be discussed in section 6.5. The polar coordinates ρ and ϕ are obtained by the transformations $x = \rho\cos\phi$ and $y = \rho\sin\phi$. Using Cartesian coordinates the position vector of P is expressed as $\vec{l} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$, then we can write

$$\vec{l} = \rho\,\cos\phi\,\hat{i} + \rho\,\sin\phi\,\hat{j} + z\,\hat{k}$$

One should also be aware of $\vec{l} = \rho\,\hat{\rho} + z\,\hat{k}$. Note that

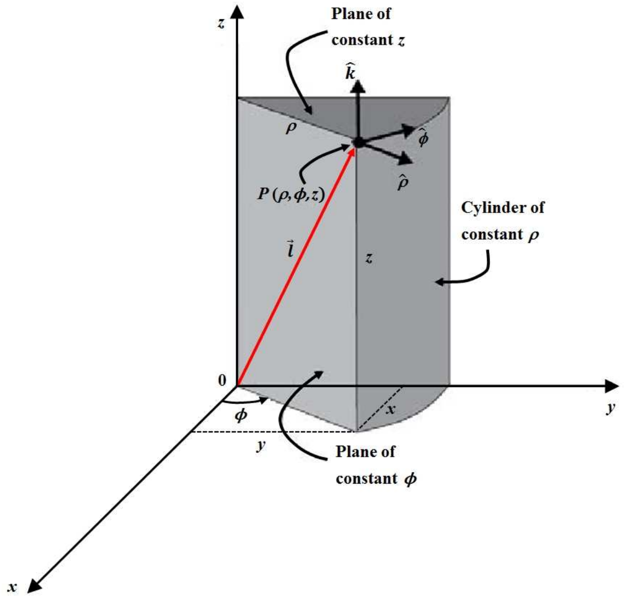


Figure 6.3: The cylindrical coordinate system

$$\begin{cases} \dfrac{\partial \vec{l}}{\partial \rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j} & \text{along } \hat{\rho} \\[2ex] \dfrac{\partial \vec{l}}{\partial \phi} = -\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j} & \text{along } \hat{\phi} \\[2ex] \dfrac{\partial \vec{l}}{\partial z} = \hat{k} & \text{along } \hat{k} \end{cases}$$

The magnitudes of them are

$$\begin{cases} \left|\dfrac{\partial \vec{l}}{\partial \rho}\right| = \sqrt{\cos^2\phi + \sin^2\phi} = 1 \\[2em] \left|\dfrac{\partial \vec{l}}{\partial \phi}\right| = \sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi} = \rho \\[2em] \left|\dfrac{\partial \vec{l}}{\partial z}\right| = 1 \end{cases}$$

Hence, we obtain the unit vectors

$$\begin{cases} \hat{\rho} = \dfrac{\partial \vec{l}}{\partial \rho} \Bigg/ \left|\dfrac{\partial \vec{l}}{\partial \rho}\right| = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath} \\[2em] \hat{\phi} = \dfrac{\partial \vec{l}}{\partial \phi} \Bigg/ \left|\dfrac{\partial \vec{l}}{\partial \phi}\right| = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath} \\[2em] \hat{k} = \dfrac{\partial \vec{l}}{\partial z} \Bigg/ \left|\dfrac{\partial \vec{l}}{\partial z}\right| = \hat{k} \end{cases}$$

One can check that $\hat{\rho}$, $\hat{\phi}$ and $\hat{k}$ are orthogonal to each other. $\hat{\rho}$ and $\hat{\phi}$ are unit vectors, but they are not constant vectors because both of them are functions of ϕ; their direction vary with ϕ. The infinitesimal change in the position vector is

$$\begin{aligned} \mathrm{d}\vec{l} &= \frac{\partial \vec{l}}{\partial \rho}\,\mathrm{d}\rho + \frac{\partial \vec{l}}{\partial \phi}\,\mathrm{d}\phi + \frac{\partial \vec{l}}{\partial z}\,\mathrm{d}z \\ \mathrm{d}\vec{l} &= \left|\frac{\partial \vec{l}}{\partial \rho}\right| \mathrm{d}\rho\,\hat{\rho} + \left|\frac{\partial \vec{l}}{\partial \phi}\right| \mathrm{d}\phi\,\hat{\phi} + \left|\frac{\partial \vec{l}}{\partial z}\right| \mathrm{d}z\,\hat{k} \end{aligned}$$

Therefore, the vectorial representation of a small change using the cylindrical coordinates is

$$\mathrm{d}\vec{l} = \mathrm{d}\rho\,\hat{\rho} + \rho\,\mathrm{d}\phi\,\hat{\phi} + \mathrm{d}z\,\hat{k}$$

6.5 Spherical Coordinates

A spherical coordinate system has coordinates (r, θ, ϕ), where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\,\pi$. One should not confuse the symbols, i.e. r and θ, adopted in polar coordinates because they represent differently in the two systems. Figure 6.4 shows the spherical coordinates as well as their associated unit vectors.

The transformations are $x = r\sin\theta\,\cos\phi$, $y = r\sin\theta\,\sin\phi$, and $z = r\cos\theta$. As the position vector of P in the Cartesian coordinates is $\vec{l} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$, then we can write

$$\vec{l} = r\sin\theta\,\cos\phi\,\hat{i} + r\sin\theta\,\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$

One should also be aware of $\vec{l} = r\hat{r}$ (some books read $\vec{r} = r\,\hat{r} = \vec{l}$). Note that

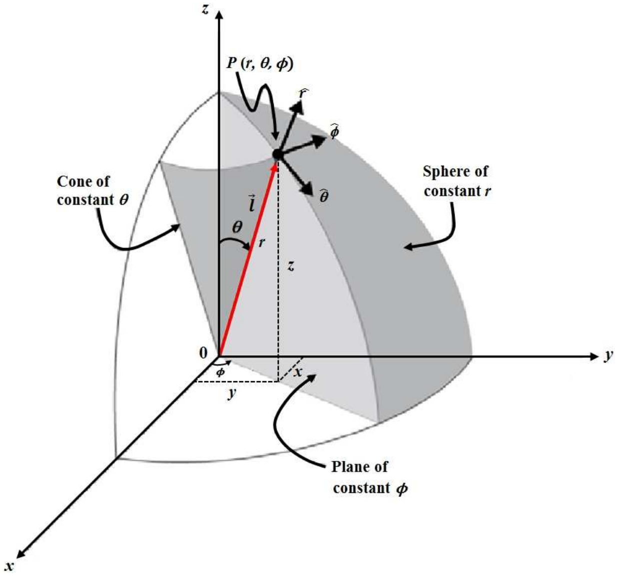


Figure 6.4: The spherical coordinate system

$$\begin{cases} \dfrac{\partial\vec{l}}{\partial r} = \sin\theta\,\cos\phi\,\hat{i} + \sin\theta\,\sin\phi\,\hat{j} + \cos\theta\,\hat{k} & \text{along } \hat{r} \\[2ex] \dfrac{\partial\vec{l}}{\partial\theta} = r\cos\theta\,\cos\phi\,\hat{i} + r\cos\theta\,\sin\phi\,\hat{j} - r\sin\theta\,\hat{k} & \text{along } \hat{\theta} \\[2ex] \dfrac{\partial\vec{l}}{\partial\phi} = -r\sin\theta\,\sin\phi\,\hat{i} + r\sin\theta\,\cos\phi\,\hat{j} & \text{along } \hat{\phi} \end{cases}$$

The magnitudes of them are

$$\begin{cases} \left|\dfrac{\partial \vec{l}}{\partial r}\right| = \sqrt{\sin^2\theta\,\cos^2\phi + \sin^2\theta\,\sin^2\phi + \cos^2\theta} = 1 \\[2ex] \left|\dfrac{\partial \vec{l}}{\partial \theta}\right| = \sqrt{r^2\cos^2\theta\,\cos^2\phi + r^2\cos^2\theta\,\sin^2\phi + r^2\sin^2\theta} = r \\[2ex] \left|\dfrac{\partial \vec{l}}{\partial \phi}\right| = \sqrt{r^2\sin^2\theta\,\sin^2\phi + r^2\sin^2\theta\cos^2\phi} = r\sin\theta \end{cases}$$

Hence, we obtain the unit vectors

$$\begin{cases} \hat{r} = \dfrac{\partial \vec{l}}{\partial r}\Bigg/\left|\dfrac{\partial \vec{l}}{\partial r}\right| = \sin\theta\,\cos\phi\,\hat{\imath} + \sin\theta\,\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k} \\[2ex] \hat{\theta} = \dfrac{\partial \vec{l}}{\partial \theta}\Bigg/\left|\dfrac{\partial \vec{l}}{\partial \theta}\right| = \cos\theta\,\cos\phi\,\hat{\imath} + \cos\theta\,\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k} \\[2ex] \hat{\phi} = \dfrac{\partial \vec{l}}{\partial \phi}\Bigg/\left|\dfrac{\partial \vec{l}}{\partial \phi}\right| = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath} \end{cases}$$

One can check that $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are orthogonal to each other. $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ are unit vectors but they are not constant vectors because $\hat{r}$ and $\hat{\theta}$ are functions of θ and ϕ and $\hat{\phi}$ is a function of ϕ. Their directions vary with the parameters. The infinitesimal change in the position vector is

$$\begin{aligned} \mathrm{d}\vec{l} &= \frac{\partial \vec{l}}{\partial r}\,\mathrm{d}r + \frac{\partial \vec{l}}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial \vec{l}}{\partial \phi}\,\mathrm{d}\phi \\ \mathrm{d}\vec{l} &= \left|\frac{\partial \vec{l}}{\partial r}\right|\,\mathrm{d}r\,\hat{r} + \left|\frac{\partial \vec{l}}{\partial \theta}\right|\,\mathrm{d}\theta\,\hat{\theta} + \left|\frac{\partial \vec{l}}{\partial \phi}\right|\,\mathrm{d}\phi\,\hat{\phi} \end{aligned}$$

Therefore, the vectorial representation of a small change using the spherical coordinates is

$$\mathrm{d}\vec{l} = \mathrm{d}r\,\hat{r} + r\,\mathrm{d}\theta\,\hat{\theta} + r\,\sin\theta\,\mathrm{d}\phi\,\hat{\phi}$$

Exercise 6.9 If $x = \dfrac{1}{2}\,(u^2 - v^2)$ and $y = uv$, where $-\infty < u < \infty$ and $v \geq 0$. Obtain $\hat{e}_u$ and $\hat{e}_v$ in terms of $\hat{\imath}$ and $\hat{\jmath}$, where $\hat{e}_u$ and $\hat{e}_v$ are unit vectors associated with u and v respectively. Show that $\hat{e}_u$ and $\hat{e}_v$ are orthogonal.

Example 6.12 A point charge q_0 is fixed at the origin, show that the work done required by the electric force to move a positive charge q from point $A(r_1, \theta_1, \phi_1)$ to point $B(r_2, \theta_2, \phi_2)$ is independent to the path choosen.

Solution

The electric force exerted on charge q by charge q_0 is $\vec{F} = k_e \dfrac{q_0 q}{r^2}\,\hat{r}$, where $\hat{r}$ is the unit vector in the radial direction and k_e is the Coulomb's constant. Recall that the representation of a small displacement in the spherical coordinate system is $\vec{\mathrm{d}l} = \mathrm{d}r\,\hat{r} + r\,\mathrm{d}\theta\,\hat{\theta} + r\sin\theta\,\mathrm{d}\phi\,\hat{\phi}$. Hence, the required work done is

$$\begin{aligned} W &= \int_{(r_1,\theta_1,\phi_1)}^{(r_2,\theta_2,\phi_2)} \vec{F}\cdot\mathrm{d}\vec{l} \\ &= \int_{(r_1,\theta_1,\phi_1)}^{(r_2,\theta_2,\phi_2)} k_e\,\frac{q_0 q}{r^2}\,\hat{r}\cdot\left(\mathrm{d}r\,\hat{r} + r\,\mathrm{d}\theta\,\hat{\theta} + r\sin\theta\,\mathrm{d}\phi\,\hat{\phi}\right) \\ &= \int_{(r_1,\theta_1,\phi_1)}^{(r_2,\theta_2,\phi_2)} k_e\,\frac{q_0 q}{r^2}\,\mathrm{d}r \\ &= -k_e\,\frac{q_0 q}{r}\Bigg|_{(r_1,\theta_1,\phi_1)}^{(r_2,\theta_2,\phi_2)} \\ &= -k_e q_0 q\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \end{aligned}$$

The answer indicates that the work done by the electric force in moving a charge q from A to B is independent to the path. ■

6.6 Del Operator

If $f(x, y, z)$ is a continuous scalar function for which its first derivatives exist, we can write the total differential

$$\begin{aligned} \mathrm{d}f &= \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z \\ &= \left(\frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath} + \frac{\partial f}{\partial z}\,\hat{k}\right)\cdot(\mathrm{d}x\,\hat{\imath} + \mathrm{d}y\,\hat{\jmath} + \mathrm{d}z\,\hat{k}) \end{aligned}$$

This is an important relation which gives the change of f whenever there is a small change of the domain point (x, y, z). Simply, we state

$$\mathrm{d}f = \nabla f\cdot\mathrm{d}\vec{l}, \qquad (6.13)$$

where $\nabla f = \dfrac{\partial f}{\partial x}\,\hat{\imath} + \dfrac{\partial f}{\partial y}\,\hat{\jmath} + \dfrac{\partial f}{\partial z}\,\hat{k}$ is a vector function which connects $\mathrm{d}f$ and $\mathrm{d}\vec{l}$. The small change of the domain point is represented by $\mathrm{d}\vec{l} = \mathrm{d}x\,\hat{\imath} + \mathrm{d}y\,\hat{\jmath} + \mathrm{d}z\,\hat{k}$. Sometimes, we call ∇ the "del" operator or "grad" operator. In Cartesian coordinates, we have

$$\nabla \equiv \hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z} \tag{6.14}$$

One can compare equation 6.13 with that obtained in a single variable scalar function $F(x)$. The total differential of F is $dF = F'(x)\,\mathrm{d}x$, where $F'(x)$ becomes the bridging function to connect $\mathrm{d}F$ and $\mathrm{d}x$.

If f is a function of two variables only, say $f(x, y)$, then the "del" operator in the 2-D Cartesian system is

$$\nabla \equiv \hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y}\,,$$

where the z component in equation 6.15 is ignored. Using the transformations in polar coordinates, we can rewrite $f(x, y)$ as $f(r\,\cos\theta, r\sin\theta)$, then

$$\mathrm{d}f = \frac{\partial f}{\partial r}\,\mathrm{d}r + \frac{\partial f}{\partial \theta}\,\mathrm{d}\theta \tag{6.15}$$

If we let $\nabla f = a\,\hat{e}_r + b\,\hat{e}_\theta$ and plug it into equation 6.13, we have

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\vec{l} = (a\,\hat{e}_r + b\,\hat{e}_\theta) \cdot (\mathrm{d}r\,\hat{e}_r + r\,\mathrm{d}\theta\,\hat{e}_\theta)$$

which gives

$$\mathrm{d}f = a\,\mathrm{d}r + br\,\mathrm{d}\theta \tag{6.16}$$

Equations 6.15 and 6.16 give

$$\frac{\partial f}{\partial r}\,\mathrm{d}r + \frac{\partial f}{\partial \theta}\,\mathrm{d}\theta = a\,\mathrm{d}r + br\,\mathrm{d}\theta \tag{6.17}$$

By comparing the coefficients of $\mathrm{d}r$ and $\mathrm{d}\theta$ on both sides of equation 6.17, we have

$$a = \frac{\partial f}{\partial r} \quad \text{and} \quad b = \frac{1}{r}\,\frac{\partial f}{\partial \theta}$$

Hence, we have

$$\nabla f = \frac{\partial f}{\partial r}\,\hat{e}_r + \frac{1}{r}\,\frac{\partial f}{\partial \theta}\,\hat{e}_\theta \tag{6.18}$$

Example 6.13 Show that the del operator for the cylindrical coordinate system is given by

$$\nabla = \hat{\rho}\,\frac{\partial}{\partial \rho} + \hat{\phi}\,\frac{1}{\rho}\,\frac{\partial}{\partial \phi} + \hat{k}\,\frac{\partial}{\partial z} \tag{6.19}$$

Solution: The infinitesimal change in a scalar function $f(\rho, \phi, z)$ is given by

$$\mathrm{d}f = \frac{\partial f}{\partial \rho}\,\mathrm{d}\rho + \frac{\partial f}{\partial \phi}\,\mathrm{d}\phi + \frac{\partial f}{\partial z}\,\mathrm{d}z$$

On the other hand, $\mathrm{d}f = \nabla f \cdot \mathrm{d}\vec{l}$ and $\mathrm{d}\vec{l} = \mathrm{d}\rho\,\hat{\rho} + \rho\,\mathrm{d}\phi\,\hat{\phi} + \mathrm{d}z\,\hat{k}$. Readers should refer to section 6.4 for the last relation. Now, let's write $\nabla f = c_1\,\hat{\rho} + c_2\,\hat{\phi} + c_3\,\hat{k}$, where c_1, c_2, and c_3 are unknowns that we are going to obtain later. Thus

$$\begin{aligned} \frac{\partial f}{\partial \rho}\,\mathrm{d}\rho + \frac{\partial f}{\partial \phi}\,\mathrm{d}\phi + \frac{\partial f}{\partial z}\,\mathrm{d}z &= (c_1\,\hat{r} + c_2\,\hat{\phi} + c_3\,\hat{k}) \cdot (\mathrm{d}\rho\,\hat{\rho} + \rho\,\mathrm{d}\phi\,\hat{\phi} + \mathrm{d}z\,\hat{k}) \\ &= c_1\,\mathrm{d}\rho + c_2\rho\,\mathrm{d}\phi + c_3\,\mathrm{d}z \end{aligned}$$

Then, we obtain $c_1 = \frac{\partial f}{\partial \rho}$, $c_2 = \frac{1}{\rho}\,\frac{\partial f}{\partial \phi}$ and $c_3 = \frac{\partial f}{\partial z}$.
Therefore, the del operator for the cylindrical coordinate system is

$$\nabla = \hat{\rho}\,\frac{\partial}{\partial \rho} + \hat{\phi}\,\frac{1}{\rho}\,\frac{\partial}{\partial \phi} + \hat{k}\,\frac{\partial}{\partial z}$$

■

Example 6.14 Obtain the del operator for the spherical coordinate system.

$$\nabla = \hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\,\frac{\partial}{\partial \theta} + \hat{\phi}\,\frac{1}{r\,\sin\theta}\,\frac{\partial}{\partial \phi} \tag{6.20}$$

Solution

The infinitesimal change in a scalar function $f(r, \theta, \phi)$ is given by

$$\mathrm{d}f = \frac{\partial f}{\partial r}\,\mathrm{d}r + \frac{\partial f}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial f}{\partial \phi}\,\mathrm{d}\phi$$

On the other hand, $\mathrm{d}f = \nabla f \cdot \mathrm{d}\vec{l}$ and $\mathrm{d}\vec{l} = \mathrm{d}r\,\hat{r} + r\,\mathrm{d}\theta\,\hat{\theta} + r\,\sin\theta\,\mathrm{d}\theta\,\hat{\phi}$. Readers should refer to section 6.5 for the last relation. Now, let's write $\nabla f = c_1\,\hat{r} + c_2\,\hat{\theta} + c_3\,\hat{\phi}$, where c_1, c_2, and c_3 are unknowns that we are going to obtain later. Thus

$$\begin{aligned} \frac{\partial f}{\partial r}\,\mathrm{d}r + \frac{\partial f}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial f}{\partial \phi}\,\mathrm{d}\phi &= (c_1\,\hat{r} + c_2\,\hat{\theta} + c_3\,\hat{\phi}) \cdot (\mathrm{d}r\,\hat{r} + r\,\mathrm{d}\theta\,\hat{\theta} + r\,\sin\theta\,\mathrm{d}\phi\,\hat{\phi}) \\ &= c_1\,\mathrm{d}r + c_2 r\,\mathrm{d}\theta + c_3\,r\,\sin\theta\,\mathrm{d}\phi \end{aligned}$$

Then, we obtain $c_1 = \frac{\partial f}{\partial r}$, $c_2 = \frac{1}{r}\,\frac{\partial f}{\partial \theta}$ and $c_3 = \frac{1}{r\,\sin\theta}\,\frac{\partial f}{\partial \phi}$.
Therefore, the del operator for the spherical coordinate system is

$$\nabla = \hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\,\frac{\partial}{\partial \theta} + \hat{\phi}\,\frac{1}{r\,\sin\theta}\,\frac{\partial}{\partial \phi}$$

■

6.6.1 Electric Field and Del Operator

Electric field is defined as the electric force acting on a unit positive test charge. It is a vector field associated to each point in the space. Then the work done by an electric field $\vec{E}$ to move this test charge by a displacement $\mathrm{d}\vec{l}$ is $\vec{E}\cdot\mathrm{d}\vec{l}$. The amount of work done relates to the change of electric potential of the test charge by

$$\begin{aligned} \mathrm{d}V &= -\vec{E}\cdot\mathrm{d}\vec{l} \\ &= -(E_x\,\hat{\imath}+E_y\,\hat{\jmath}+E_z\,\hat{k})\cdot(\mathrm{d}x\,\hat{\imath}+\mathrm{d}y\,\hat{\jmath}+\mathrm{d}z\,\hat{k}) \\ &= -(E_x\,\mathrm{d}x+E_y\,\mathrm{d}y+E_z\,\mathrm{d}z) \end{aligned}$$

However, the total differential of V gives

$$\mathrm{d}V=\frac{\partial V}{\partial x}\,\mathrm{d}x+\frac{\partial V}{\partial y}\,\mathrm{d}y+\frac{\partial V}{\partial z}\,\mathrm{d}z \tag{6.21}$$

Thus, we have

$$\begin{cases} E_x=-\dfrac{\partial V}{\partial x} \\[2ex] E_y=-\dfrac{\partial V}{\partial y} \\[2ex] E_z=-\dfrac{\partial V}{\partial z} \end{cases}$$

Therefore, we can express the electric field in terms of the partial derivatives of the electric potential, e.g.

$$\vec{E}=E_x\,\hat{\imath}+E_y\,\hat{\jmath}+E_z\,\hat{k}=-\left(\hat{\imath}\,\frac{\partial V}{\partial x}+\hat{\jmath}\,\frac{\partial V}{\partial y}+\hat{k}\,\frac{\partial V}{\partial z}\right)$$

Obviously,

$$\vec{E}=-\left(\hat{\imath}\,\frac{\partial}{\partial x}+\hat{\jmath}\,\frac{\partial}{\partial y}+\hat{k}\,\frac{\partial}{\partial z}\right)V$$

Hence, we have $\vec{E}=-\nabla V$, where $\nabla\equiv\hat{\imath}\,\frac{\partial}{\partial x}+\hat{\jmath}\,\frac{\partial}{\partial y}+\hat{k}\,\frac{\partial}{\partial z}$. Thus equation 6.21 can be rewritten as $\mathrm{d}V=\nabla V\cdot\mathrm{d}\vec{l}$, where ∇V links up $\mathrm{d}V$ and $\mathrm{d}\vec{l}$. One can always compare the case in single variable function. Let $y=f(x)$, then the total differential of y is $\mathrm{d}y=f'(x)\,\mathrm{d}x$, where $f'(x)$ links up $\mathrm{d}y$ and $\mathrm{d}x$.

Example 6.15 Point P is at a distance r from a point charge q_0 at the origin of a 3-D space. If the **electric potential** at P is given by a scalar function

$$V(r) = k\,\frac{q_0}{r}\,,$$

verify that the **electric field** at P due to the charge is given by $\vec{E} = -\nabla V$, where k is the Coulomb's constant.

Solution

It is known that the charge radiates electric field in all directions and the field strength is given by an inverse square law. The mathematical expression of the electric field at point P is $\vec{E} = k\,\frac{q_0}{r^2}\,\hat{r}$. It is the Coulomb's force exerted on a unit positive test charge at P. On the other hand, the operator given in equation 6.20 gives

$$\begin{aligned}
-\nabla V &= -\left(\hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\,\frac{\partial}{\partial \theta} + \hat{\phi}\,\frac{1}{r^2\,\sin\theta}\,\frac{\partial}{\partial \phi}\right) V \\
&= -\left(\hat{r}\,\frac{\partial V}{\partial r} + \hat{\theta}\,\frac{1}{r}\,\frac{\partial V}{\partial \theta} + \hat{\phi}\,\frac{1}{r^2\,\sin\theta}\,\frac{\partial V}{\partial \phi}\right) \\
&= -\left(-k\,\frac{q_0}{r^2}\,\hat{r} + 0\,\hat{\theta} + 0\,\hat{\phi}\right) \\
&= k\,\frac{q_0}{r^2}\,\hat{r}
\end{aligned}$$

Therefore, the statement is verified as $\vec{E} = -\nabla V$. ■

6.7 Properties of Del Operator

6.7.1 Directional Derivative

Consider the function $f(x, y, z)$ and two points $P(x, y, z)$ and $Q(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$ in the 3-D space, the infinitesimally small distance between PQ is $\mathrm{d}s$, where $\mathrm{d}s = |\mathrm{d}\vec{l}|$. The infinitesimal change in f is

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z\,,$$

which implies

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}s} &= \frac{\partial f}{\partial x}\,\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial f}{\partial y}\,\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial f}{\partial z}\,\frac{\mathrm{d}z}{\mathrm{d}s} \\
\frac{\mathrm{d}f}{\mathrm{d}s} &= \left(\frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath} + \frac{\partial f}{\partial z}\,\hat{k}\right) \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}s}\,\hat{\imath} + \frac{\mathrm{d}y}{\mathrm{d}s}\,\hat{\jmath} + \frac{\mathrm{d}z}{\mathrm{d}s}\,\hat{k}\right) \\
\frac{\mathrm{d}f}{\mathrm{d}s} &= \nabla f \cdot \frac{\mathrm{d}\vec{l}}{\mathrm{d}s} \\
\frac{\mathrm{d}f}{\mathrm{d}s} &= \nabla f \cdot \hat{u}
\end{aligned}$$

Hence, the rate of change of f along the unit vector $\hat{u}$ is given by $\dfrac{\mathrm{d}f}{\mathrm{d}s} = \nabla f \cdot \hat{u}$. It is labelled by $D_{\hat{u}} f(x, y, z)$ and is also called as the directional derivative of f at the point (x, y, z) along $\hat{u}$.

Example 6.16 Given a surface S: $z = f(x, y)$ for which its first order derivatives exist, where x and y are independent variables, find the directional derivative at $P_0(x_0, y_0)$ along the unit vector $\hat{u} = u_1 \hat{i} + u_2 \hat{j}$. What is it's geometrical meaning?

Solution

As shown in figure 6.5, P lies on the surface S and has coordinates (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$. Obviously, $P_0(x_0, y_0)$ is the projection of P on the xy-plane. The vertical plane which passes through P and aligns with $\hat{u}$ cuts the surface S by a curve C. The slope of the curve C at P equals to the directional derivative at P_0 along the direction $\hat{u}$, thus

$$\begin{aligned} D_{\hat{u}} f(x_0, y_0) &= \nabla f(x_0, y_0) \cdot \hat{u} \\ &= \Big(f_x(x_0, y_0)\, \hat{i} + f_y(x_0, y_0)\, \hat{j} \Big) \cdot (u_1 \hat{i} + u_2 \hat{j}) \\ &= f_x(x_0, y_0)\, u_1 + f_y(x_0, y_0)\, u_2 \end{aligned}$$

■

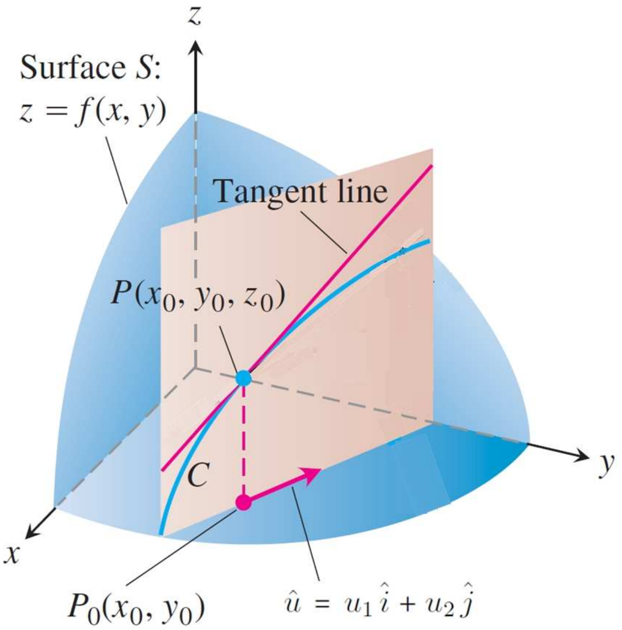


Figure 6.5: The directional derivative of $z = f(x, y)$

6.7.2 Maximum Derivative

As the rate $\dfrac{\mathrm{d}f}{\mathrm{d}s} = \nabla f \cdot \hat{u} = |\nabla f|\,|\hat{u}|\,\cos\theta = |\nabla f|\,\cos\theta$, where θ is the angle between the two vectors. The maximum of $\dfrac{\mathrm{d}f}{\mathrm{d}s}$ is obtained when $\theta = 0$. That is to say, the maximum rate of change of f is found along ∇f and the value is $|\nabla f|$.

Example 6.17 Given that $g(x, y) = e^{-y}\sin x + \dfrac{1}{3}\,e^{-3y}\sin 3x$.

(a) Find the rate of change of $g(x, y)$ at $(\frac{\pi}{3}, 0)$ in the direction $-\hat{\imath} + \sqrt{3}\,\hat{\jmath}$.

(b) Find also the maximum derivative at $(\frac{\pi}{3}, 0)$.

Solution

(a) The del operator of a two-variable function is $\nabla \equiv \hat{\imath}\,\dfrac{\partial}{\partial x} + \hat{\jmath}\,\dfrac{\partial}{\partial y}$. Hence,

$$\begin{aligned}\nabla g &= \left(\hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y}\right)(e^{-y}\sin x + \frac{1}{3}\,e^{-3y}\sin 3x)\\ &= \frac{\partial}{\partial x}(e^{-y}\sin x + \frac{1}{3}\,e^{-3y}\sin 3x)\,\hat{\imath} + \frac{\partial}{\partial y}(e^{-y}\sin x + \frac{1}{3}\,e^{-3y}\sin 3x)\,\hat{\jmath}\\ &= (e^{-y}\cos x + e^{-3y}\cos 3x)\,\hat{\imath} - (e^{-y}\sin x + e^{-3y}\sin 3x)\,\hat{\jmath}\end{aligned}$$

Substituting the point $(\frac{\pi}{3}, 0)$ into ∇g, we have

$$\begin{aligned}\nabla g(\frac{\pi}{3}, 0) &= \left(\cos\frac{\pi}{3} + \cos\pi\right)\,\hat{\imath} - \left(\sin\frac{\pi}{3} + \sin\pi\right)\,\hat{\jmath}\\ &= -\frac{1}{2}\,\hat{\imath} - \frac{\sqrt{3}}{2}\,\hat{\jmath}\end{aligned}$$

The unit vector that points along $-\hat{\imath} + \sqrt{3}\,\hat{\jmath}$ is $\hat{u} = -\dfrac{1}{2}\,\hat{\imath} + \dfrac{\sqrt{3}}{2}\,\hat{\jmath}$. The rate of change of $g(x, y)$ at $(\frac{\pi}{3}, 0)$ along $\hat{u}$ is

$$\begin{aligned}\nabla g(\frac{\pi}{3}, 0)\cdot\hat{u} &= \left(-\frac{1}{2}\,\hat{\imath} - \frac{\sqrt{3}}{2}\,\hat{\jmath}\right)\cdot\left(-\frac{1}{2}\,\hat{\imath} + \frac{\sqrt{3}}{2}\,\hat{\jmath}\right)\\ &= \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}\end{aligned}$$

(b) The maximum value of the directional derviative at $(\frac{\pi}{3}, 0)$ is

$$|\nabla g(\frac{\pi}{3}, 0)| = \sqrt{(-\frac{1}{2})^2 + (-\frac{\sqrt{3}}{2})^2} = 1$$

This value is the maximum slope of $g(x,y)$ at $(\frac{\pi}{3},0)$ and it can be found in the direction

$$\nabla g(\frac{\pi}{3},0)=-\frac{1}{2}\,\hat{\imath}-\frac{\sqrt{3}}{2}\,\hat{\jmath}$$

■

6.7.3 Normal of a Surface

A surface in the 3-D space is described by $f(x,y,z)=c$, where c is a constant. Obviously,

$$\begin{aligned} \mathrm{d}f &= 0 \\ \nabla f\cdot \mathrm{d}\vec{l} &= 0 \end{aligned}$$

Since $\mathrm{d}\vec{l}$ lies on the surface, $\mathrm{d}\vec{l}\perp\nabla f$. Thus, ∇f is along the normal vector of the surface. The unit normal vector of the surface is represented by $\hat{n}=\dfrac{\nabla f}{|\nabla f|}$.

Example 6.18 A point $P\,(0.6,0.8,1)$ lies on the double cone $x^2+y^2=z^2$ as shown in figure 6.6. Find the unit normal vector of the surface $x^2+y^2=z^2$ at P. Hence, find the equation of the tangent plane at P.

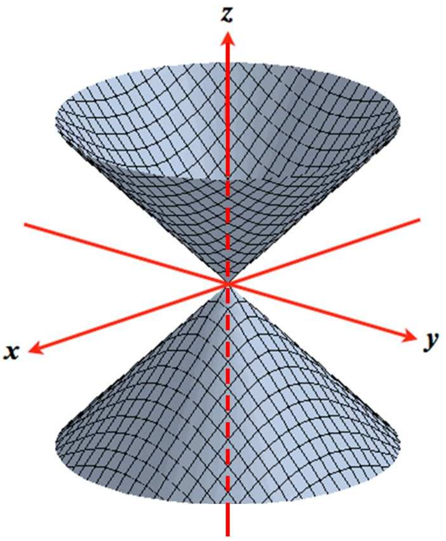


Figure 6.6: The double cone $x^2+y^2=z^2$

Solution

Define a scalar function $f(x,y,z)=x^2+y^2-z^2$. Obviously,

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x}\,\hat{\imath}+\frac{\partial f}{\partial y}\,\hat{\jmath}+\frac{\partial f}{\partial z}\,\hat{k}=2x\,\hat{\imath}+2y\,\hat{\jmath}-2z\,\hat{k} \\ |\nabla f| &= 2\sqrt{x^2+y^2+z^2} \end{aligned}$$

Since $f(x,y,z)=0$, the unit normal to the surface at P is given by

$$\begin{aligned}\hat{n}=\left.\frac{\nabla f}{|\nabla f|}\right|_{(0.6,0.8,1)} &= \left.\frac{1}{\sqrt{x^2+y^2+z^2}}\,(x\,\hat{\imath}+y\,\hat{\jmath}-z\,\hat{k})\right|_{(0.6,0.8,1)}\\ &= 0.3\,\hat{\imath}+0.4\,\hat{\jmath}-0.5\,\hat{k}\end{aligned}$$

Let (x,y,z) be a point on the tangent plane. Thus the equation of the tangent plane at P is

$$\Big\{(x-0.6)\,\hat{\imath}+(y-0.8)\,\hat{\jmath}+(z-1)\,\hat{k}\Big\}\cdot\hat{n}=0$$

After simplification, we get $z=0.6\,x+0.8\,y$. ■

Exercise 6.10 A fixed point $A(a,b,c)$ and a moving point $P(x,y,z)$ form a vector $\vec{R}=\overrightarrow{AP}$ with length R. Show that ∇R is a unit vector along $\vec{R}$.

Exercise 6.11 Obtain the del operator for the spherical coordinate system.

6.8 Lagrange's Undetermined Multiplier

Consider a function $f(x,y,z)$ of three independent variables. The funciton f has an extremum value when

$$\mathrm{d}f=0 \tag{6.22}$$

Thus

$$\mathrm{d}f=\frac{\partial f}{\partial x}\,\mathrm{d}x+\frac{\partial f}{\partial y}\,\mathrm{d}y+\frac{\partial f}{\partial z}\,\mathrm{d}z=0 \tag{6.23}$$

and

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=\frac{\partial f}{\partial z}=0 \tag{6.24}$$

If there is a constraint equation $g(x,y,z)=0$, then equation (6.24) is no longer valid because x, y and z are not independent variables simultaneously. The constraint equation also gives

$$\mathrm{d}g=\frac{\partial g}{\partial x}\,\mathrm{d}x+\frac{\partial g}{\partial y}\,\mathrm{d}y+\frac{\partial g}{\partial z}\,\mathrm{d}z=0 \tag{6.25}$$

Obviously,

$$\mathrm{d}f+\lambda\,\mathrm{d}g=\left(\frac{\partial f}{\partial x}+\lambda\,\frac{\partial g}{\partial x}\right)\mathrm{d}x+\left(\frac{\partial f}{\partial y}+\lambda\,\frac{\partial g}{\partial y}\right)\mathrm{d}y+\left(\frac{\partial f}{\partial z}+\lambda\,\frac{\partial g}{\partial z}\right)\mathrm{d}z=0 \tag{6.26}$$

for any real number λ, where λ is called the Lagrange's multiplier. If we choose λ such that $\dfrac{\partial f}{\partial z} + \lambda \dfrac{\partial g}{\partial z} = 0$, then x and y are independent variables now. So, we have $\dfrac{\partial f}{\partial x} + \lambda \dfrac{\partial g}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} + \lambda \dfrac{\partial g}{\partial y} = 0$.

To summarize everything, if $f(x, y, z)$ has an extremum value and $g(x, y, z) = 0$, then the stationary point (x, y, z) can be found by setting

$$\begin{cases} \dfrac{\partial f}{\partial x} + \lambda \dfrac{\partial g}{\partial x} = 0 \\ \\ \dfrac{\partial f}{\partial y} + \lambda \dfrac{\partial g}{\partial y} = 0 \\ \\ \dfrac{\partial f}{\partial z} + \lambda \dfrac{\partial g}{\partial z} = 0 \end{cases} \tag{6.27}$$

Example 6.19 Let A be the point $(3, 4, 12)$. Find the maximum and minimum distance between A and the sphere $x^2 + y^2 + z^2 = 1$.

Solution

Let $P(x, y, z)$ be any point of the sphere and $f(x, y, z) = (AP)^2$, where

$$(AP)^2 = (x-3)^2 + (y-4)^2 + (z-12)^2 \tag{6.28}$$

Then, we have $f(x, y, z) = (x-3)^2 + (y-4)^2 + (z-12)^2$

Let $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ which is the constraint equation. Thus f attains an extremum value when

$$\begin{cases} \dfrac{\partial f}{\partial x} + \lambda \dfrac{\partial g}{\partial x} = 0 \\ \\ \dfrac{\partial f}{\partial y} + \lambda \dfrac{\partial g}{\partial y} = 0 \\ \\ \dfrac{\partial f}{\partial z} + \lambda \dfrac{\partial g}{\partial z} = 0 \end{cases} \tag{6.29}$$

So, we have

$$\begin{cases} 2\,(x-3) + 2\,\lambda\, x = 0 \\ \\ 2\,(y-4) + 2\,\lambda\, y = 0 \\ \\ 2\,(z-12) + 2\,\lambda\, z = 0 \end{cases} \tag{6.30}$$

which gives

$$x = \frac{3}{\lambda + 1}, \qquad y = \frac{4}{\lambda + 1}, \qquad \text{and} \qquad z = \frac{12}{\lambda + 1}$$

Substituting the results to $g(x, y, z) = 0$, we have

$$\left(\frac{3}{\lambda + 1}\right)^2 + \left(\frac{4}{\lambda + 1}\right)^2 + \left(\frac{12}{\lambda + 1}\right)^2 = 1$$

which gives $(\lambda + 1)^2 = 169$, and thus $\lambda = 12, -14$.

When $\lambda = 12$, we get $(x, y, z) = \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$.

When $\lambda = -14$, we get $(x, y, z) = \left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$.

Using equation (6.28), AP has two extreme distances:

$$\sqrt{\left(\frac{3}{13} - 3\right)^2 + \left(\frac{4}{13} - 4\right)^2 + \left(\frac{12}{13} - 12\right)^2} = 12 \qquad \text{(the minimum)}$$

and

$$\sqrt{\left(-\frac{3}{13} - 3\right)^2 + \left(-\frac{4}{13} - 4\right)^2 + \left(-\frac{12}{13} - 12\right)^2} = 14 \qquad \text{(the maximum)}$$

■

Example 6.20 Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution

Let $2x$, $2y$, and $2z$ be the length, breadth, and height of the rectangular parallelepiped, then the volume of the parallelepiped is

$$V(x, y, z) = 8xyz \tag{6.31}$$

subject to the contraint equation

$$g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \tag{6.32}$$

For stationary points of V, we have $\mathrm{d}V + \lambda\,\mathrm{d}g = 0$, so

$$\begin{cases} \dfrac{\partial V}{\partial x} + \lambda \dfrac{\partial g}{\partial x} = 0 \\[2ex] \dfrac{\partial V}{\partial y} + \lambda \dfrac{\partial g}{\partial y} = 0 \\[2ex] \dfrac{\partial V}{\partial z} + \lambda \dfrac{\partial g}{\partial z} = 0 \end{cases} \tag{6.33}$$

which gives

$$\begin{cases} 8\,yz + \dfrac{2\,\lambda\,x}{a^2} = 0 & \Longrightarrow \quad 8\,xyz + \dfrac{2\,\lambda\,x^2}{a^2} = 0 \\[2ex] 8\,zx + \dfrac{2\,\lambda\,y}{b^2} = 0 & \Longrightarrow \quad 8\,xyz + \dfrac{2\,\lambda\,y^2}{b^2} = 0 \\[2ex] 8\,xy + \dfrac{2\,\lambda\,z}{c^2} = 0 & \Longrightarrow \quad 8\,xyz + \dfrac{2\,\lambda\,z^2}{c^2} = 0 \end{cases} \tag{6.34}$$

Summing the three equations stated above, we have

$$24\,xyz + 2\,\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0 \tag{6.35}$$

Thus,

$$\lambda = -12\,xyz \tag{6.36}$$

Substituting into equation (6.34), we obtain

$$x = \frac{a}{\sqrt{3}}, \qquad y = \frac{b}{\sqrt{3}}, \qquad \text{and} \qquad z = \frac{c}{\sqrt{3}}$$

Therefore, the maximum volume is $V_{\max} = \dfrac{8\,abc}{3\sqrt{3}}$. ■

Example 6.21 Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius a. What is the largest surface area?

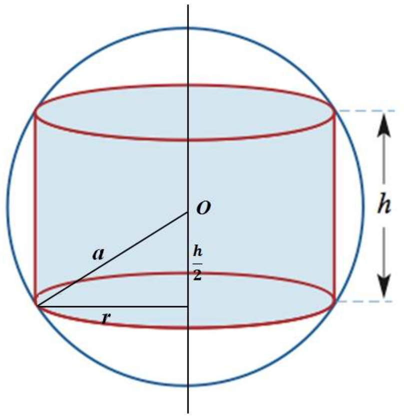


Figure 6.7: A cylinder is inscribed in a sphere

Solution

Let r and h be the radius and the height of the cylinder respectively. The constraint equation is $r^2 + \left(\frac{h}{2}\right)^2 = a^2$. Then, we define $g(r,h) = r^2 + \left(\frac{h}{2}\right)^2 - a^2$, where $g(r,h) = 0$. The surface area of the cylinder is $S = 2\,\pi r h$. If the surface area is an extremum, we have

$$\frac{\partial S}{\partial r} + \lambda\,\frac{\partial g}{\partial r} = 0 \qquad \text{and} \qquad \frac{\partial S}{\partial h} + \lambda\,\frac{\partial g}{\partial h} = 0$$

where λ is the Lagrange's multiplier. Then we have

$$2\,\pi h = 2\,\lambda r \qquad \text{and} \qquad 2\,\pi r = \frac{\lambda h}{2}$$

Eliminating the multiplier, we have $h^2 = 4\,r^2$. Then we solve for r and h with the use of the constraint equation: $g(x,y) = 0$. We get $r = \frac{a}{\sqrt{2}}$ and $h = a\sqrt{2}$. Hence, the largest surface area $S = 2\,\pi\left(\frac{a}{\sqrt{2}}\right)\left(a\sqrt{2}\right) = 2\,\pi a^2$. ■

Exercise 6.12 Use Lagrange's undetermined multiplier to find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

Exercise 6.13 The temperature T at any point (x,y,z) in space is $T = 400\,xyz^2$. Find the highest temperature at the surface of a unit sphere $x^2 + y^2 + z^2 = 1$.

Exercise 6.14 Let a_1, a_2, a_3, $\ldots$, a_n be n positive numbers and the sum of them is l. Find the maximum value of the product $a_1\,a_2\,a_3\,\ldots\,a_n$. Show further that

$$\sqrt[n]{a_1\,a_2\,a_3\,\ldots\,a_n} \le \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}$$

6.9 Functional Dependence

Let $u = f(x,y)$ and $v = g(x,y)$ be two given differentiable functions of two independent variables x and y. Suppose u and v are connected by a relation $F(u,v) = 0$, where $F(u,v)$ is differentiable. Then u and v are said to be functionally dependent on each other if the partial derivatives u_x, u_y, v_x, and v_y are not all zero simultaneously. The condition for functional dependence can be expressed in terms of a determinant as shown below. Differentiating $F(u,v) = 0$ with respect to x and y respectively, we have

$$\begin{cases} \dfrac{\partial F}{\partial u}\,\dfrac{\partial u}{\partial x} + \dfrac{\partial F}{\partial v}\,\dfrac{\partial v}{\partial x} = 0 \\[2ex] \dfrac{\partial F}{\partial u}\,\dfrac{\partial u}{\partial y} + \dfrac{\partial F}{\partial v}\,\dfrac{\partial v}{\partial y} = 0 \end{cases} \tag{6.37}$$

For non-trivial solutions of $\dfrac{\partial F}{\partial u}$ and $\dfrac{\partial F}{\partial v}$, we have

$$J(u,v) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0 \tag{6.38}$$

where $J(u,v)$ is the Jacobian of functions u and v. If $J(u,v) \neq 0$, then u and v are functionally independent.

Remark:

Knowing that $\det A = \det A^T$, we have

$$J(u,v) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Example 6.22 Let $u = e^x \sin y$ and $v = x + \ln \sin y$. Show that u and v are functionally dependent.

Solution

Let's check the Jacobian of functions u and v.

$$J(u,v) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \sin y & 1 \\ e^x \cos y & \dfrac{\cos y}{\sin y} \end{vmatrix} = 0$$

Therefore, u and v are functionally dependent and there exist a function $F(u,v)$ such that $F(u,v) = 0$. One can verify that $v = \ln u$ and $F(u,v) = \ln u - v = 0$. ■

Example 6.23 Let $x = r \cos\theta$ and $y = r \sin\theta$. Show that x and y are functionally independent.

Solution

Let's check the Jacobian of functions x and y.

$$J(x,y) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r \sin\theta & r \cos\theta \end{vmatrix} = r \neq 0$$

Therefore, u and v are functionally independent. ■

Exercise 6.15 If $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$, find $J(u,v)$. Hence, determine whether u and v are functionally dependent. **Useful formula:** $\dfrac{\mathrm{d}}{\mathrm{d}z}\left(\tan^{-1} z\right) = \dfrac{1}{1+z^2}$.

6.10 Taylor Series of Two Variables

Function of two variables can be expanded in power series and this series is called the Taylor series of two variables. Suppose that all the k^{th} partial derivatives of $f(x,y)$ are continuous, then $f(x,y)$ can be expressed as a power series about (x_0,y_0).

$$f(x,y) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^k f(x_0,y_0) \tag{6.39}$$

where $h = x - x_0$, $k = y - y_0$. Alternatively, we can write

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + [h\, f_x(x_0,y_0) + k\, f_y(x_0,y_0)] + \\ &\quad \frac{1}{2!}\left[h^2 f_{xx}(x_0,y_0) + 2\,hk\, f_{xy}(x_0,y_0) + k^2 f_{yy}(x_0,y_0)\right] + \cdots \end{aligned}$$

where $f_x = \dfrac{\partial f}{\partial x}$, $f_y = \dfrac{\partial f}{\partial y}$, $f_{xx} = \dfrac{\partial^2 f}{\partial x^2}$, $f_{xy} = \dfrac{\partial^2 f}{\partial x\,\partial y}$, and $f_{yy} = \dfrac{\partial^2 f}{\partial y^2}$. The derivatives of f at (x_0,y_0) are represented by $f_x(x_0,y_0)$, $f_y(x_0,y_0)$, $f_{xx}(x_0,y_0)$, $f_{xy}(x_0,y_0)$, and $f_{yy}(x_0,y_0)$ respectively.

Proof:

Let $F(t) = f(x,y) = f(x_0+ht, y_0+kt)$, then $F(1) = f(x_0+h, y_0+k)$ and $F(0) = f(x_0,y_0)$. Taylor expansion of a single variable function $F(1)$ about a point at $t = 0$ gives

$$\begin{aligned} F(1) &= F(0) + (1-0)\,F'(0) + \frac{1}{2!}\,(1-0)^2\,F\text{"}(0) + \frac{1}{3!}\,(1-0)^3\,F^{(3)}(0) + \cdots \\ &= F(0) + F'(0) + \frac{1}{2!}\,F\text{"}(0) + \frac{1}{3!}\,F^{(3)}(0) + \cdots \end{aligned} \tag{6.40}$$

To obtain $F'(0)$ and $F\text{"}(0)$, we write down $F'(t)$ first. By the chain rule, we have

$$F'(t) = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = h\,\frac{\partial f}{\partial x} + k\,\frac{\partial f}{\partial y}$$

Thus

$$\begin{aligned}
F\text{"}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(h\,\frac{\partial f}{\partial x} + k\,\frac{\partial f}{\partial y}\right) \\
&= h\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial x}\right) + k\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial y}\right) \\
&= h\,\left(\frac{\partial^2 f}{\partial x^2}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial^2 f}{\partial x\,\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}\right) + k\,\left(\frac{\partial^2 f}{\partial x\,\partial y}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial^2 f}{\partial y^2}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}\right) \\
&= h\,\left(h\,\frac{\partial^2 f}{\partial x^2} + k\,\frac{\partial^2 f}{\partial x\,\partial y}\right) + k\,\left(h\,\frac{\partial^2 f}{\partial x\,\partial y} + k\,\frac{\partial^2 f}{\partial y^2}\right) \\
&= h^2\,\frac{\partial^2 f}{\partial x^2} + 2\,hk\,\frac{\partial^2 f}{\partial x\,\partial y} + k^2\,\frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

Therefore,

$$F'(0) = \left(h\,\frac{\partial}{\partial x} + k\,\frac{\partial}{\partial y}\right)\,f(x_0,y_0)$$

and

$$F\text{"}(0) = \left(h^2\,\frac{\partial^2}{\partial x^2} + 2\,hk\,\frac{\partial^2}{\partial x\,\partial y} + k^2\,\frac{\partial^2}{\partial y^2}\right)\,f(x_0,y_0) = \left(h\,\frac{\partial}{\partial x} + k\,\frac{\partial}{\partial y}\right)^2\,f(x_0,y_0)$$

Similary, we have

$$F^{(3)}(0) = \left(h\,\frac{\partial}{\partial x} + k\,\frac{\partial}{\partial y}\right)^3\,f(x_0,y_0)$$

Since $F(1) = f(x,y)$, the Taylor series appears after plugging all derivatives at $t = 0$ into equation (6.40).

Example 6.24 Find the Taylor series of the function $f(x,y) = e^{x+2y}$ to the second order about a point $(0,0)$.

Solution

Let's consider the Taylor series of $f(x,y)$.

$$\begin{aligned}
f(x,y) &= f(x_0,y_0) + [h\,f_x(x_0,y_0) + k\,f_y(x_0,y_0)] + \\
&\quad \frac{1}{2!}\,\left[h^2\,f_{xx}(x_0,y_0) + 2\,hk\,f_{xy}(x_0,y_0) + k^2\,f_{yy}(x_0,y_0)\right] + \cdots
\end{aligned}$$

Now, we obtain the derivatives of f, where $f(x,y) = e^{x+2y}$. Obviously
$f_x = e^{x+2y}$, $f_y = 2\,e^{x+2y}$, $f_{xx} = e^{x+2y}$, $f_{xy} = 2\,e^{x+2y}$, $f_{yy} = 4\,e^{x+2y}$.
Let $(x_0,y_0) = (0,0)$, then $f(0,0) = 1$ and
$f_x(0,0) = 1$, $f_y(0,0) = 2$,
$f_{xx}(0,0) = 1$, $f_{xy}(0,0) = 2$, $f_{yy}(0,0) = 4$.

So

$$\begin{aligned} e^{x+2y} &= 1+[x\,(1)+y\,(2)]+\frac{1}{2!}\,[x^2\,(1)+2xy\,(2)+y^2\,(4)]+\,\cdots \\ &= 1+x+2\,y+\frac{x^2}{2}+2\,xy+2\,y^2+\,\cdots \end{aligned}$$

■

Example 6.25 Find the Taylor series of the function $f(x,y)=e^x\sin y$ to the third order about the point $(0,0)$.

Solution

Let's consider the Taylor series of $f(x,y)$.

$$\begin{aligned} f(x,y) &= f(x_0,y_0)+[h\,f_x(x_0,y_0)+k\,f_y(x_0,y_0)]+ \\ &\quad \frac{1}{2!}\,\left[h^2\,f_{xx}(x_0,y_0)+2\,hk\,f_{xy}(x_0,y_0)+k^2\,f_{yy}(x_0,y_0)\right]+ \\ &\quad \frac{1}{3!}\,\left[h^3\,f_{xxx}(x_0,y_0)+3\,h^2k\,f_{xxy}(x_0,y_0)+3\,hk^2\,f_{xyy}(x_0,y_0)+k^3\,f_{yyy}(x_0,y_0)\right]+\,\cdots \end{aligned}$$

Now, we obtain the derivatives of f, where $f(x,y)=e^x\,\sin y$.

$f_x=e^x\sin y,\ f_y=e^x\cos y,$

$f_{xx}=e^x\sin y,\ f_{xy}=e^x\cos y,\ f_{yy}=-e^x\sin y,$

$f_{xxx}=e^x\sin y,\ f_{xxy}=e^x\cos y,\ f_{xyy}=-e^x\sin y,\ f_{yyy}=-e^x\cos y.$

Let $(x_0,y_0)=(0,0)$, then $f(0,0)=0$ and

$f_x(0,0)=0,\ f_y(0,0)=1,$

$f_{xx}(0,0)=0,\ f_xy(0,0)=1,\ f_{yy}(0,0)=0,$

$f_{xxx}(0,0)=0,\ f_{xxy}(0,0)=1,\ f_{xyy}(0,0)=0,\ f_{yyy}(0,0)=-1.$

So

$$\begin{aligned} e^x\,\sin y &= 0+[x\,(0)+y\,(1)]+\frac{1}{2!}\,[x^2\,(0)+2xy\,(1)+y^2\,(0)]+ \\ &\quad \frac{1}{3!}\,[x^3\,(0)+3x^2y\,(1)+3\,xy^2\,(0)+y^3(-1)]+\,\cdots \\ &= y+xy+\frac{1}{2}\,x^2y-\frac{1}{6}\,y^3+\,\cdots \end{aligned}$$

■

Example 6.26 The function $z=f(x,y)$ solves the equation $x+2\,y+z+e^{2z}=1$. Find the Taylor expansion of $f(x,y)$ in powers of x and y to degree two.

Solution

Differentiating both sides of $x + 2\,y + z + e^{2z} = 1$ with respect to x, we have

$$1 + 0 + \frac{\partial z}{\partial x} + 2\,e^{2z}\,\frac{\partial z}{\partial x} = 0$$

which gives

$$\frac{\partial z}{\partial x} = -\frac{1}{1 + 2\,e^{2z}} \qquad \text{and} \qquad \frac{\partial^2 z}{\partial x^2} = -\frac{4\,e^{2z}}{(1 + 2\,e^{2z})^3}$$

Differentiating both sides of $x + 2\,y + z + e^{2z} = 1$ again with respect to y, we have

$$0 + 2 + \frac{\partial z}{\partial y} + 2\,e^{2z}\,\frac{\partial z}{\partial y} = 0$$

which gives

$$\frac{\partial z}{\partial y} = -\frac{2}{1 + 2\,e^{2z}}, \qquad \frac{\partial^2 z}{\partial x\,\partial y} = -\frac{8\,e^{2z}}{(1 + 2\,e^{2z})^3}, \qquad \text{and} \qquad \frac{\partial^2 z}{\partial y^2} = -\frac{16\,e^{2z}}{(1 + 2\,e^{2z})^3}$$

Knowing that the Taylor expansion of f about $(0,0)$ is given by

$$\begin{aligned} f(x,y) &= f(0,0) + x\,f_x(0,0) + y\,f_y(0,0) + \\ &\quad \frac{1}{2!}\left[x^2\,f_{xx}(0,0) + 2\,xy\,f_{xy}(0,0) + y^2\,f_{yy}(0,0)\right] + \cdots \end{aligned}$$

With the fact that $f(0,0) = 0$ and $f_x(0,0) = -\frac{1}{3}$, $f_y(0,0) = -\frac{2}{3}$, $f_{xx}(0,0) = -\frac{4}{27}$, $f_{xy}(0,0) = -\frac{8}{27}$, and $f_{yy}(0,0) = -\frac{16}{27}$, we can write

$$f(x,y) = -\frac{1}{3}\,x - \frac{2}{3}\,y - \frac{2}{27}\,x^2 - \frac{8}{27}\,xy - \frac{8}{27}\,y^2 + \cdots$$

■

Exercise 6.16 Compute the second order Taylor formula for the function $f(x,y) = xy + x^2 + y^2$ about the point $(1,1)$.

Exercise 6.17 Compute the second order Taylor formula for the function $f(x,y) = \ln(x^2 + y^2)$ about the point $(1,0)$.

Example 6.27 Evaluate $\{(10.97) + (4.02)^2\}^{1/3}$ by using Taylor expansion to the second order approximation.

Solution

Let $f(x,y) = (x + y^2)^{1/3}$, then the derivatives of it becomes

$$f_x = \frac{1}{3}\,(x+y^2)^{-2/3}, \qquad\qquad f_y = \frac{2\,y}{3}\,(x+y^2)^{-2/3},$$
$$f_{xx} = -\frac{2}{9}\,(x+y^2)^{-5/3}, \quad f_{xy} = -\frac{4}{9}\,y\,(x+y^2)^{-5/3}, \ \text{ and } \ f_{yy} = \frac{2}{9}\,(3\,x-y^2)\,(x+y^2)^{-5/3}.$$

Set $x = 10.97$, $y = 4.02$, $x_0 = 11$, $y_0 = 4$, $h = -0.03$, $k = 0.02$, then

$$\begin{aligned} f(x,y) &\approx f(x_0,y_0) + h\,f_x(x_0,y_0) + k\,f_y(x_0,y_0) + \\ &\quad \frac{1}{2!}\left[h^2\,f_{xx}(x_0,y_0) + 2\,hk\,f_{xy}(x_0,y_0) + k^2\,f_{yy}(x_0,y_0)\right] \end{aligned}$$

where $f(x_0,y_0) = 3$. The derivatives of f at (x_0,y_0) are stated as follows.

$$f_x(11,4) = \frac{1}{3^3}, \ f_y(11,4) = \frac{8}{3^3}, \ f_{xx}(11,4) = -\frac{2}{3^7}, \ f_{xy}(11,4) = -\frac{16}{3^7}, \ f_{yy}(11,4) = \frac{34}{3^7}.$$

So, we have

$$\begin{aligned} \left\{(10.97)+(4.02)^2\right\}^{1/3} &\approx 3 + (-0.03)\left(\frac{1}{3^3}\right) + (0.02)\left(\frac{8}{3^3}\right) + \\ &\quad \frac{1}{2!}\left[(-0.03)^2\left(-\frac{2}{3^7}\right) + 2\,(-0.03)\,(0.02)\left(-\frac{16}{3^7}\right) + (0.02)^2\left(\frac{34}{3^7}\right)\right] \\ &= 3.004821902 \end{aligned}$$

■

Exercise 6.18 Evaluate $\{(1.02)^3 + (1.97)^3\}^{1/2}$ by using Taylor expansion to the second order approximation.

6.11 Extreme Values and Saddle Points

Before we discuss the calculus of extreme values and saddle points, let us review the definition of such points first. Let $f(x,y)$ be defined on a region R containg the point (a,b). Then

(1) $f(a,b)$ is a local maximum value of f if $f(a,b) \geq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b). See figure 6.8.

(2) $f(a,b)$ is a local minimum value of f if $f(a,b) \leq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b). See figure 6.8.

(3) A differentiable function $f(x,y)$ has a saddle point if in every open disk centered at (a,b) there are domain points (x,y) where $f(x,y) > f(a,b)$ and domain points (x,y) where $f(x,y) < f(a,b)$. The corresponding point $(a,b,f(a,b))$ on the surface $z = f(x,y)$ is called a saddle point of the surface. It is neither a maximum nor a minimum for the function. See figure 6.9.

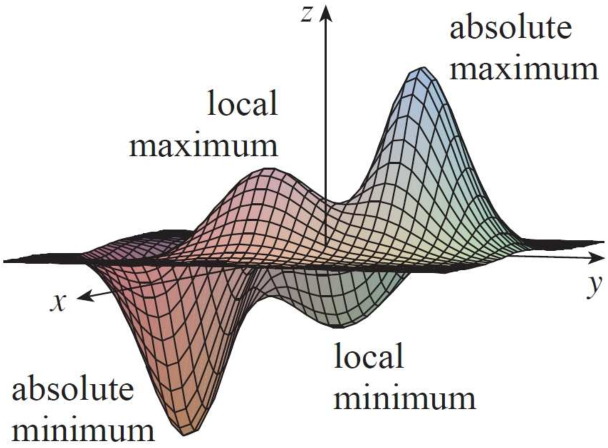


Figure 6.8: The maximum and minimum values of a surface

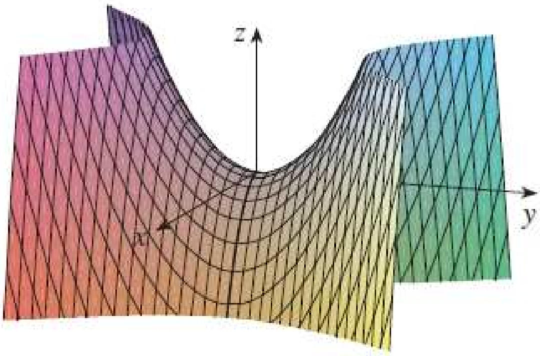


Figure 6.9: The saddle point of $z = y^2 - x^2$ at the origin

Clearly, we have the following conclusions. If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$. In order to determine the nature of the extreme values, we have the following rules. Suppose that $f(x, y)$ and its first and second partial derivatives are continuous though a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$, see figure 6.10. Let $\delta = f_{xx}f_{yy} - f_{xy}^2$ be the discriminant of f. Then

(i) f has a local maximum at (a, b) if $\delta > 0$ and $(f_{xx} < 0$ or $f_{yy} < 0)$ at (a, b).

(ii) f has a local minimum at (a, b) if $\delta > 0$ and $(f_{xx} > 0$ or $f_{yy} > 0)$ at (a, b).

(iii) f has a saddle point at (a, b) if $\delta < 0$ at (a, b).

(iv) The test is inconclusive at (a, b) if $\delta = 0$ at (a, b).

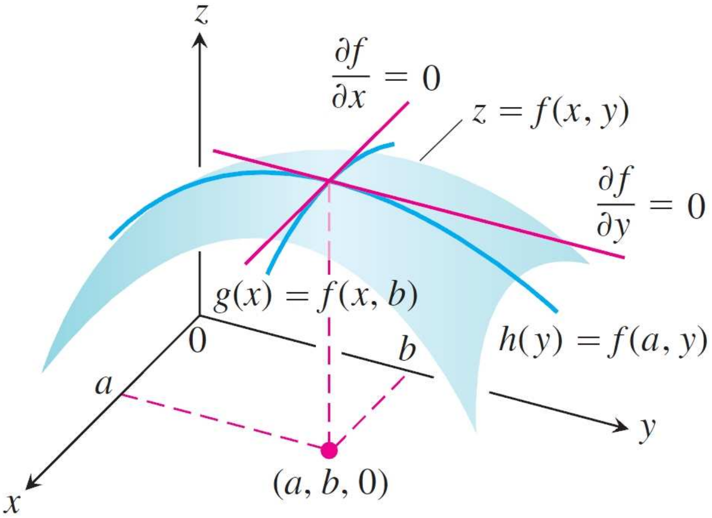


Figure 6.10: The partial derivatives of $f(x, y)$ in various directions

The discriminant δ is also called Hessian of f, where

$$\delta = f_{xx}\, f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Remarks:

Statements (i) to (iv) provide tests to determine the nature of extreme values. The derivation of them start by considering the difference of $f(x_0 + h, y_0 + k)$ and $f(x_0, y_0)$.

$$\begin{aligned} & f(x_0 + h, y_0 + k) - f(x_0, y_0) \\ = \;& f(x_0, y_0) + h\, f_x(x_0, y_0) + k\, f_y(x_0, y_0) + \\ & \quad \frac{1}{2!}\left[h^2\, f_{xx}(x_0, y_0) + 2\, hk\, f_{xy}(x_0, y_0) + k^2\, f_{yy}(x_0, y_0)\right] + \cdots - f(x_0, y_0) \end{aligned}$$

Choose (x, y) such that h and k are so small that the higher order terms are ignored.

$$\begin{aligned} & f(x_0 + h, y_0 + k) - f(x_0, y_0) \\ = \;& \frac{1}{2!}\left[h^2\, f_{xx}(x_0, y_0) + 2\, hk\, f_{xy}(x_0, y_0) + k^2\, f_{yy}(x_0, y_0)\right] \end{aligned}$$

Completing the square, we have

$$\begin{aligned} & f(x_0 + h, y_0 + k) - f(x_0, y_0) \\ = \;& \frac{1}{2!}\, f_{xx}\left[\left(h + \frac{f_{xy}}{f_{xx}}\, k\right)^2 + \left(\frac{f_{xx}\, f_{yy} - f_{xy}^2}{f_{xx}^2}\right) k^2\right]\Bigg|_{(x_0, y_0)} \end{aligned}$$

We define the discriminant $\delta = f_{xx}\, f_{yy} - f_{xy}^2$. Hence, there are four cases.

(C1) For relative maximum value at (x_0, y_0): $\delta > 0$ and ($f_{xx} < 0$ or $f_{yy} < 0$)

(C2) For relative minimum value at (x_0, y_0): $\delta > 0$ and ($f_{xx} > 0$ or $f_{yy} > 0$)

(C3) For neither a relative maximum nor a relative minimum at (x_0, y_0), i.e. a saddle point: $\delta < 0$.

(C4) It is inconclusive if $\delta = 0$.

Example 6.28 Find the critical points of $f(x, y) = x^3 + y^3 - 3\,x - 12\,y + 20$.

Solution

Let's write down the derivatives of f.

$$f_x = 3\,x^2 - 3, \qquad f_y = 3\,y^2 - 12, \qquad f_{xx} = 6\,x, \qquad f_{xy} = 0, \qquad f_{yy} = 6\,y$$

The discriminant $\delta = f_{xx}\,f_{yy} - f_{xy}^2 = 36\,xy$.

For $f_x = 0$, $x = \pm 1$.

For $f_y = 0$, $y = \pm 2$.

The critical points are $P(1, 2)$, $Q(-1, 2)$, $R(1, -2)$, $S(-1, -2)$.

$P(1, 2)$: $\delta > 0$, $f_{xx} > 0$ or $f_{yy} > 0$, P is a relative minimum.

$Q(-1, 2)$: $\delta < 0$, $f_{xx} > 0$, Q is a saddle point.

$R(1, -2)$: $\delta < 0$, $f_{xx} > 0$, R is a saddle point.

$S(-1, -2)$: $\delta > 0$, $f_{xx} < 0$ or $f_{yy} < 0$, S is a relative maximum. ■

Exercise 6.19 Find the local extreme values of $f(x, y) = x^2 + y^2 - 4\,y + 9$.

Example 6.29 Find the critical points of $f(x, y) = y^2 - x^2$.

Solution

Let's write down the derivatives of f.

$$f_x = -2\,x, \qquad f_y = 2\,y, \qquad f_{xx} = -2 > 0, \qquad f_{xy} = 0, \qquad f_{yy} = 2 > 0$$

The discriminant $\delta = f_{xx}\,f_{yy} - f_{xy}^2 = -4 < 0$.

If we set $f_x = 0$ and $f_y = 0$, we obtain $x = 0$ and $y = 0$, so the critical point is found at $(0, 0)$. Therefore, f has a saddle point at $(0, 0, 0)$. See figure 6.9. ■

Exercise 6.20 Find the local extreme values of $f(x, y) = 3\,y^2 - 2\,y^3 - 3\,x^2 + 6\,xy$.