Ballistic Pendulum – Theory

Overview

The ballistic pendulum is a classic method of determining the velocity of a projectile. It is also a good demonstration of some of the basic principles of physics.

The ball is fired into the pendulum, which then swings up a measured amount. From the height reached by the pendulum, we can calculate its potential energy. This potential energy is equal to the kinetic energy of the pendulum at the swing, just after the collision with the ball.

We can’t equate the kinetic energy of the pendulum after the collision with the kinetic energy of the ball before the swing, since the collision between the ball and pendulum is inelastic and kinetic energy is not conserved in inelastic collisions. Momentum in conserved in all forms of collision, though; so we know that the momentum of the ball before the collision is equal to the momentum of the pendulum after the collision. Once we know the momentum of the ball and its mass, we can determine the initial velocity.

There are two ways of calculating the velocity of the ball. The first method (approximate method) assumes that the pendulum and ball together act as a point mass located at their combined center of mass. This method does not take rotational inertia into account. It is somewhat quicker and easier than the second method, but not as accurate.

The second method (exact method) uses the actual rotational inertia of the pendulum in the calculations. The equations are slightly more complicated, and it is necessary to take more data in order to find the moment of inertia of the pendulum; but the results obtained are generally better.

Approximate method

Begin with the potential energy of the pendulum at the top of its swing:

\[
\Delta PE = M g \Delta h_{cm}
\]

Where M is the combined mass of pendulum and ball, g is the acceleration of gravity, and \( \Delta h_{cm} \) is the change in height. Substitute for the height:

\[
\Delta h_{cm} = R (1 - \cos \theta)
\]

\[
\Delta PE = M g R_{cm} (1 - \cos \theta)
\]

Here \( R_{cm} \) is the distance from the pivot point to the center of mass of the pendulum/ball system. This potential energy is equal to the kinetic energy of the pendulum immediately after the collision:

\[
KE = \frac{1}{2} M v_p^2
\]

The momentum of the pendulum after collision is just \( P_p = M v_p \), which we substitute into the previous equation to give:

\[
KE = \frac{P_p^2}{2M}
\]

Solving this equation for the pendulum momentum gives:

\[
P_p = \sqrt{2M(KE)}
\]
This momentum is equal to the momentum of the ball before the collision: \( P_b = m v_b \). Setting these two equations equal to each other and replacing KE with our known potential energy gives us:

\[
mv_b = \sqrt{2M^2gR_{cm}(1 - \cos \theta)}
\]

Solving this for the ball velocity and simplify to get \( v_b = \frac{M}{m} \sqrt{2gR_{cm}(1 - \cos \theta)} \).

\[\text{Figure 1} \quad \text{Figure 2}\]

**Exact Method**

The potential energy is found in a way identical to the way shown previously:

\[
\Delta PE = M g R_{cm} (1 - \cos \theta)
\]

For the kinetic energy, we use the equation for angular kinetic energy instead of linear, and substitute into it the equation for angular momentum.

\[
 KE = I \omega^2 / 2  \\
 L_p = I \omega  \\
 KE = \frac{L_p^2}{2I}
\]

Here \( I \) is the moment of inertia of the pendulum/ball combination, and \( \omega \) is the angular velocity immediately after the collision.

As we did previously, solve this last equation for angular momentum:

\[
 L_p = \sqrt{2I(KE)}
\]

This angular momentum is equal to the angular momentum of the ball before the collision, as measured from the pendulum pivot point:

\[
 L_b = m R_b^2 \omega = m R_b v
\]

\( R_b \) is the distance from the pendulum pivot to the ball. (This radius is not in general equal to
$R_{cm}$, which is the distance from the pivot point to the center of mass for the pendulum/ball system.) These two angular momentum are equal to each other, so:

$$mR_b v = \sqrt{2IMgR_{cm}(1 - \cos \theta)}$$

Solve for $v$:

$$v = \frac{1}{mR_b} \sqrt{2IMgR_{cm}(1 - \cos \theta)}$$

Now we need to find $I$, the moment of inertia of the pendulum and ball. To do this, we start with the rotational equivalent of Newton’s second law,

$$\tau = I \alpha$$

where $\tau$ is torque, $I$ is moment of inertia, and $\alpha$ is angular acceleration. The force on the center of mass of the pendulum is just $Mg$, and the component of that force directed towards the center of the pendulum swing is (see Fig. 2):

$$F = -Mg \sin \theta$$

The torque on the pendulum is thus:

$$I \alpha = -R_{cm} Mg \sin \theta$$

For small angles, $\sin \theta \approx \theta$, so if we make this substitution and solve for $\alpha$ we get:

$$\alpha \approx -\frac{R_{cm} Mg}{I} \theta$$

This angular equation is the same form as the equation for linear simple harmonic motion:

$$\alpha = -\frac{kx}{m} = -\omega^2 x$$

So if we compare these two equations, linear and angular, we can see that the pendulum exhibits simple harmonic motion. And that the square of the angular frequency ($\omega^2$) for this motion is just:

$$\omega^2 = \frac{MgR_{cm}}{I}$$

Solving this for $I$ gives us the desired result:

$$I = \frac{MgR_{cm}}{\omega^2} = \frac{MgR_{cm}T^2}{4\pi^2}$$

Where $T$ is the period of the pendulum.

Note: We have made a small-angle approximation to find this equation for $I$; but I does not depend on $\theta$. This means that we must measure the period $T$ using small oscillations; but once
we have calculated I with that period, we may use that value regardless of the amplitude reached during other parts of the experiment.

**Experiment 1: Projectile Range Versus Angle**

**Equipment Needed**

- Projectile launcher and plastic ball
- Plumb bob
- Measuring tape or meter stick
- Carbon paper
- Box to make elevation same as muzzle
- White paper
- Graph paper

**Purpose**

The purpose of this experiment is to find how the range of the ball depends on the angle at which it is launched. The angle that gives the greatest range is determined for two cases: for shooting on level ground and for shooting off a table.

**Theory**

The range is the horizontal distance, \( x \), between the muzzle of the launcher and the place where the ball hits, given by

\[
x = (v_0 \cos \theta) t,
\]

where \( v_0 \) is the initial speed of the ball as it leaves the muzzle, \( \theta \) is the angle of inclination above horizontal, and \( t \) is the time of flight. See Fig. 1.1.

![Figure 1.1 Shooting on a level surface](image)

For the case in which the ball hits on a place that is at the same level as the level of the muzzle of the launcher, the time of flight of the ball will be twice the time it takes the ball to reach the peak of its trajectory. At the peak, the vertical velocity is zero so

\[
v_y = 0 = v_0 \sin \theta - gt_{peak}
\]

Therefore, solving for the time gives that the total time of flight is

\[
t = 2t_{peak} = 2v_0 \sin \theta / g.
\]

For the case in which the ball is shot off at an angle off a table onto the floor (See Fig. 1.2), the time of flight is found using the equation for the vertical motion:

\[
y = y_0 + (v_0 \sin \theta)t - \frac{1}{2}(gt^2)
\]

where \( y_0 \) is the initial height of the ball and \( y \) is the position of the ball when it hits the floor.
Setup

1) Clamp the Projectile Launcher to a sturdy table near one end of the table with the launcher aimed so the ball will land on the table.
2) Adjust the angle of the Projectile Launcher to ten degrees.
3) Put the plastic ball into the projectile launcher and cock it to the medium or long range position.

Note: In general, this experiment will not work as well on the short range setting because the muzzle velocity is more variable with change angle.

Fire one shot to locate where the ball hits. Place a box at that location so the ball will hit at the same level as the muzzle of the launcher. See Fig. 1.3.

Procedure

Shooting on a level surface

1) Fire one shot to locate where the ball hits the box. At this position, tape a piece of white paper to the box, place a piece of carbon paper (carbon-side down) on top of this paper and face it down. When the ball hits the box, it will leave a mark on the white paper.
2) Fire three shots.
3) Use a plumb bob to find the point on the table that is directly beneath the release point on the barrel. Measure the horizontal distance along the table from the release point to the leading
edge of the paper. Record in Table 1.1.
4) Measure from the leading edge of the paper to each of the three dots and record these 
distances in Table 1.1.
5) Increase the angle by 10 degrees and repeat all the steps.
6) Repeat for angles up to and including 80 degrees.

**Shooting off the table**

Aim the projectile launcher so the ball will hit the floor. Repeat the procedure and record 
the data in Table 1.2.

**Analysis**

1) Find the average of the three distances in each case and record in Tables 1.1 and 1.2.
2) Add the average distance to the distance to the leading edge of the paper to find the total 
distance (range) in each case. Record in Tables 1.1 and 1.2.
3) For each data table, plot the range vs. angle and draw a smooth curve through the points.

**Experiment 2: Projectile Motion**

**Equipment needed**

- Projectile Launcher and plastic ball
- Plumb bob
- Meter stick
- Carbon paper
- White paper

**Purpose**

The purpose of this experiment is to predict and verify the range of a ball launched at an 
angle. The initial velocity of the ball is determined by shooting it horizontally and measuring the 
range and the height of the launcher.

**Theory**

To predict where a ball will land on the floor when it is shot off a table at some angle 
above the horizontal distances through which the ball travels. Then the initial velocity can be 
used to calculate where the ball land when the ball is shot at an angle.

**Horizontal initial velocity**

For a ball shot horizontally off a table with an initial speed, \( v_0 \), the horizontal distance 
traveled by the ball is given by \( x=v_0t \), where \( t \) is the time the ball is in the air. Air friction is 
assumed to be negligible.

The vertical distance the ball drops in time \( t \) is given by \( y = \frac{1}{2} gt^2 \)

The initial velocity of the ball can be determined by measuring \( x \) and \( y \). The time of flight 
of the ball can be found by using:

\[
t = \sqrt{\frac{2y}{g}}
\]
and then the initial velocity can be found using  \( v_0 = \frac{x}{t} \)

**Initial Velocity at an angle**

To predict the range, \( x \), of a ball shot off with an initial velocity at an angle, \( \theta \), above the horizontal, first predict the time of flight using the equation for the vertical motion:

\[
y = y_0 + (v_0 \sin \theta)t - \frac{1}{2} (gt^2)
\]

where \( y_0 \) is the initial height of the ball and \( y \) is the position of the ball when it hits the floor. Then use  \( x = (v_0 \cos \theta)t \) to find the range.

**Setup**

1) Clamp the projectile launcher to a sturdy table near one end of the table.
2) Adjust the angle of the projectile launcher to zero degrees so the ball will be shot off horizontally.

**Procedure**

**Part A: Determining the Initial Velocity of the Ball**

1) Put the plastic ball into the projectile launcher and cock it to the long range position. Fire one shot to locate where the ball hits the floor. At this position, tape a piece of white paper to the floor. Place a piece of carbon paper (carbon-side down) on top of this paper and tape it down. When the ball hits the floor, it will leave a mark on the white paper.
2) Fire five shots.
3) Measure the vertical distance from the bottom of the ball as it leaves the barrel (this position is marked on the side of the barrel) to the floor. Record this distance in Table 2.1.
4) Use a plumb bob to find the point on the floor that is directly beneath the release point on the barrel. Measure the horizontal distance along the floor from the release point to the leading edge of the paper. Record in Table 2.1.
5) Measure from the leading edge of the paper to each of the five dots and record these distances in Table 2.1.
6) Find the average of the five distances and record in Table 2.1
7) Using the vertical distance and the average horizontal distance, calculate the time of flight and the initial velocity of the ball. Record in Table 2.1.

**Part B: Predicting the Range of the Ball Shot at an Angle**

1) Adjust the angle of the Projectile Launcher to an angle between 30 and 60 degrees and record this angle in Table 2.2.
2) Using the initial velocity and vertical distance found in the first part of this experiment, assume the ball is shot off at the new angle you have just selected and calculate the new time of flight and the new horizontal distance. Record in Table 2.2.
3) Draw a line across the middle of a white piece of paper and tape the paper on the floor so the line is at the predicted horizontal distance from the Projectile Launcher. Cover the paper with carbon paper.
4) Shoot the ball five times.

5) Measure the five distances and take the average. Record in Table 2.2.

**Analysis**

1) Calculate the difference between the predicted value and the resulting average distance when shot at an angle.

2) Estimate the precision of the predicted range. How many of the final 10 shots landed within this range? Precision is usually specified as 2 standard deviations, so that 95.4% of the shots will fall within this range. (Hint: Determine the uncertainties of the initial velocity, $v_0$ and the launching angle, $\theta$, then calculate the uncertainty of $x$).

**Experiment 3: Projectile velocity – Exact Method**

**Equipment needed**

<table>
<thead>
<tr>
<th>Launcher</th>
<th>Steel ball</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clamp (optional)</td>
<td>Mass balance</td>
</tr>
<tr>
<td>String</td>
<td>Ruler</td>
</tr>
<tr>
<td>Stopwatch</td>
<td></td>
</tr>
</tbody>
</table>

**Purpose**

The muzzle velocity of the projectile launcher is determined by shooting the ball into the pendulum and observing the angle to which the pendulum swings. The exact equation for ball velocity, as derived earlier in this manual, is

$$v = \frac{1}{mR_b} \sqrt{2IMgR_{cm}(1 - \cos \theta)}$$

where $M$ is the mass of the pendulum and ball combined, $m$ is the mass of the ball, $g$ is the acceleration of gravity, $R_{cm}$ is the distance from the pivot to the center of mass of the pendulum, $R_b$ is the distance from the pivot to the ball, $\theta$ is the angle reached by the pendulum, and $I$ is the moment of inertia of the pendulum with the ball in the catcher.

The value of $I$ can be found by measuring the period of small oscillations of the pendulum and ball and using the equation

$$I = \frac{MgR_{cm}T^2}{4\pi^2}$$

where $T$ is the period.

**Setup**

1) Attach the projectile launcher to the ballistic pendulum mount at the level of the ball catcher.
2) Clamp the pendulum base to the table, if a clamp is available. Make sure that the pendulum can hang vertically without touching the launcher.

**Procedure**

1) Latch the pendulum at $90^\circ$ so it is out of the way, then load the projectile launcher. Allow the pendulum to hang freely, and move the angle indicator to zero degrees.
2) Fire the launcher and record the angle reached. If you want to do the experiment with a lower or higher angle, add or remove mass to the pendulum. Repeat these test measurements until you are satisfied with the mass of the pendulum.

3) Once you have chosen the mass to use for your experiment, remove the pendulum from the base by unscrewing and removing the pivot axle. Using the mass balance, find the mass of the pendulum and the ball together. Record this value as M in Table 3.1.

4) Measure the mass of the ball, and record this as m.

5) Tie a loop in the string, and hang the pendulum from the loop. (See Fig. 3.1) with the ball latched in position in the ball catcher, adjust the position of the pendulum in this loop until it balances. Measure the distance from the pivot point to the balance point, and record it as \( R_{cm} \). You may find it easier to do this by balancing the pendulum on the edge of a ruler or similar object.

6) Measure the distance between the pivot point and the center of the ball. Record this as \( R_b \).

7) Replace the pendulum in the base, making sure that it is facing the right way. Be sure that the angle indicator is to the right of the pendulum rod.

8) Remove the launcher so that the pendulum can swing freely. With the ball in the pendulum, give it an initial displacement of 5° or less. Using the stopwatch, time how long it takes to go through at least ten oscillations. Divide this time by the number of oscillations, and record your result as \( T \) in Table 3.1.

9) Calculate the value of I, and record it in Table 3.1.

10) Load the launcher, then set the angle indicator to an angle 1-2° less than that reached in step 2. This will nearly eliminate the drag on the pendulum caused by the indicator, since the pendulum will only move the indicator for the last few degrees.

11) Fire the launcher, and record the angle reached by the pendulum in Table 3.1. Repeat this several times, setting the angle indicator to a point 1-2° below the previous angle reached by the pendulum each time.

**Calculations**

1) Find the average angle reached by the pendulum. Record this value in Table 3.1.

2) Calculate the muzzle velocity of the projectile launcher.