Kavli Institute for Theoretical Sciences
University of Chinese Academy of Sciences (UCAS)

Unconventional SC & Non-Fermi liquids
-- Eliashberg theory

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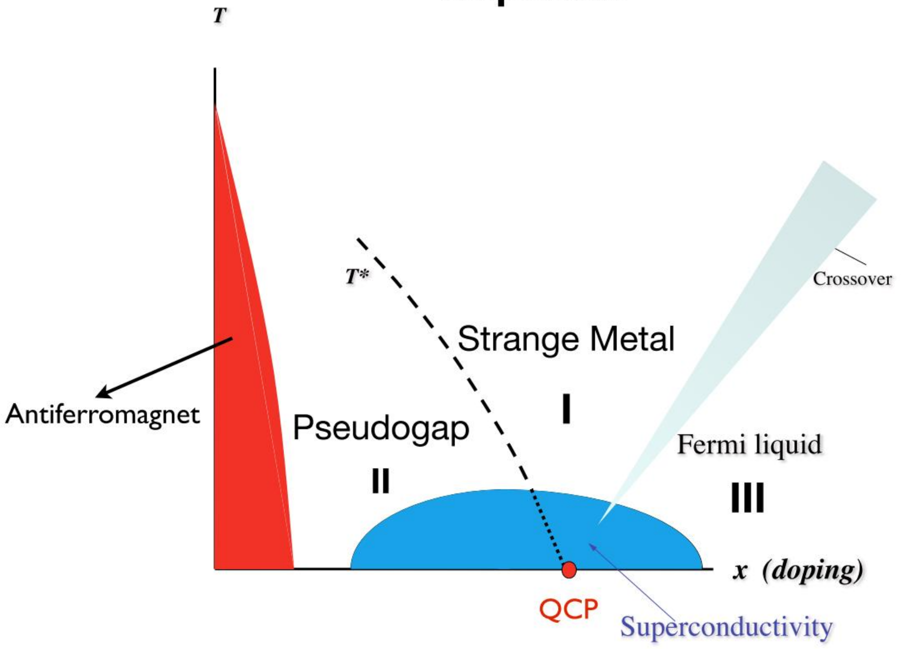
UCAS, KITS

2022-12-16

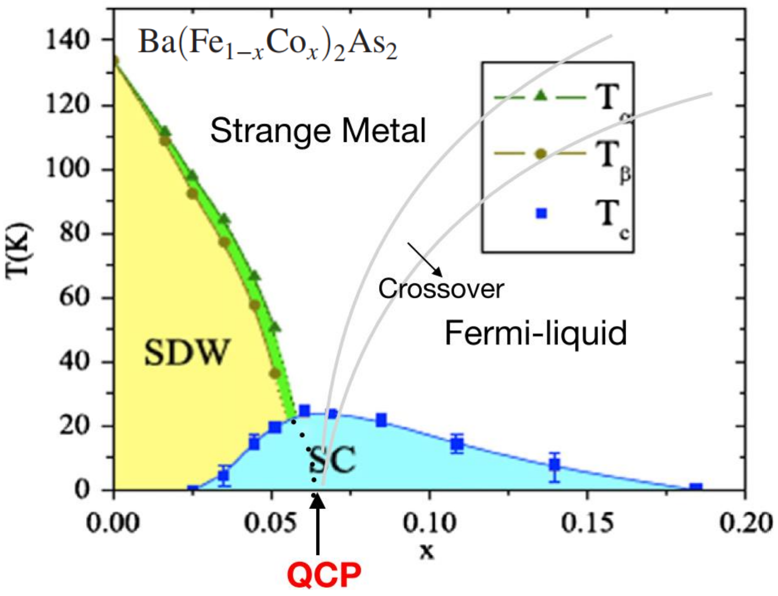
Unconventional superconductivity v.s. Non-Fermi liquids

Cuprates



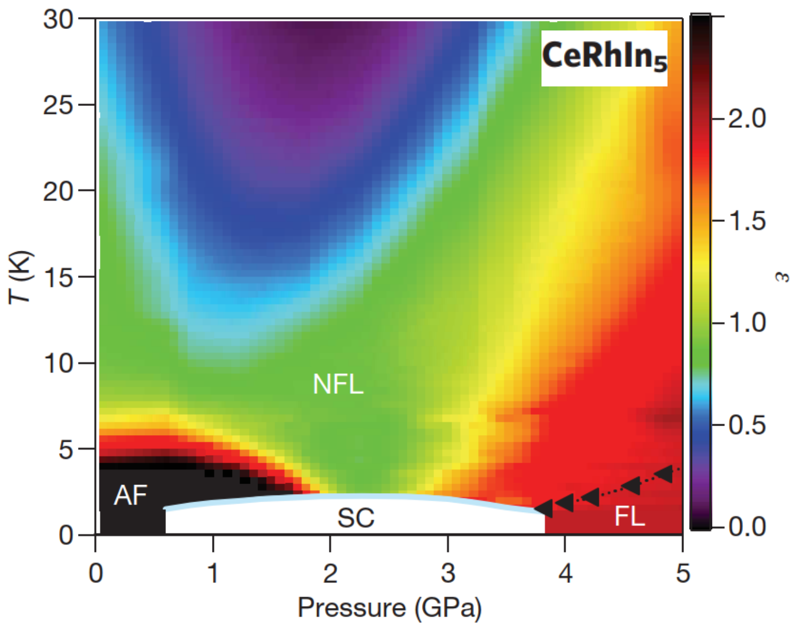
C. Varma, RMP **92**, 031001 (2020)

An Fe-based compound



J. –H. Chu et al., PRB **79**, 014506 (2009)

Heavy-fermion SC



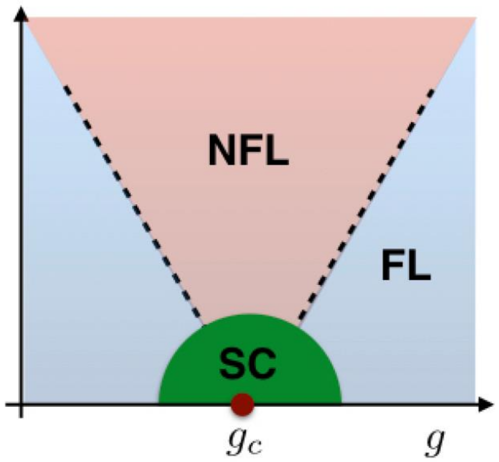
T. Park et al., Nature **456**, 366 (2008).

- **Unconventional SC usually emerges out near the QCP and accompanying with non-Fermi liquid behaviors.**
- **Non-Fermi liquid (NFL)**: Anomalous power-law metallic transport properties deviating from the Landau's Fermi liquid theory ($\Delta\rho \propto T^2$, $C_V \propto T$).

Outline

Part 1 (this week)

- **I. Introduction to unconventional superconductivity**
- **II. Eliashberg theory and its applications**
- **III. Non-Fermi-liquids & advance in its pairing**



Part 2 (next week)

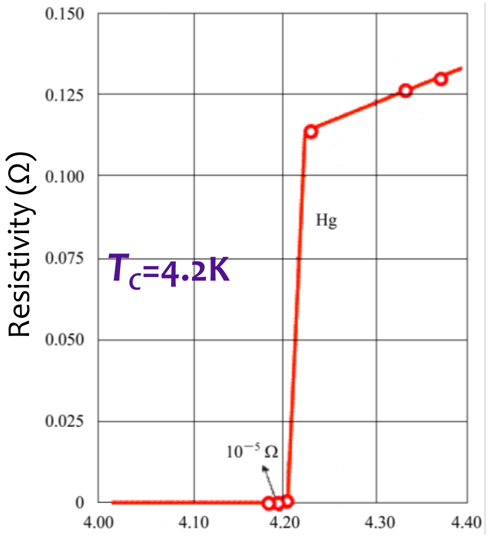
- **Two-stage SC in Hatsugai-Kohmoto-BCS model**

I. Introduction to unconventional SC

- Phenomenology of conventional & unconventional SC
- Generalized BCS theory & Symmetry classification of pairing symmetry

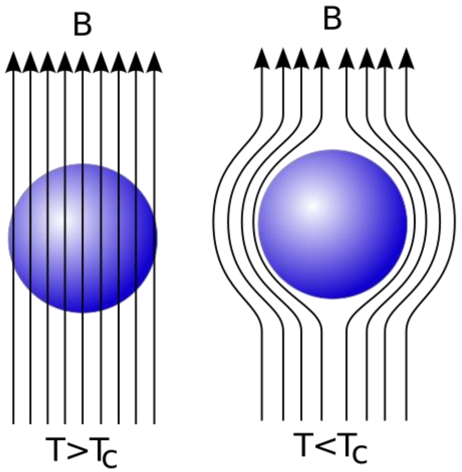
Superconductivity & BCS theory

zero resistance



(H. K. Onnes, 1911)

Meissner effect (perfect diagmanetism)

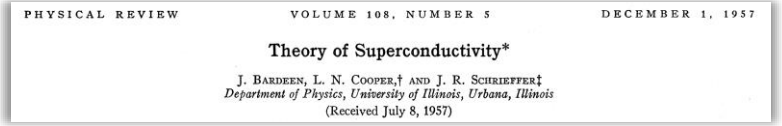


(F. W. Meissner, R. Ochsenfeld, 1933)

BCS theory (1957)

The Nobel Prize in Physics 1972

John Bardeen, Leon N. Cooper, Robert Schrieffer



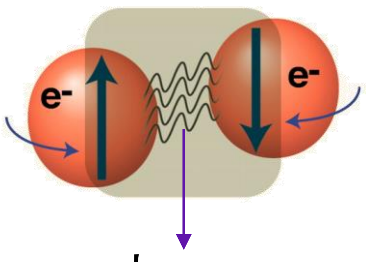
PHYSICAL REVIEW VOLUME 108, NUMBER 5 DECEMBER 1, 1957

Theory of Superconductivity*

J. Bardeen, L. N. Cooper,† and J. R. Schrieffer‡
Department of Physics, University of Illinois, Urbana, Illinois
(Received July 8, 1957)

(J. Bardeen, L. N. Cooper, J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957))

Cooper pair



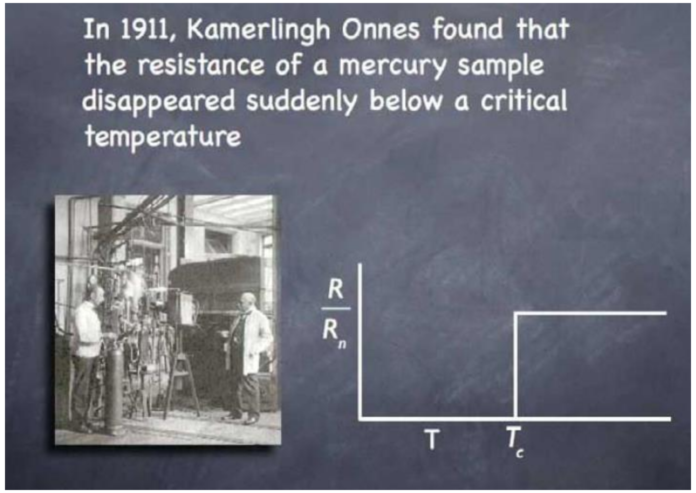
$$|\psi\rangle = \prod_{|\vec{k}|\leq k_F} \{u_{\vec{k}} + v_{\vec{k}} c^{\dagger}_{\vec{k}\uparrow} c^{\dagger}_{-\vec{k}\downarrow}\} |0\rangle$$

Superconductivity = zero resistance + Meissner effect = a macroscopic condensate resulted from phase coherence of Cooper pairs.

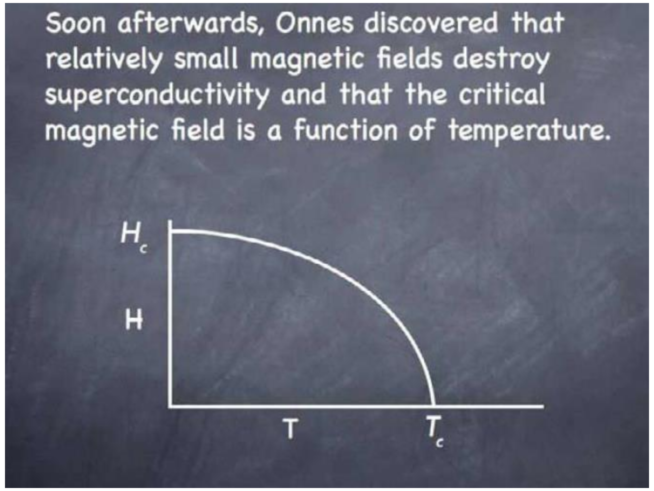
- The electrons with opposite momentum and opposite spin pair together under the attractive interaction, which formed a pair bound state, i.e., ***Cooper pair***.
- In conventional SCs, the effective **attractive interaction** induced by the electron-phonon coupling.

Phenomenology of conventional SC

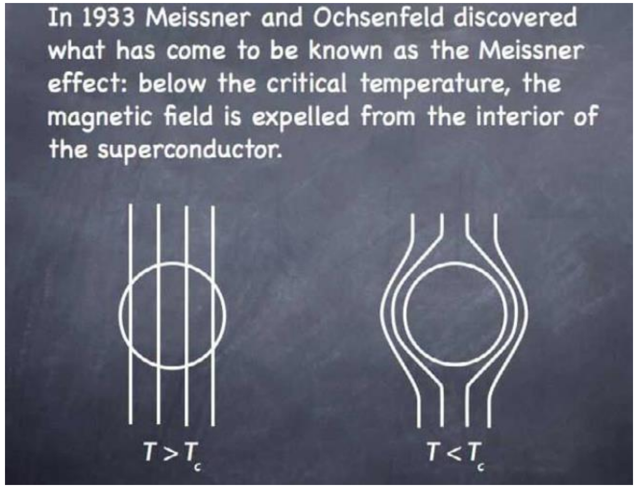
zero resistance



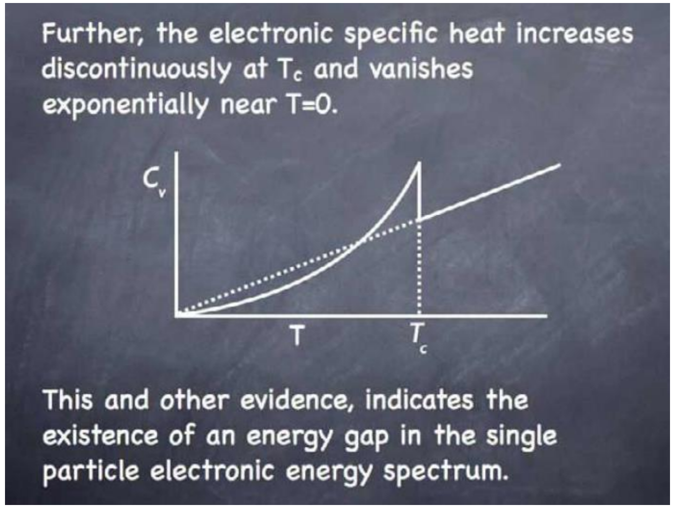
Upper critical field



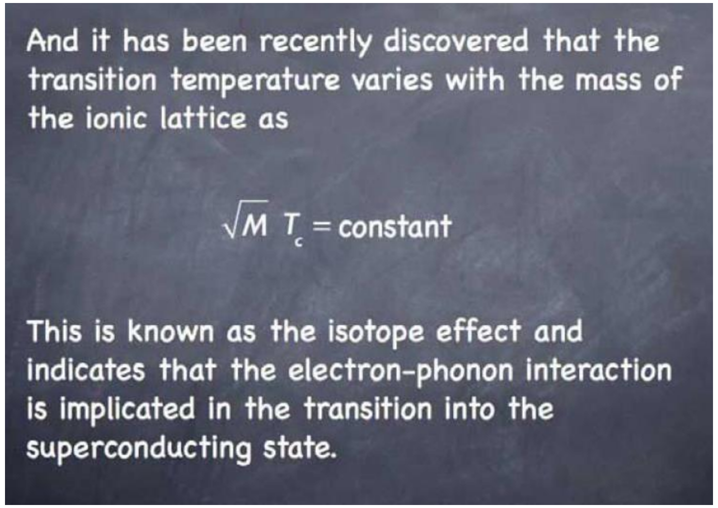
Meissner effect

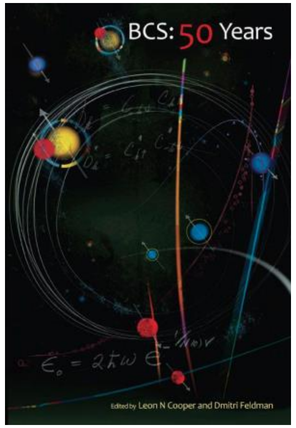


Specific heat



Isotope effect





(L. N. Cooper and D. Feldman (eds.), *BCS: 50 years* (2011).)

Elemental superconductors

Periodic Table of Superconductivity

(dedicated to the memory of Bernd Matthias; compiled by James S. Schilling)

30 elements superconduct at ambient pressure, 23 more superconduct at high pressure.

ambient pressure superconductor: T_c(K), T_c^{max}(K), P(GPa)

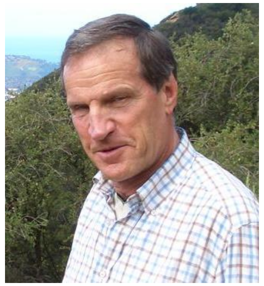
high pressure superconductor: T_c^{max}(K), P(GPa)

H																	He
Li 0.0004 14 30	Be 0.026 3.7 30											B 11 250	C	N	O 0.6 100	F	Ne
Na	Mg											Al 1.14	Si 8.2 15.2	P 13 30	S 17.3 190	Cl	Ar
K	Ca 29 217	Sc 19.6 106	Ti 0.39 3.35 56.0	V 5.38 16.5 120	Cr	Mn	Fe 2.1 21	Co	Ni	Cu	Zn 0.875	Ga 1.091 7 1.4	Ge 5.35 11.5	As 2.4 32	Se 8 150	Br 1.4 100	Kr
Rb	Sr 7 50	Y 19.5 115	Zr 0.546 11 30	Nb 9.50 9.9 10	Mo 0.92	Tc 7.77	Ru 0.51	Rh .00033	Pd	Ag	Cd 0.56	In 3.404	Sn 3.722 5.3 11.3	Sb 3.9 25	Te 7.5 35	I 1.2 25	Xe
Cs 1.3 12	Ba 5 18	insert La-Lu	Hf 0.12 8.6 62	Ta 4.483 4.5 43	W 0.012	Re 1.4	Os 0.655	Ir 0.14	Pt	Au	Hg-α 4.153	Tl 2.39	Pb 7.193	Bi 8.5 9.1	Po	At	Rn
Fr	Ra	insert Ac-Lr	Rf	Ha													

La-fcc 6.00 13 15	Ce 1.7 5	Pr	Nd	Pm	Sm	Eu 2.75 142	Gd	Tb	Dy	Ho	Er	Tm	Yb	Lu 12.4 174
Ac	Th 1.368	Pa 1.4	U 0.8(β) 2.4(α) 1.2	Np	Pu	Am 0.79 2.2 6	Cm	Bk	Cf	Es	Fm	Md	No	Lr

M. Debessai, T. Matsuoka, J.J. Hamlin, W. Bi, Y. Meng, K. Shimizu, and J.S. Schilling, J. Phys.: Conf. Series **215**, 012034 (2010).
High pressure data for Ca and Be: K. Shimizu email from 9 Dec 2013.

$CeCu_2Si_2$： 1^{st} unconventional superconductor (1979)



F. Steglich

VOLUME 43, NUMBER 25 PHYSICAL REVIEW LETTERS 17 DECEMBER 1979

Superconductivity in the Presence of Strong Pauli Paramagnetism: $CeCu_2Si_2$

F. Steglich
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and

J. Aarts, C. D. Bredl, W. Lieke, D. Meschede, and W. Franz
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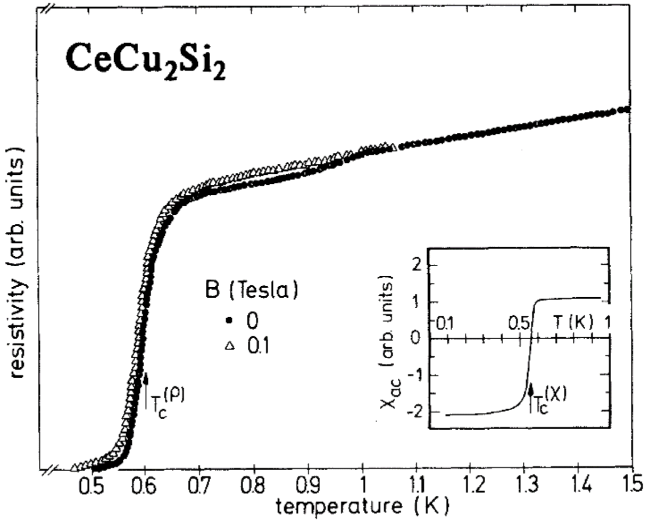
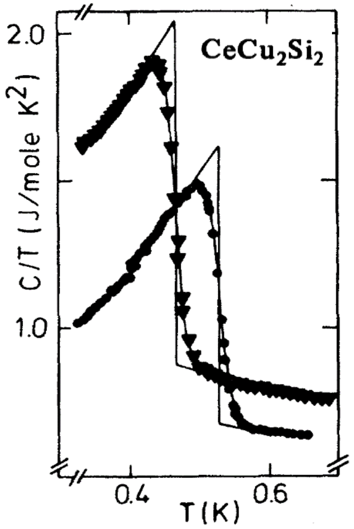
(Received 10 August 1979; revised manuscript received 7 November 1979)

The size of the specific-heat jump at T_c, in proportion to γT_c, suggests that Cooper-pair states are formed by these heavy fermions. Since the Debye temperature, Θ, is of the order of 200 K,[5] we find $T_c < T_F < \Theta$ with $T_c/T_F \simeq T_F/\Theta \simeq 0.05$. This suggests that $CeCu_2Si$ (i) behaves as a "high-temperature superconductor" and (ii) cannot be described by conventional theory of superconductivity which assumes a typical phonon frequency $k_B\Theta/h \ll k_BT_F/h$, the characteristic frequency of the fermions.





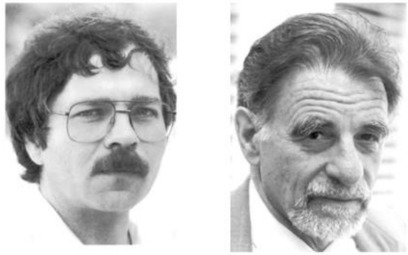
- The 1^{st} unconventional superconductor.
- Specific heat jump $(\Delta C/\gamma T)_{T_c} \sim O(1)$: heavy quasiparticle pairing.
- $T_c/T_F \approx 0.05$, much higher than BCS case ($\sim 10^{-5}$): high-density participation in SC.
- Debye temperature $\Theta > T_F$: beyond BSC theory (in BCS case, $\Theta/T_F \ll 1$).

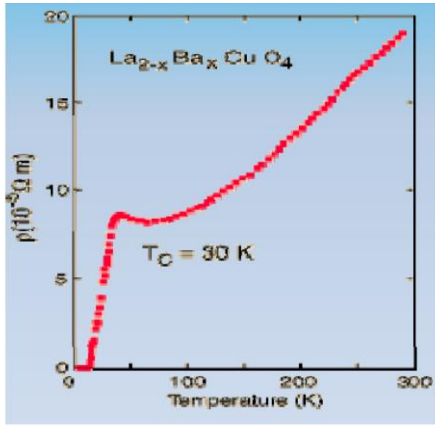
F. Steglich *et al*, Phys. Rev. Lett. **43**, 1892 (1979)

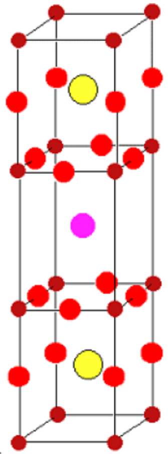
Cuprates & Iron-pnictides

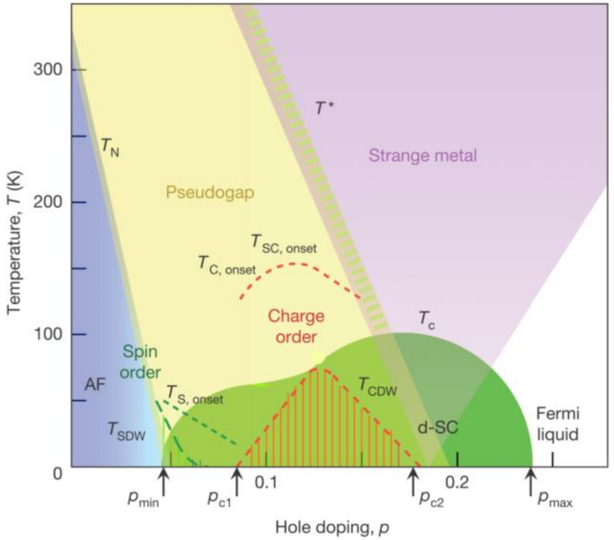
Cuprates (1986)

The Nobel Prize in Physics 1987
J. Georg Bednorz, K. Alex Müller







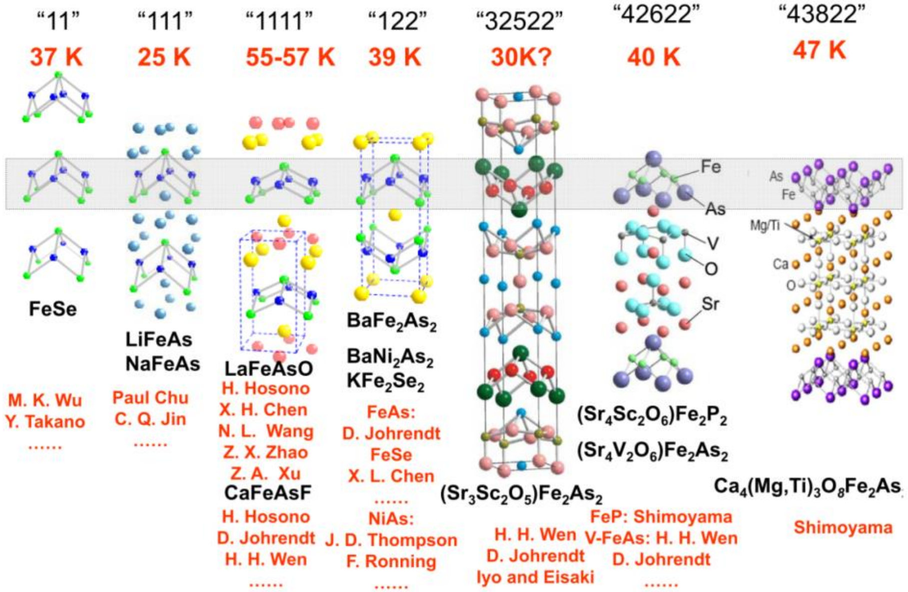


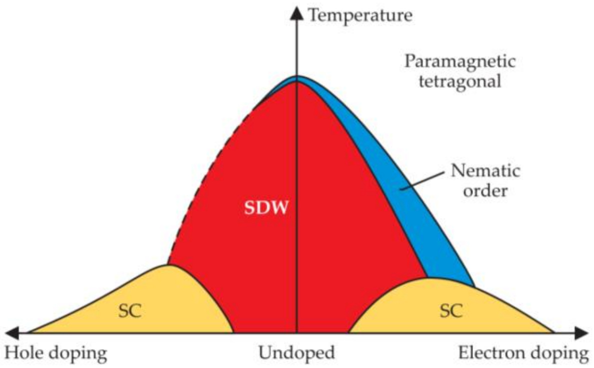
B. Keimer, *et al*, nature, **518**, 179 (2015)

- d-wave SC
- **Various competing orders:** Pseudogap? CDW? SDW? PDW? Nematicity?...
- **Pairing mechanism:** spin-fluctuation? RVB? Loop-current? Polaron?...
- ……

Iron-pnictides (2008)

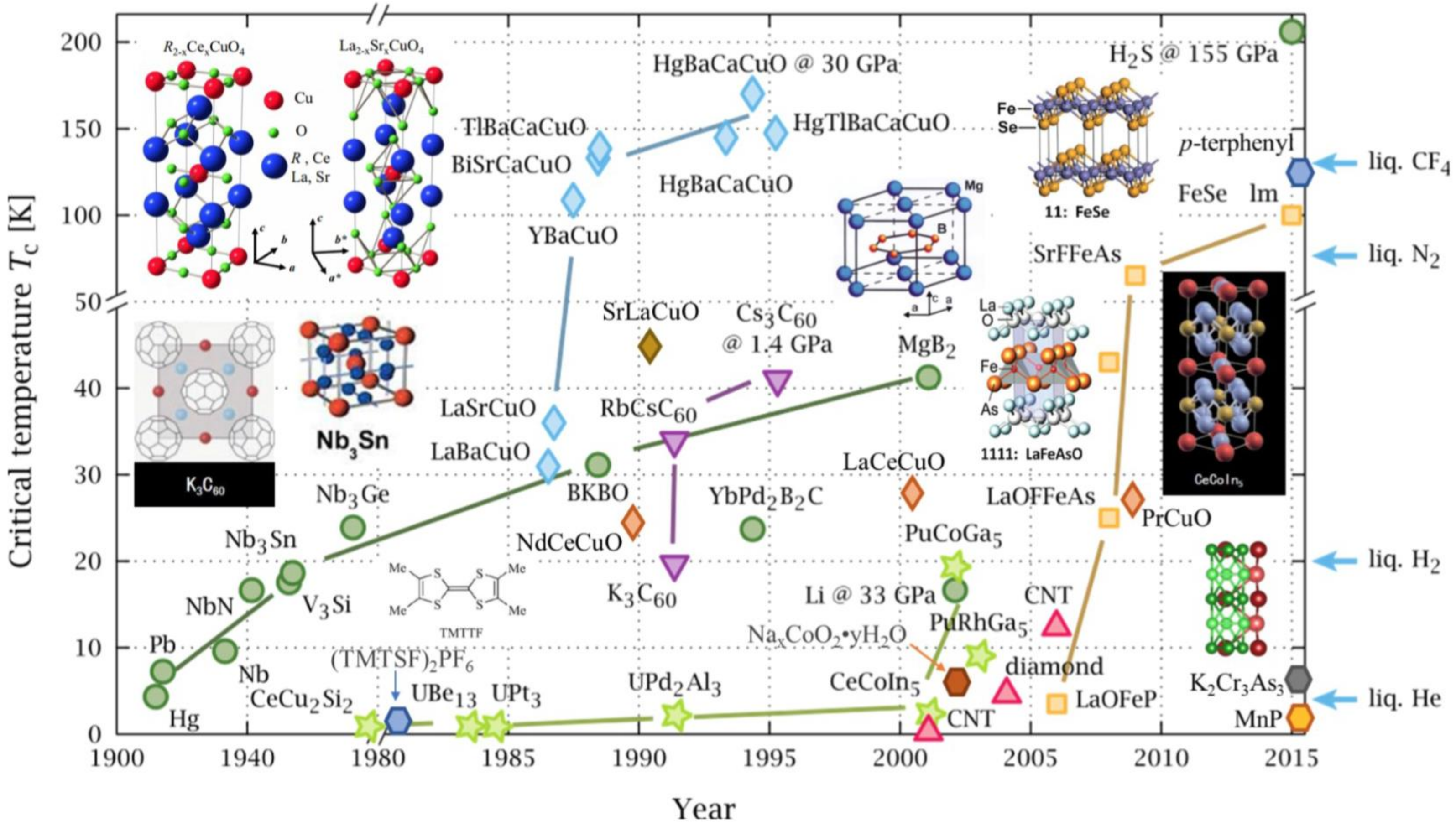
H. Hosono





- $s^{\pm}$-wave SC
- **Competing orders:** SDW, Nematicity, ...
- **Pairing mechanism:** spin-fluctuation? Orbital-selective?...
- ……

Discovery history of typical superconductors



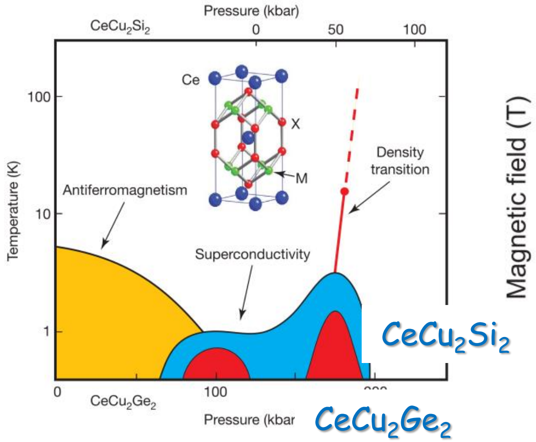
- Twisted bilayer/trilayer/double-bilayer graphene
- Nickelates
- Kagome SCs
- ……

Conventional SC v.s. Unconventional SC

	Conventional SC	Unconventional SC
Normal states	**Metal** (Fermi liquid)	**Bad metal** (Non-Fermi liquid)
Pairing mechanism	**Electron-phonon interaction**	**No consensus (Strong electron-electron interaction)**
Pairing symmetry	**s-wave (BCS)**	**d_{x2-y2}-wave (Cuprates, $CeCoIn_5$, …)** **$s^{\pm}$-wave (Iron-pnictides)** **p-wave (Sr_2RuO_4?, …)**
Dimensionality	3D	Usually quasi-2D
Relation to magnetism	**No magnetism (exclusive)**	SC usually emerging near the AFM boundary, and sometimes can **coexist**.
$\Delta(k)$	**s** 	**d_{x2-y2}**  **$s^{\pm}$**  **p_x** 

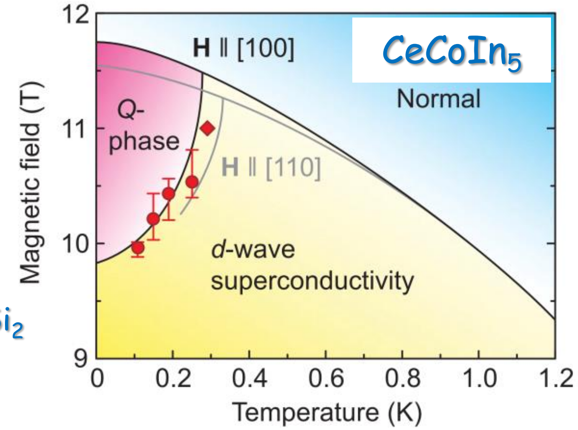
Diversity of phase diagrams in HF SCs

2 SC domes



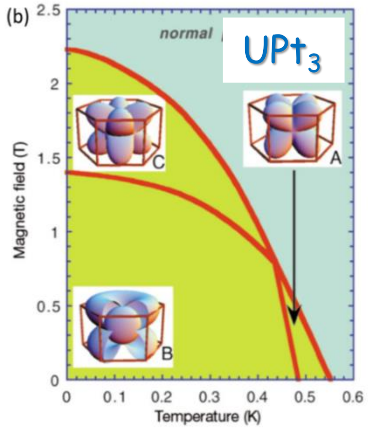
P. Monthoux *et al*, nature **450**, 1177 (2007)

high field-induced Q-phase



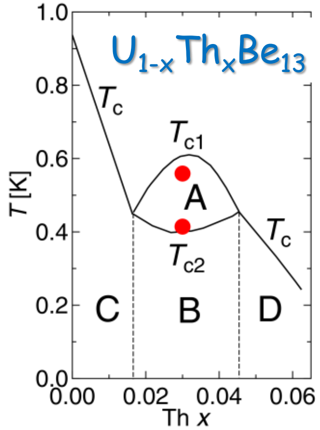
D. Y. Kim et al, Phys. Rev. X **6**, 041059 (2016)

Multiple SCs



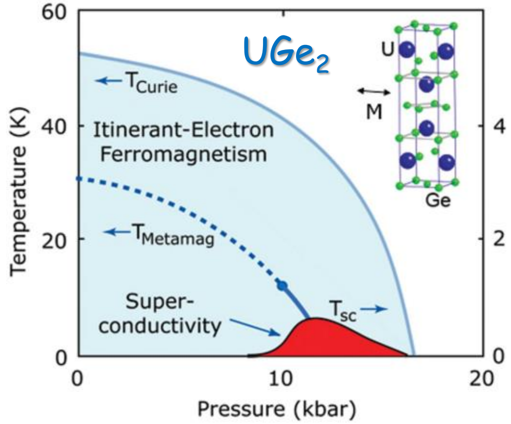
A. Huxley *et al*, nature **406**, 160 (2000)

Multiple SCs



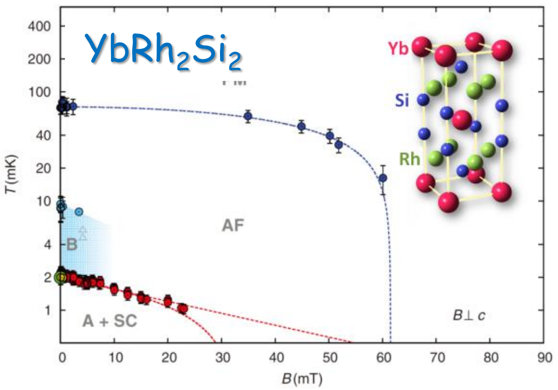
Y. Shimizu *et al*, Phys. Rev. B **96**, 100505(R) (2017)

Ferromagnetic SC



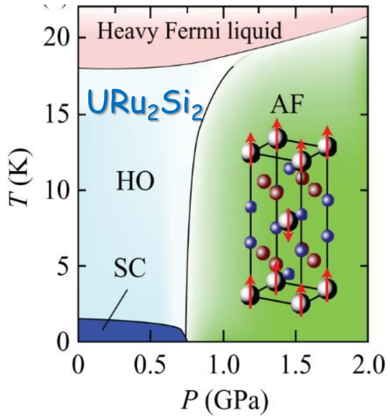
P. Monthoux *et al*, nature **450**, 1177 (2007)

Ultra-low SC (2mK)



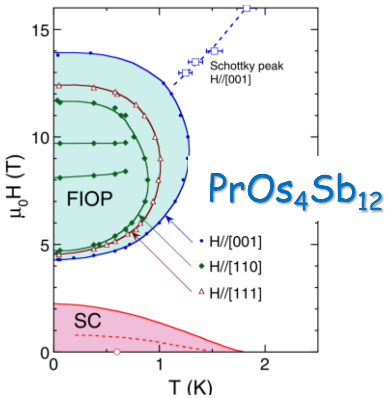
E. Schuberth *et al*, Science **351**, 485 (2016)

Hidden order



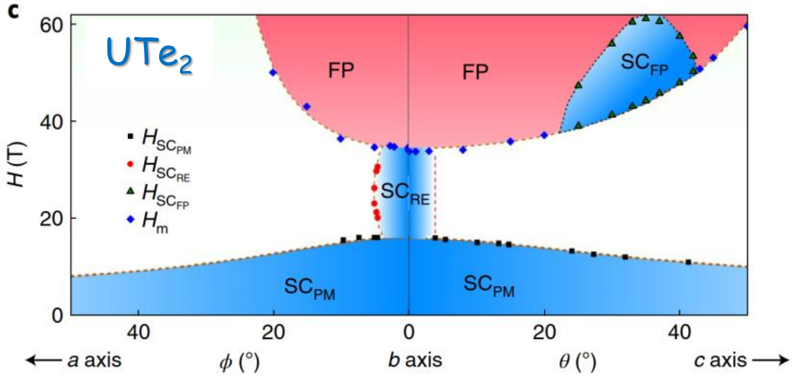
Y. Kasahara *et al*, New J. Phys. **11**, 055061 (2009)

Field-induce quadrupolar phase



Y. Aoki *et al*, J. Phys. Soc. Jpn. **75**, 051006 (2007)

Field-boosted SC



S. Ran et al. Nat. Phys. **15**, 1250 (2019)

… …

杨义峰, 李宇, 物理学报 **64**, 217401 (2015)

李宇, 盛玉韬, 杨义峰, 物理学报 **70**, 017402 (2021)

Generalized BCS mean-field theory

- Hamiltonian with generalized pairing interaction $H=\sum_{\boldsymbol{k},\sigma}\xi_{\boldsymbol{k}}c^{\dagger}_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}\sigma}+\frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{k}'}\sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}V^{\sigma_1\sigma_2\sigma_3\sigma_4}_{\boldsymbol{k},\boldsymbol{k}'}c^{\dagger}_{\boldsymbol{k}\sigma_1}c^{\dagger}_{-\boldsymbol{k}\sigma_2}c_{-\boldsymbol{k}'\sigma_3}c_{\boldsymbol{k}'\sigma_4},$
- Introduce the mean-field $\langle c_{-\boldsymbol{k}\sigma_1}c_{\boldsymbol{k}\sigma_2}\rangle$,

$$H=\sum_{\boldsymbol{k},\sigma}\xi_{\boldsymbol{k}}c^{\dagger}_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}\sigma}-\frac{1}{2}\sum_{\boldsymbol{k}}\sum_{\sigma_1,\sigma_2}\left(\Delta^{\sigma_1\sigma_2}_{\boldsymbol{k}}c^{\dagger}_{\boldsymbol{k}\sigma_1}c^{\dagger}_{-\boldsymbol{k}\sigma_2}+\Delta^{\sigma_1\sigma_2*}_{\boldsymbol{k}}c_{-\boldsymbol{k}\sigma_2}c_{\boldsymbol{k}\sigma_1}\right)+K,$$

where $K=-\frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{k}'}\sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}\left\langle c^{\dagger}_{\boldsymbol{k}\sigma_1}c^{\dagger}_{-\boldsymbol{k}\sigma_2}\right\rangle\langle c_{-\boldsymbol{k}'\sigma_3}c_{\boldsymbol{k}'\sigma_4}\rangle.$

- Define the SC gap function $\Delta^{\sigma_1\sigma_2}_{\boldsymbol{k}}=-\sum_{\boldsymbol{k}'}\sum_{\sigma_3,\sigma_4}V^{\sigma_1\sigma_2\sigma_3\sigma_4}_{\boldsymbol{k},\boldsymbol{k}'}\langle c_{-\boldsymbol{k}'\sigma_3}c_{\boldsymbol{k}'\sigma_4}\rangle.$
- Writing in the matrix form and expanding in Pauli's matrices

$$\hat{\Delta}_{\boldsymbol{k}}=\begin{pmatrix}\Delta^{\uparrow\uparrow}_{\boldsymbol{k}} & \Delta^{\uparrow\downarrow}_{\boldsymbol{k}}\\ \Delta^{\downarrow\uparrow}_{\boldsymbol{k}} & \Delta^{\downarrow\downarrow}_{\boldsymbol{k}}\end{pmatrix}=\left[\left(\frac{\Delta^{\uparrow\downarrow}_{\boldsymbol{k}}-\Delta^{\downarrow\uparrow}_{\boldsymbol{k}}}{2}\right)\hat{\sigma}_0+\left(\frac{-\Delta^{\uparrow\uparrow}_{\boldsymbol{k}}+\Delta^{\downarrow\downarrow}_{\boldsymbol{k}}}{2}\right)\hat{\sigma}_x+\left(-i\frac{\Delta^{\uparrow\uparrow}_{\boldsymbol{k}}+\Delta^{\downarrow\downarrow}_{\boldsymbol{k}}}{2}\right)\hat{\sigma}_y+\left(\frac{\Delta^{\uparrow\downarrow}_{\boldsymbol{k}}+\Delta^{\downarrow\uparrow}_{\boldsymbol{k}}}{2}\right)\hat{\sigma}_z\right]i\hat{\sigma}_y=(\phi_{\boldsymbol{k}}\hat{\sigma}_0+\boldsymbol{d}_{\boldsymbol{k}}\cdot\boldsymbol{\sigma})\,i\hat{\sigma}_y,$$

$$\hat{\Delta}_{\boldsymbol{k}}=\begin{cases}\phi_{\boldsymbol{k}}i\hat{\sigma}_y, & \text{(spin-singlet pairing)}\\ (\boldsymbol{d}_{\boldsymbol{k}}\cdot\boldsymbol{\sigma})\,i\hat{\sigma}_y, & \text{(spin-triplet pairing)}\end{cases}$$

Pauli's exclusion principle ($\hat{\Delta}_{\boldsymbol{k}}=-\hat{\Delta}^{T}_{-\boldsymbol{k}}$) → $\phi_{\boldsymbol{k}}=\phi_{-\boldsymbol{k}}$, (even-parity) $\boldsymbol{d}_{\boldsymbol{k}}=-\boldsymbol{d}_{-\boldsymbol{k}}$. (odd-parity)

- In Nambu's basis, $\Psi^{\dagger}_{\boldsymbol{k}}=\left(c^{\dagger}_{\boldsymbol{k}\uparrow},\ c^{\dagger}_{\boldsymbol{k}\downarrow},\ c_{-\boldsymbol{k}\uparrow},\ c_{-\boldsymbol{k}\downarrow}\right)$,

$$H=\frac{1}{2}\sum_{\boldsymbol{k}}\Psi^{\dagger}_{\boldsymbol{k}}\begin{pmatrix}\xi_{\boldsymbol{k}}\hat{\sigma}_0 & \hat{\Delta}_{\boldsymbol{k}}\\ -\hat{\Delta}^{*}_{-\boldsymbol{k}} & -\xi_{\boldsymbol{k}}\hat{\sigma}_0\end{pmatrix}\Psi_{\boldsymbol{k}}+K.$$

Unitary transformation $\Psi_{\boldsymbol{k}}=\hat{U}_{\boldsymbol{k}}\Phi_{\boldsymbol{k}}$, $\hat{U}_{\boldsymbol{k}}=\begin{pmatrix}u_{\boldsymbol{k}}\hat{\sigma}_0 & \hat{v}_{\boldsymbol{k}}\\ \hat{v}^{*}_{-\boldsymbol{k}} & u^{*}_{-\boldsymbol{k}}\hat{\sigma}_0\end{pmatrix}$ →

$$H=\frac{1}{2}\sum_{\boldsymbol{k}}\Phi^{\dagger}_{\boldsymbol{k}}\begin{pmatrix}\hat{E}_{\boldsymbol{k}} & 0\\ 0 & -\hat{E}_{-\boldsymbol{k}}\end{pmatrix}\Phi_{\boldsymbol{k}}+K,$$

- **Generalized gap equation**

$$\Delta^{\sigma_1\sigma_2}_{\boldsymbol{k}}=-\sum_{\boldsymbol{k}'}\sum_{\sigma_3,\sigma_4}V^{\sigma_1\sigma_2\sigma_3\sigma_4}_{\boldsymbol{k},\boldsymbol{k}'}\frac{\tanh\left(\frac{\beta E_{\boldsymbol{k}'}}{2}\right)}{2E_{\boldsymbol{k}'}}\Delta^{\sigma_3\sigma_4}_{\boldsymbol{k}'}.$$

where $E_{\boldsymbol{k}}=\sqrt{\xi^2_{\boldsymbol{k}}+\frac{1}{2}\mathrm{tr}\left(\hat{\Delta}_{\boldsymbol{k}}\hat{\Delta}^{\dagger}_{\boldsymbol{k}}\right)},$

Symmetry classification of pairing symmetry (Δ_k)

- **Symmetry breaking in superconductors**

$$\mathcal{G} = G \otimes SO(3) \otimes U(1) \otimes T,$$

Crystal point group — Spin rotation group — Gauge phase group — Time-reversal symmetry group

Only break U(1) symmetry: conventional SC
s-wave (A_{1g})

Break U(1)+other symmetries: unconventional SC
d-wave (Cuprates, $CeCoIn_5$), p-wave (UTe_2?), d+id-wave (URu_2Si_2), ……

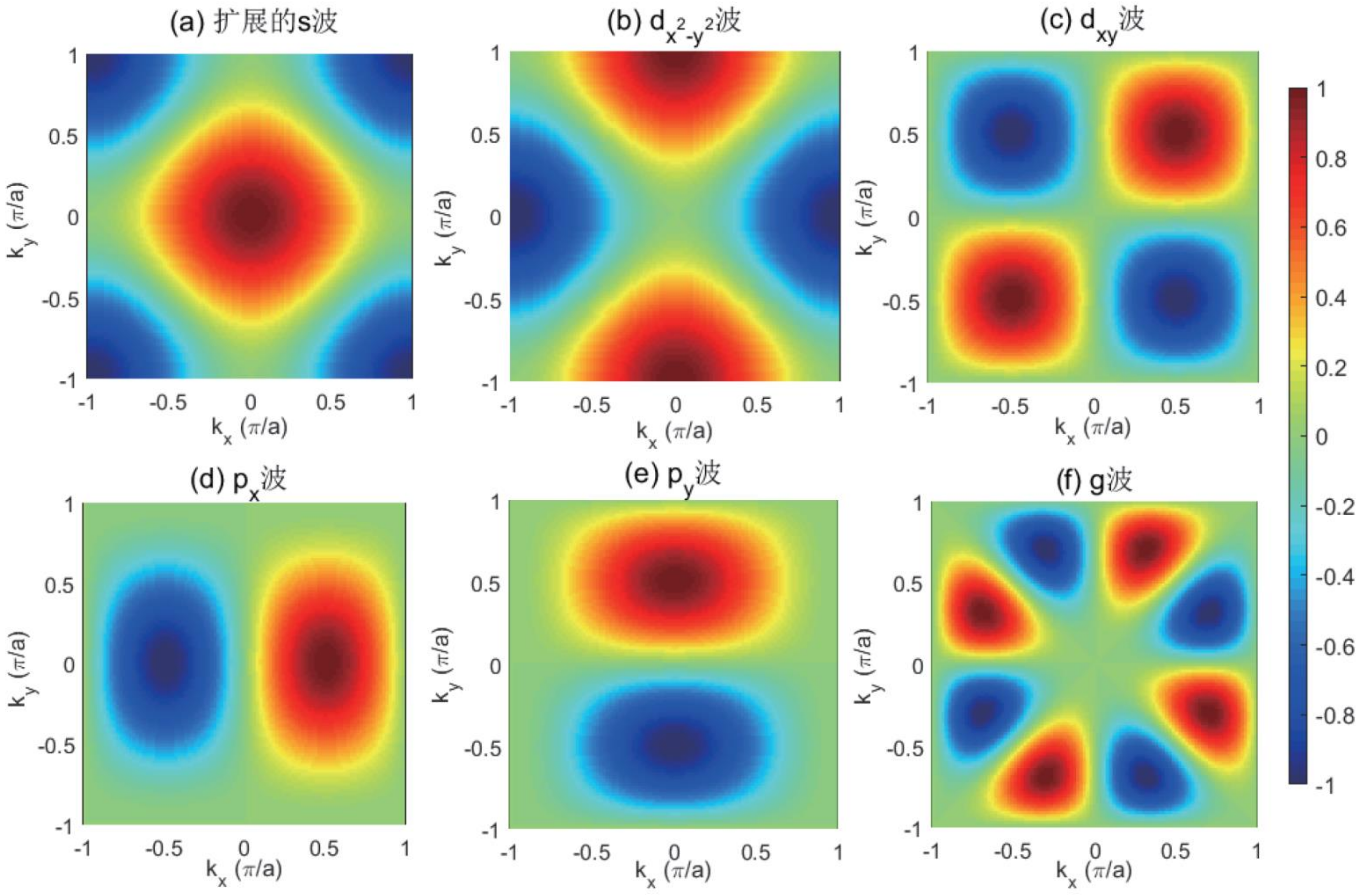
	even parity	odd parity
Fermion exchange	$\psi(\boldsymbol{k}) = \psi(-\boldsymbol{k})$	$\boldsymbol{d}(\boldsymbol{k}) = -\boldsymbol{d}(-\boldsymbol{k})$
Orbital rotation	$\hat{g}\psi(\boldsymbol{k}) = \psi(R_g\boldsymbol{k})$	$\hat{g}\boldsymbol{d}(\boldsymbol{k}) = \boldsymbol{d}(R_g\boldsymbol{k})$
Spin rotation	$\hat{g}\psi(\boldsymbol{k}) = \psi(\boldsymbol{k})$	$\hat{g}\boldsymbol{d}(\boldsymbol{k}) = R_g\boldsymbol{d}(\boldsymbol{k})$
Time-reversal	$\hat{K}\psi(\boldsymbol{k}) = \psi^*(-\boldsymbol{k})$	$\hat{K}\boldsymbol{d}(\boldsymbol{k}) = -\boldsymbol{d}^*(-\boldsymbol{k})$
Inversion	$\hat{I}\psi(\boldsymbol{k}) = \psi(-\boldsymbol{k})$	$\hat{I}\boldsymbol{d}(\boldsymbol{k}) = \boldsymbol{d}(-\boldsymbol{k})$
$U(1)$-gauge	$\hat{\Phi}\psi(\boldsymbol{k}) = e^{i\phi}\psi(\boldsymbol{k})$	$\hat{\Phi}\boldsymbol{d}(\boldsymbol{k}) = e^{i\phi}\boldsymbol{d}(\boldsymbol{k})$

M. Sigrist, AIP Cof. Proc. **78**, 165 (2005).

Character table for D_{4h} group

Γ	E	$2C_4$	C_2	$2C_2'$	$2C_2''$	I	$2S_4$	σ_h	$2\sigma_v$	$2\sigma_d$	Basis function	
A_{1g}	1	1	1	1	1	1	1	1	1	1	$\psi = 1$	s
A_{2g}	1	1	1	-1	-1	1	1	1	-1	-1	$\psi = k_xk_y(k_x^2 - k_y^2)$	g
B_{1g}	1	-1	1	1	-1	1	-1	1	1	-1	$\psi = k_x^2 - k_y^2$	d_{x2-y2}
B_{2g}	1	-1	1	-1	1	1	-1	1	-1	1	$\psi = k_xk_y$	d_{xy}
E_g	2	0	-2	0	0	2	0	-2	0	0	$\psi = \{k_xk_z, k_yk_z\}$	$d_{xz}+id_{yz}$
A_{1u}	1	1	1	1	1	-1	-1	-1	-1	-1	$\vec{d} = \hat{x}k_x + \hat{y}k_y$	p
A_{2u}	1	1	1	-1	-1	-1	-1	-1	1	1	$\vec{d} = \hat{x}k_y - \hat{y}k_x$	p
B_{1u}	1	-1	1	1	-1	-1	1	- 1	- 1	1	$\vec{d} = \hat{x}k_x - \hat{y}k_y$	p
B_{2u}	1	-1	1	-1	1	-1	1	-1	1	- 1	$\vec{d} = \hat{x}k_y + \hat{y}k_x$	p
E_u	2	0	-2	0	0	-2	0	2	0	0	$\vec{d} = \{\hat{z}k_x, \hat{z}k_y\}$	p+ip

Visualization of typical pairing states & nodes



Symmetry-enforced nodes for typical states

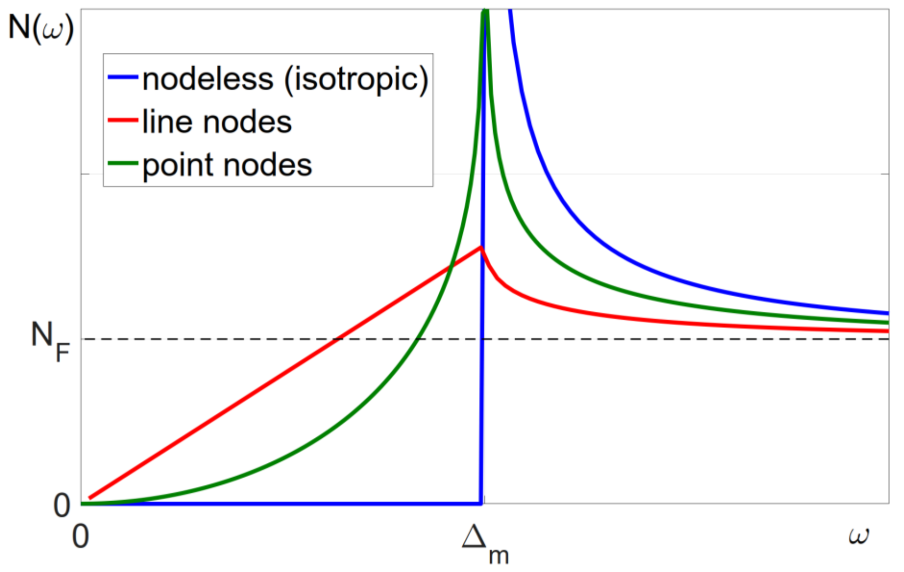
states	Nodes (2D)
s-wave	nodeless
d_{x2-y2}-wave	$k_x=\pm k_y$ (line nodes)
d_{xy}-wave	$k_x=0$, $k_y=0$ (line nodes)
p_x-wave	$k_y=0$ (line nodes)
(p_x+ip_y)-wave	$k_x=k_y=0$ (point node)
g-wave	$k_x=\pm k_y$, $k_x=0$, $k_y=0$ (line nodes)

*(Note, the above classified nodes will be modified since the pairing function Δ_k usually only involving the **k** points near the Fermi surfaces.)*

Physical consequences of the nodes in Δ_k

Density of states

$$N(\omega) = \frac{2}{V}\sum_{\boldsymbol{k}} \delta(E_{\boldsymbol{k}} - \omega) = 2\int \frac{d\boldsymbol{k}_{//}}{(2\pi)^3 v_{k_F}} \frac{\omega}{\sqrt{\omega^2 - \Delta_{\boldsymbol{k}}^2}},$$

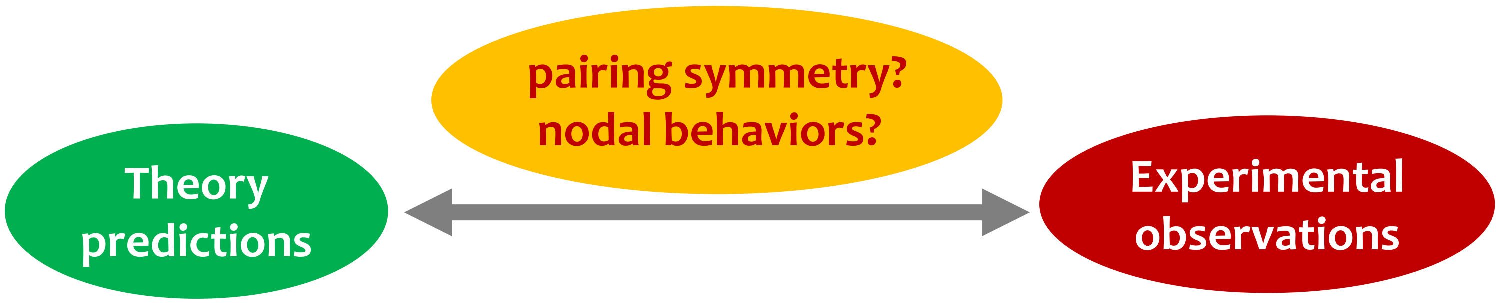


Below Δ_m,

$$N(\omega) = \begin{cases} 0, & \text{(nodeless)} \\ \propto \omega, & \text{(line nodes)} \\ \propto \omega^2, & \text{(point nodes)} \end{cases}$$

Temperature behaviors of typical physical quantities for different nodes in SC ($T \ll T_c$)

Physical quantities	Nodeless	Line nodes	Point nodes
Specific heat $C(T)$	$T^{-3/2}e^{-\Delta/T}$	T^2	T^3
London penetration depth $\lambda(T)$	$T^{-3/2}e^{-\Delta/T}$	T	T^2
NMR $1/T_1$	T	T^3	T^5
Thermal conductivity $\kappa(T)$	$T^{-3/2}e^{-\Delta/T}$	T^2	T^3



II. Eliashberg theory & its applications

- Eliashberg theory
- Quantum critical spin-fluctuation mechanism
- Pairing symmetry in heavy-fermion SC

Eliashberg theory (1)

- Electron-phonon coupled Hamiltonian (including screened Coulomb interactions)

$$H=\sum_{\boldsymbol{k}}\xi_{\boldsymbol{k}}\psi^{\dagger}_{\boldsymbol{k}}\tau_3\psi_{\boldsymbol{k}}+\sum_{\boldsymbol{q},\nu}\Omega_{\boldsymbol{q}\nu}b^{\dagger}_{\boldsymbol{q}\nu}b_{\boldsymbol{q}\nu}+\sum_{\boldsymbol{k},\boldsymbol{k}',\nu}g_{\boldsymbol{k}\boldsymbol{k}'\nu}\phi_{\boldsymbol{k}-\boldsymbol{k}'\nu}\psi^{\dagger}_{\boldsymbol{k}'}\tau_3\psi_{\boldsymbol{k}}$$

$$+\sum_{\substack{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4\\ \sigma,\sigma'}}\langle\boldsymbol{k}_1,\boldsymbol{k}_2|\,V_c\,|\boldsymbol{k}_3,\boldsymbol{k}_4\rangle\left(\psi^{\dagger}_{\boldsymbol{k}_1}\tau_3\psi_{\boldsymbol{k}_4}\right)\left(\psi^{\dagger}_{\boldsymbol{k}_2}\tau_3\psi_{\boldsymbol{k}_3}\right)\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4}$$

where the Nambu spinor $\psi^{\dagger}_{\boldsymbol{k}}=\left(c^{\dagger}_{\boldsymbol{k}\uparrow},c_{-\boldsymbol{k}\downarrow}\right)$, and the phonon field operator $\phi_{\boldsymbol{q}\nu}=b_{\boldsymbol{q}\nu}+b^{\dagger}_{-\boldsymbol{q}\nu}$

- **Green's functions**

Electron $\mathcal{G}(\boldsymbol{k},\tau)=-\left\langle PT_{\tau}\left[\psi_{\boldsymbol{k}}(\tau)\,\psi^{\dagger}_{\boldsymbol{k}}(0)\right]\right\rangle=\begin{pmatrix}G_{\boldsymbol{k}\uparrow}(\tau) & F_{\boldsymbol{k},\uparrow\downarrow}(\tau)\\ \bar{F}_{\boldsymbol{k},\downarrow\uparrow}(\tau) & -G_{-\boldsymbol{k}\downarrow}(-\tau)\end{pmatrix},$

Phonon $D_{\nu}(\boldsymbol{q},\tau)=-\left\langle T_{\tau}\left[\phi_{\boldsymbol{k}\nu}(\tau)\,\phi^{\dagger}_{\boldsymbol{k}\nu}(0)\right]\right\rangle,$

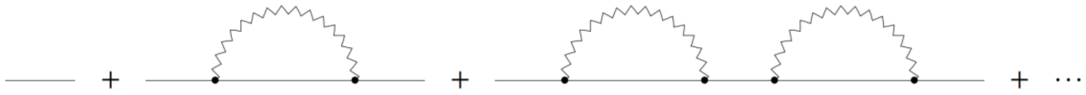
- **Perturbation expansion**

$$\mathcal{G}(\boldsymbol{k},\tau)=-\frac{\int D\psi D\bar{\psi}\psi_{\boldsymbol{k}}(\tau)\,\bar{\psi}_{\boldsymbol{k}}(0)\,e^{-S}}{\int D\psi D\bar{\psi}e^{-S}}=-\frac{\frac{1}{Z_0}\int D\psi D\bar{\psi}\psi_{\boldsymbol{k}}(\tau)\,\bar{\psi}_{\boldsymbol{k}}(0)\,e^{-S_0-S_{int}}}{\frac{1}{Z_0}\int D\psi D\bar{\psi}e^{-S_0-S_{int}}}$$

$$=-\left\langle PT_{\tau}\left[\psi_{\boldsymbol{k}}(\tau)\,\psi^{\dagger}_{\boldsymbol{k}}(0)\,e^{-S_{int}}\right]\right\rangle_{0,\mathrm{connected}}=-\left\langle PT_{\tau}\left[\psi_{\boldsymbol{k}}(\tau)\,\psi^{\dagger}_{\boldsymbol{k}}(0)\sum_{n=0}\frac{(-S_{int})^n}{n!}\right]\right\rangle_{0,\mathrm{connected}}$$

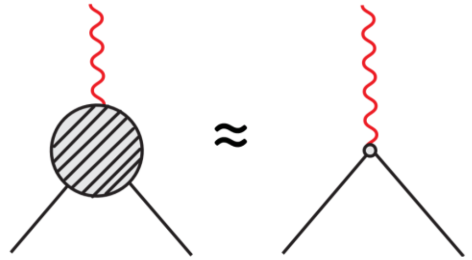
- **Dyson's equation**

$$\left[\mathcal{G}(\boldsymbol{k},i\omega_n)\right]^{-1}=\left[\mathcal{G}_0(\boldsymbol{k},i\omega_n)\right]^{-1}-\hat{\Sigma}(\boldsymbol{k},i\omega_n)$$



Migdal's theorem

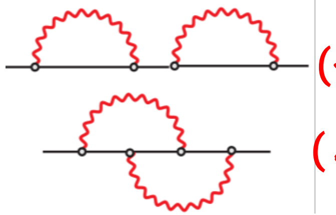
A. B. Migdal, JETP **34**, 1438 (1958).



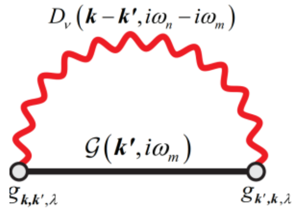
- Vertex corrections can be neglected due to the small prefactors ($\sim O\left(\sqrt{m/M}\right)\ll 1$).
- Only non-crossing diagrams are counted in Feynman diagram expansions.

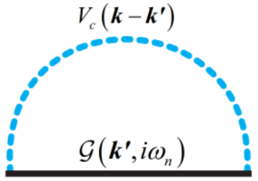
Non-crossing: (✓)

Crossing: (✗)



Eliashberg theory (2)





- **Self-energy** $\hat{\Sigma}(\boldsymbol{k}, i\omega_n) = \hat{\Sigma}_{ph}(\boldsymbol{k}, i\omega_n) + \hat{\Sigma}_C(\boldsymbol{k}, i\omega_n)\,,$

$$\hat{\Sigma}_{ph}(\boldsymbol{k}, i\omega_n) = -\frac{1}{\beta}\sum_{\boldsymbol{k}', i\omega_m, \nu} |g_{\boldsymbol{k},\boldsymbol{k}',\nu}|^2 D_\nu(\boldsymbol{k}-\boldsymbol{k}', i\omega_n - i\omega_m)\,\tau_3\mathcal{G}(\boldsymbol{k}', i\omega_m)\,\tau_3,$$

$$\hat{\Sigma}_C(\boldsymbol{k}, i\omega_n) = -\frac{1}{\beta}\sum_{\boldsymbol{k}', i\omega_m} V_C(\boldsymbol{k}-\boldsymbol{k}')\,\tau_3\mathcal{G}(\boldsymbol{k}', i\omega_m)\,\tau_3,$$

Non-interacting Green's function

$$\mathcal{G}_0(\boldsymbol{k}, i\omega_n) = [i\omega_n\tau_0 - \xi_{\boldsymbol{k}}\tau_3]^{-1}\,,$$

- Expanding the self-energy in Pauli matrices as

$$\hat{\Sigma}(\boldsymbol{k}, i\omega_n) = [1 - Z(\boldsymbol{k}, i\omega_n)]\,i\omega_n\tau_0 + \eta(\boldsymbol{k}, i\omega_n)\,\tau_3 + \phi(\boldsymbol{k}, i\omega_n)\,\tau_1 + \bar{\phi}(\boldsymbol{k}, i\omega_n)\,\tau_2$$

- Then, $\mathcal{G}_k^{-1} = \mathcal{G}_{0,k}^{-1} - \hat{\Sigma}_k$

$$= i\omega_n Z_k\tau_0 - (\xi_{\boldsymbol{k}} + \eta_k)\,\tau_3 - \phi_k\tau_1 - \bar{\phi}_k\tau_2$$

$$= \begin{pmatrix} i\omega_n Z_k - (\xi_{\boldsymbol{k}} + \eta_k) & -\phi_k + i\bar{\phi}_k \\ -\phi_k - i\bar{\phi}_k & i\omega_n Z_k + (\xi_{\boldsymbol{k}} + \eta_k) \end{pmatrix},$$

$$\mathcal{G}_k = \frac{1}{\Theta_k}\begin{pmatrix} i\omega_n Z_k + (\xi_{\boldsymbol{k}} + \eta_k) & \phi_k - i\bar{\phi}_k \\ \phi_k + i\bar{\phi}_k & i\omega_n Z_k - (\xi_{\boldsymbol{k}} + \eta_k) \end{pmatrix}$$

$$\Theta_k = \det\mathcal{G}_k^{-1} = (i\omega_n Z_k)^2 - (\xi_{\boldsymbol{k}} + \eta_k)^2 - \phi_k^2 - \bar{\phi}_k^2.$$

- Physical meaning of the matrix elements in self-energy

Quasiparticle dispersion

$$[\det\mathcal{G}_k^{-1}]_{i\omega_n\to E_{\boldsymbol{k}}} = 0,$$

$$E_{\boldsymbol{k}} = \pm\sqrt{\left(\frac{\xi_{\boldsymbol{k}} + \eta_{\boldsymbol{k}}}{Z_{\boldsymbol{k}}}\right)^2 + \left|\frac{\phi_{\boldsymbol{k}} - i\bar{\phi}_{\boldsymbol{k}}}{Z_{\boldsymbol{k}}}\right|^2} = \pm\sqrt{\xi_{\boldsymbol{k}}^{*2} + |\Delta_{\boldsymbol{k}}|^2}.$$

$Z_{\boldsymbol{k}}$ -- Renormalization function (≥1); $\eta_{\boldsymbol{k}}$ -- Chemical potential shift; $\phi_{\boldsymbol{k}} - i\bar{\phi}_{\boldsymbol{k}}$ -- Anomalous self-energy.

Eliashberg equations

- **Eliashberg equations**

$$(1-Z_k)\, i\omega_n = -\frac{1}{\beta}\sum_{k'} i\omega_m Z_{k'} \frac{\left[\sum_\nu |g_{\boldsymbol{k},\boldsymbol{k}',\nu}|^2 D_\nu(\boldsymbol{k}-\boldsymbol{k}', i\omega_n - i\omega_m) + V_C(\boldsymbol{k}-\boldsymbol{k}')\right]}{(i\omega_m Z_{k'})^2 - (\xi_{\boldsymbol{k}'}+\eta_{k'})^2 - \phi_{k'}^2 - \bar{\phi}_{k'}^2}, \quad (1)$$

$$\eta_k = -\frac{1}{\beta}\sum_{k'} (\xi_{\boldsymbol{k}'}+\eta_{k'}) \frac{\left[\sum_\nu |g_{\boldsymbol{k},\boldsymbol{k}',\nu}|^2 D_\nu(\boldsymbol{k}-\boldsymbol{k}', i\omega_n - i\omega_m) + V_C(\boldsymbol{k}-\boldsymbol{k}')\right]}{(i\omega_m Z_{k'})^2 - (\xi_{\boldsymbol{k}'}+\eta_{k'})^2 - \phi_{k'}^2 - \bar{\phi}_{k'}^2}, \quad (2)$$

$$\phi_k = \frac{1}{\beta}\sum_{k'} \phi_{k'} \frac{\left[|g_{\boldsymbol{k},\boldsymbol{k}',\nu}|^2 D_\nu(\boldsymbol{k}-\boldsymbol{k}', i\omega_n - i\omega_m) + V_C(\boldsymbol{k}-\boldsymbol{k}')\right]}{(i\omega_m Z_{k'})^2 - (\xi_{\boldsymbol{k}'}+\eta_{k'})^2 - \phi_{k'}^2 - \bar{\phi}_{k'}^2}, \quad (3)$$

$$\bar{\phi}_k = \frac{1}{\beta}\sum_{k'} \bar{\phi}_{k'} \frac{\left[\sum_\nu |g_{\boldsymbol{k},\boldsymbol{k}',\nu}|^2 D_\nu(\boldsymbol{k}-\boldsymbol{k}', i\omega_n - i\omega_m) + V_C(\boldsymbol{k}-\boldsymbol{k}')\right]}{(i\omega_m Z_{k'})^2 - (\xi_{\boldsymbol{k}'}+\eta_{k'})^2 - \phi_{k'}^2 - \bar{\phi}_{k'}^2}. \quad (4)$$

- Reductions

-- Assume the chemical potential shift is unimportant, Eq.(2) vanish due to $\sum_{\boldsymbol{k}'} \xi_{\boldsymbol{k}'}(\ldots) \approx N_F \int_{-\omega_D}^{\omega_D} d\xi\, \xi(\ldots) = 0$

-- Assume the anomalous self energy is real, Eq.(3)(4) reduce into one.

Eliashberg equations

$$i\omega_n\left[1 - Z(\boldsymbol{k}, i\omega_n)\right] = -\frac{1}{\beta}\sum_{\boldsymbol{k}', i\omega_m} \frac{i\omega_m Z(\boldsymbol{k}', i\omega_m)\, V_{\boldsymbol{k},\boldsymbol{k}'}(i\omega_n - i\omega_m)}{\omega_m^2 Z^2(\boldsymbol{k}', i\omega_m) + \xi_{\boldsymbol{k}'}^2 + \Delta^2(\boldsymbol{k}', i\omega_m)},$$

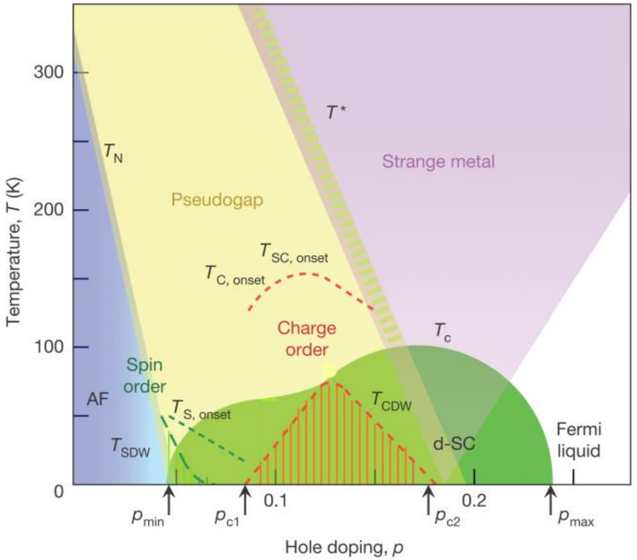
$$Z(\boldsymbol{k}, i\omega_n)\,\Delta(\boldsymbol{k}, i\omega_n) = -\frac{1}{\beta}\sum_{\boldsymbol{k}', i\omega_m} \frac{Z(\boldsymbol{k}', i\omega_m)\,\Delta(\boldsymbol{k}', i\omega_m)\, V_{\boldsymbol{k},\boldsymbol{k}'}(i\omega_n - i\omega_m)}{\omega_m^2 Z^2(\boldsymbol{k}', i\omega_m) + \xi_{\boldsymbol{k}'}^2 + \Delta^2(\boldsymbol{k}', i\omega_m)}$$

→ (SC gap equation)

where $V_{\boldsymbol{k},\boldsymbol{k}'}(i\omega_n - i\omega_m) = \sum_\nu |g_{\boldsymbol{k},\boldsymbol{k}',\nu}|^2 D_\nu(\boldsymbol{k}-\boldsymbol{k}', i\omega_n - i\omega_m) + V_C(\boldsymbol{k}-\boldsymbol{k}')$

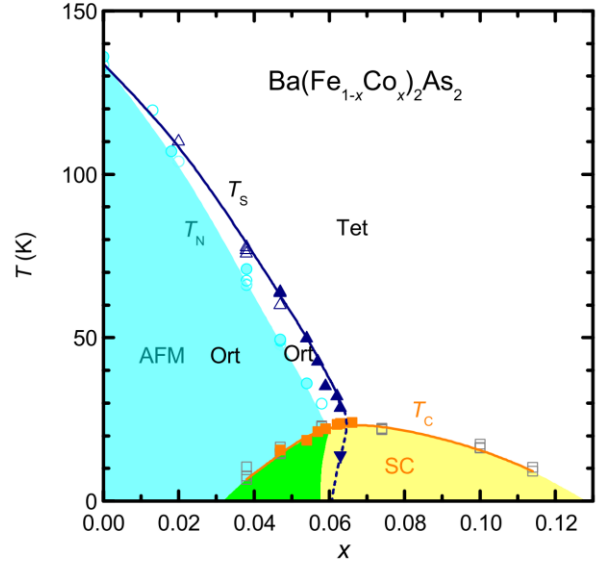
Spin-fluctuation mechanism

Cuprates



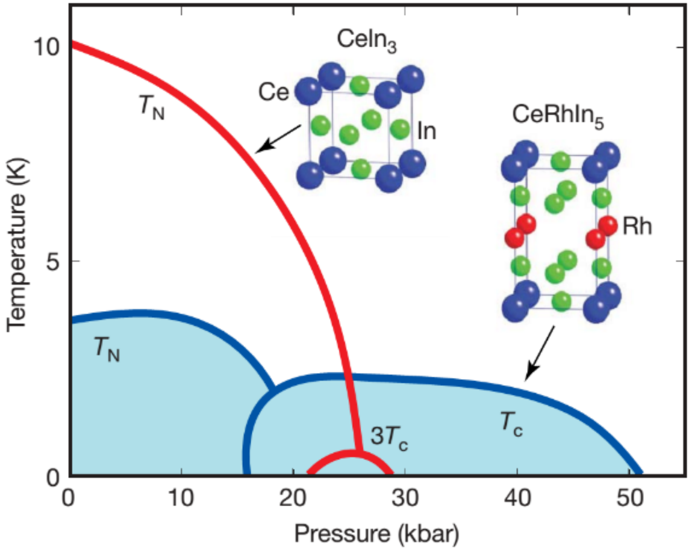
B. Keimer, *et al*, nature, **518**, 179 (2015)

Iron-pnictides



D. J. Scalapino, Rev. Mod. Phys. **84**, 1383 (2012)

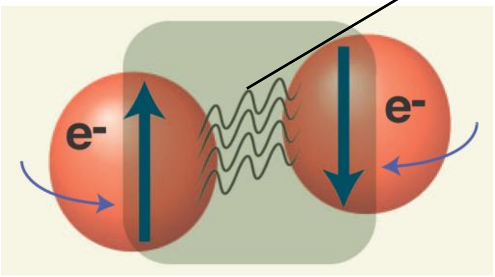
Heavy fermion SC



P. Monthoux, D. Pines, and G. G. Lonzarich, Nature **450**, 1177 (2007)

Many unconventional superconductivity emerging out on the border of AFM. ⇨ **spin-fluctuation-mediated pairing.**

Cooper pair



Spin fluctuations (e.g. paramagnons)

Quasiparticle-spin-fluctuation coupling: $\mathcal{H}_{\text{int}} = \frac{1}{\Omega}\sum_{\mathbf{q}} \bar{g}(\mathbf{q})\mathbf{s}(\mathbf{q})\cdot\mathbf{S}(-\mathbf{q})$, $\left(s_q = \frac{1}{2}\sum_{k,\alpha,\beta} c^\dagger_{k\alpha}\sigma_{\alpha\beta}c_{k\beta}\right)$

$$S_{eff} = \sum_{\mathbf{p},\alpha}\int_0^\beta d\tau \psi^\dagger_{\mathbf{p},\alpha}(\tau)(\partial_\tau + \epsilon_{\mathbf{p}} - \mu)\psi_{\mathbf{p},\alpha}(\tau) - \frac{g^2}{6}\sum_{\mathbf{q}}\int_0^\beta d\tau \int_0^\beta d\tau' \chi(\mathbf{q},\tau-\tau')\mathbf{s}(\mathbf{q},\tau)\cdot\mathbf{s}(-\mathbf{q},\tau').$$

P. Monthoux, A. V. Balatsky, and D. Pines, Phys. Rev. Lett. **67**, 3448(1991)   P. Monthoux and G. G. Lonzarich, Phys. Rev. B **59**, 14598 (1999)

Early history of spin-fluctuation mechanism

- 1959 : magnon-mediate electron-electron pairing predict a spin-triplet pairing.

  A. I. Akhiezer, I. Ya. Pomeranchunk, JETP 36, 859 (1959).

- 1965: repulsive interaction can also induce pairing (**Kohn-Luttinger mechanism**).

  W. Kohn, J. Luttinger, PRL **15**, 524 (1965).

- 1966: ferromagnetic spin fluctuation induces a repulsive interaction, which suppress the spin-singlet pairing in transition metals (e.g. Pd).

  N. Berk, J. Schrieffer, PRL **17**, 433 (1966).

- 1970s : triplet pairing in He-3 mediated from FM spin fluctuations.

  A. Layzer, D. Fay, Int. J Magn. **1**, 135 (1971); P. W. Anderson, W. Brinkman, PRL **30**, 1108 (1973). A. J. Leggett, RMP **47**, 331 (1975).

- 1985-1986: Antiferromagnetic spin-fluctuation-mediated d-wave pairing from Hubbard model, t-J model

  J. E. Hirsch, PRL **54**, 1317 (1985); D. Scalapino, E. Loh Jr, J. Hirsch, PRB **34**, 8190 (1986); K. Miyake, S. Schmitt-Rink, C. Varma, PRB **34**, 6554 (1986); M. T. Beal-Monod, C. Bourbonnais, V. J. Emery, PRB 34, 7716 (1986).

- 1990s: spin-fluctuation-induced high-T_c SC from weak-coupling theory & various universal factors affect Tc from strong-coupling theory (Eliashberg theory).

  P. Monthoux, A. Balatsky, D. Pines, PRL **67**, 3448 (1991); PRB **46**, 12803 (1992). P. Monthoux, D. Pines, PRL **69**, 961 (1992); PRB **47**, 6069 (1993); PRB 49, 4261 (1994).

- ……

Spin-fluctuation mechanism: physical picture & anisotropic pairing

- **Spin fluctuation mechanism**

spin $\boldsymbol{s}(\boldsymbol{r}, t)$ → effective local field $\boldsymbol{H}(\boldsymbol{r}, t) = g\boldsymbol{s}(\boldsymbol{r}, t)$

(linear response)

Induced magnetization

$$\boldsymbol{M}(\boldsymbol{r}, t) = g\int d^3\boldsymbol{r}' dt' \boldsymbol{s}(\boldsymbol{r}', t')\,\chi(\boldsymbol{r}-\boldsymbol{r}', t-t'),$$

Induced effective field

$$\boldsymbol{H}'(\boldsymbol{r}, t) = g\boldsymbol{M}(\boldsymbol{r}, t)$$

Effective spin-spin interaction

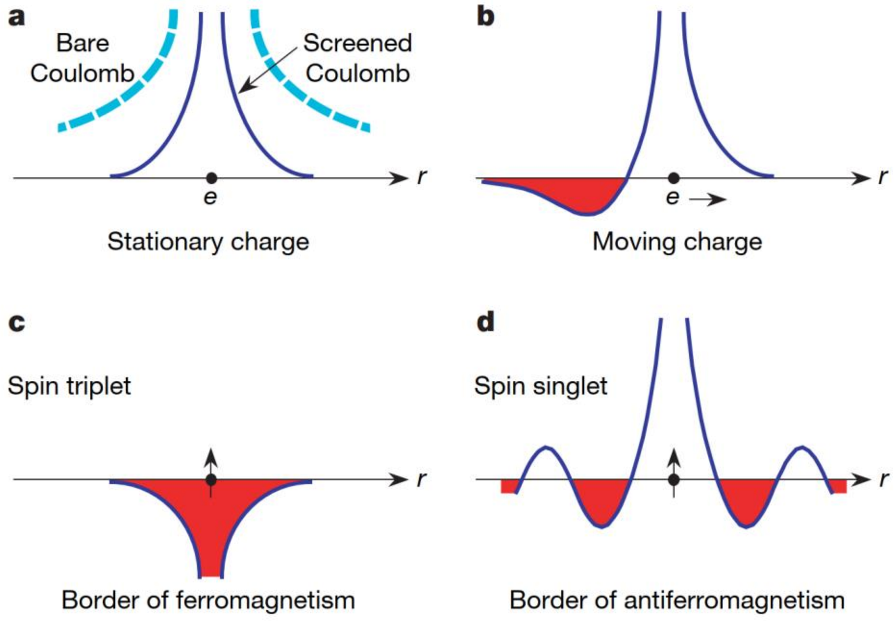
$$V_{eff} = -\int d^3\boldsymbol{r} dt\, \boldsymbol{s}(\boldsymbol{r}, t)\cdot\boldsymbol{H}'(\boldsymbol{r}, t)$$
$$= -g^2\int d^3\boldsymbol{r} dt d^3\boldsymbol{r}' dt' \chi(\boldsymbol{r}-\boldsymbol{r}', t-t')\,\boldsymbol{s}(\boldsymbol{r}, t)\cdot\boldsymbol{s}(\boldsymbol{r}', t').$$

P. Monthoux, D. Pines, G. G. Lonzarich, nature **450**, 1177(2007).

- **Gap equation analysis**

$$\Delta_k = -g_{eff}^2\sum_{k'}\chi(\mathbf{k}\text{-}\mathbf{k}')\frac{\tanh\frac{E_{k'}}{2T}}{E_{k'}}\Delta_{k'}$$

For magnetic fluctuation spectrum, $\chi > 0$, the only nontrivial solution required the sign changes for Δ_k. ⇨ **No s-wave solution!**



Spin interaction oscillating in space, produce an effecting attraction for electrons, and form anisotropic pairing of electrons with antiparallel/parallel spins for AFM/FM case.

Spin-fluctuation mechanism: predictions

Effective action

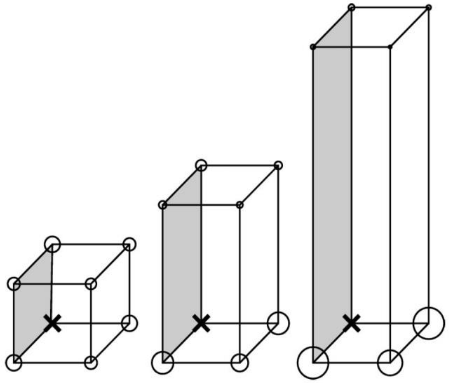
$$S_{\text{eff}}=\sum_{\mathbf{p},\alpha}\int_0^\beta d\tau\psi^\dagger_{\mathbf{p},\alpha}(\tau)(\partial_\tau+\epsilon_{\mathbf{p}}-\mu)\psi_{\mathbf{p},\alpha}(\tau)-\frac{g^2}{6}\sum_{\mathbf{q}}\int_0^\beta d\tau\int_0^\beta d\tau'\chi(\mathbf{q},\tau-\tau')\mathbf{s}(\mathbf{q},\tau)\cdot\mathbf{s}(-\mathbf{q},\tau')$$

Strong-coupling calculations: **Eliashberg equations**

$$\Sigma(\mathbf{p},i\omega_n)=g^2\frac{T}{N}\sum_{\Omega_n}\sum_{\mathbf{k}}\chi(\mathbf{p}-\mathbf{k},i\omega_n-i\Omega_n)G(\mathbf{k},i\Omega_n),$$

$$G(\mathbf{p},i\omega_n)=\frac{1}{i\omega_n-(\epsilon_{\mathbf{p}}-\mu)-\Sigma(\mathbf{p},i\omega_n)},$$

$$\Lambda(T)\Phi(\mathbf{p},i\omega_n)=\begin{bmatrix}\frac{g^2}{3}\\-g^2\end{bmatrix}\frac{T}{N}\sum_{\Omega_n}\sum_{\mathbf{k}}\chi(\mathbf{p}-\mathbf{k},i\omega_n-i\Omega_n)|G(\mathbf{k},i\Omega_n)|^2\Phi(\mathbf{k},i\Omega_n)$$



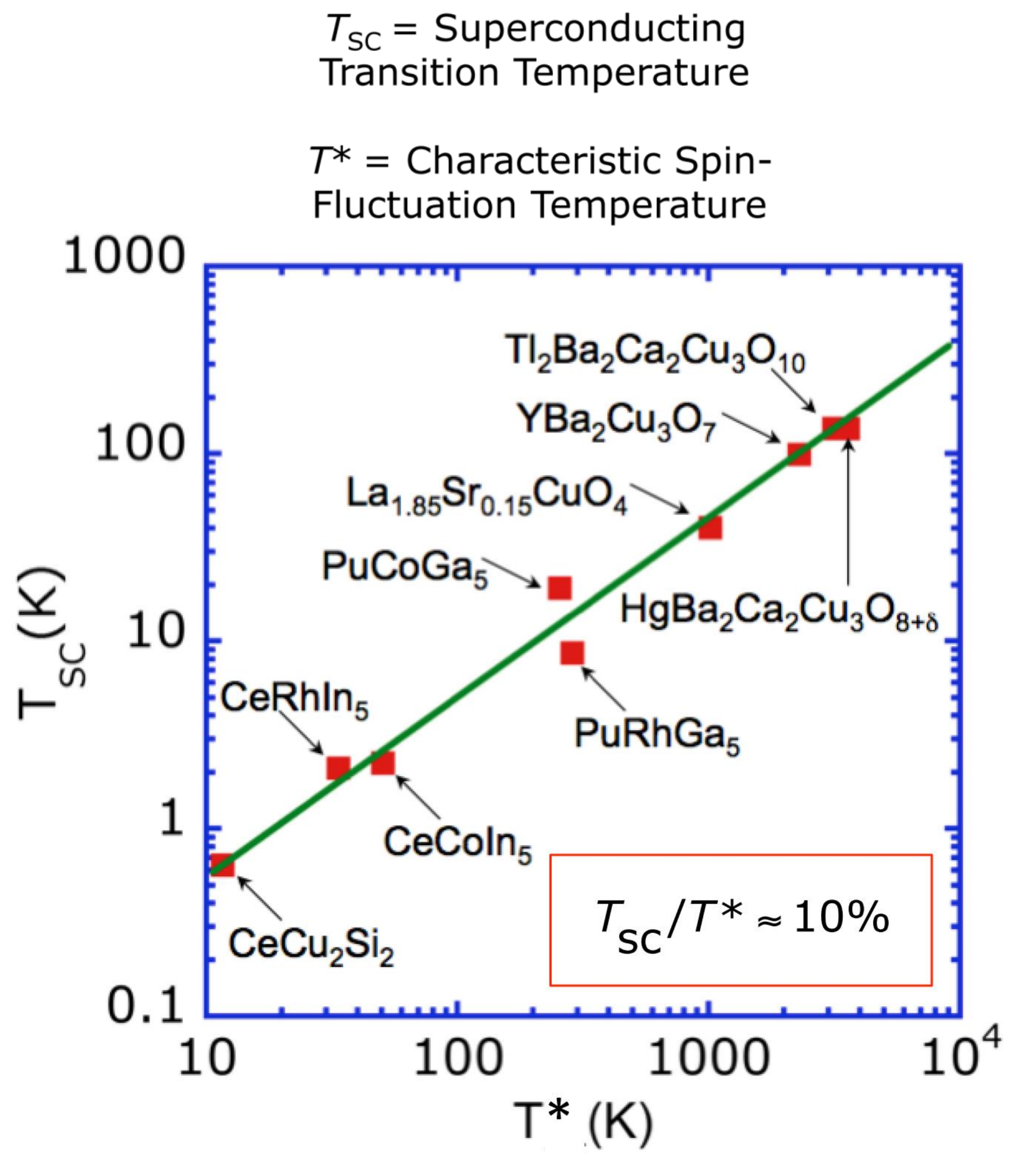
→ *predict the discovery of SC in many HFs, such as, UGe_2, $Ce_nX_mIn_{3n+2m}$, …*

Predictions for the enhancement of T_c

- **AFM d-wave is more robust than FM p-wave in quasi-2D case**
- **Quasi-2D > 3D**
- **Anisotropic > isotropic**
- **Good Fermi surface nesting.**

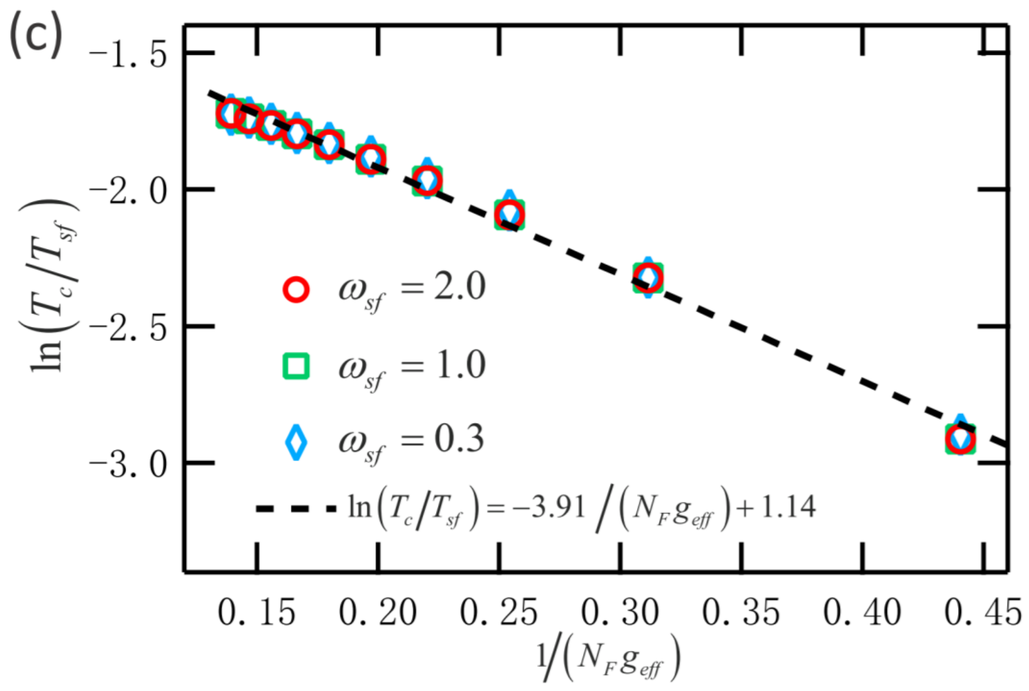
P. Mounthoux, G. G. Lonzarich, PRB **59**, 14598 (1999)
P. Mounthoux, G. G. Lonzarich, PRB **63**, 054529 (2001)
P. Mounthoux, G. G. Lonzarich, PRB **66**, 224504 (2002)
S. Nishiyama, K. Miyake, C. M, Varma, PRB **88**, 014510 (2013)

Universal scaling of T_c



(G. G. Lonzarich, 2014)

- **Prediction:** find a material with large magnetic exchange coupling ⇨ potential for a high T_c.



$$T_c = AT_{sf}exp\left(-\frac{1}{BN_F g_{eff}}\right),$$

Y. Li, and Y.-F. Yang, Chin. Sci. Bull. **62**, 4068 (2017)

Eliashberg framework

Eliashberg equations

$$i\omega_n\left[1 - Z\left(\boldsymbol{k}, i\omega_n\right)\right] = -\frac{1}{\beta}\sum_{\boldsymbol{k}', i\omega_m}\frac{i\omega_m Z\left(\boldsymbol{k}', i\omega_m\right)V_{\boldsymbol{k},\boldsymbol{k}'}\left(i\omega_n - i\omega_m\right)}{\omega_m^2 Z^2\left(\boldsymbol{k}', i\omega_m\right) + \xi_{\boldsymbol{k}'}^2 + \Delta^2\left(\boldsymbol{k}', i\omega_m\right)},$$

$$Z\left(\boldsymbol{k}, i\omega_n\right)\Delta\left(\boldsymbol{k}, i\omega_n\right) = -\frac{1}{\beta}\sum_{\boldsymbol{k}', i\omega_m}\frac{Z\left(\boldsymbol{k}', i\omega_m\right)\Delta\left(\boldsymbol{k}', i\omega_m\right)V_{\boldsymbol{k},\boldsymbol{k}'}\left(i\omega_n - i\omega_m\right)}{\omega_m^2 Z^2\left(\boldsymbol{k}', i\omega_m\right) + \xi_{\boldsymbol{k}'}^2 + \Delta^2\left(\boldsymbol{k}', i\omega_m\right)}$$

Near T_c ($\Delta \rightarrow 0$)

Integrating over **k**

(Linearized) Eliashberg equations

$$Z\left(\boldsymbol{k}, i\omega_n\right) = 1 + \frac{1}{\beta\omega_n}\oint_{FS}\frac{d\boldsymbol{k}'_{//}}{\left(2\pi\right)^3 v_{\boldsymbol{k}'_F}}\sum_{i\omega_m}\mathrm{sgn}\left(\omega_m\right)V_{\boldsymbol{k},\boldsymbol{k}'}\left(i\omega_n - i\omega_m\right),$$

$$Z\left(\boldsymbol{k}, i\omega_n\right)\Delta\left(\boldsymbol{k}, i\omega_n\right) = -\frac{1}{\beta}\oint_{FS}\frac{d\boldsymbol{k}'_{//}}{\left(2\pi\right)^3 v_{\boldsymbol{k}'_F}}\sum_{i\omega_m}\frac{V_{\boldsymbol{k},\boldsymbol{k}'}\left(i\omega_n - i\omega_m\right)}{\left|\omega_m\right|}\Delta\left(\boldsymbol{k}', i\omega_m\right).$$

- Interplay effects of Fermi-surface topology and interactions on pairing symmetry for real materials?
- Explore novel pairing states (such as odd-ω pairing, …)
- …

Frequency-dependent Eliashberg equations

$$Z\left(i\omega_n\right) = 1 + \frac{\pi}{\beta\omega_n}\sum_{i\omega_m}\lambda\left(i\omega_n - i\omega_m\right)\frac{\omega_m}{\sqrt{\omega_m^2 + \Delta^2\left(i\omega_m\right)}},$$

$$Z\left(i\omega_n\right)\Delta\left(i\omega_n\right) = \frac{\pi}{\beta}\sum_{i\omega_m}\lambda\left(i\omega_n - i\omega_m\right)\frac{\Delta\left(i\omega_m\right)}{\sqrt{\omega_m^2 + \Delta^2\left(i\omega_m\right)}},$$

- T_c-calculations.
- Effects of the dynamics of quantum critical fluctuations?
- Competition between incoherence and SC?
- Explore enhancement factors & pair-breaking factors for T_c.
- …

A phenomenological framework to HF SC

Strong correlations → **Electronic structures**
- Expt. fitted TB model
- LDA+X (X=U, DMFT, …)

Quantum critical fluctuations → **Phenomenological interactions**

$$V(\boldsymbol{q},\omega)=\frac{V_0}{1+\xi^2(\boldsymbol{q}-\boldsymbol{Q})^2-i\,\omega/\omega_{sf}}$$

Multiband/Multi-orbital characters → **Multiband case**

interband/intraband interactions

(Linearized) Eliashberg equations

$$Z_\mu(\mathbf{k},i\omega_n) = 1+\frac{\pi T}{\omega_n}\sum_{\nu,m}\int_{\mathrm{FS}_\nu}\frac{d\mathbf{k}'_\parallel}{(2\pi)^3 v_{\mathbf{k}'_\mathrm{F}}}\mathrm{sgn}(\omega_m)V^{\mu\nu}(\mathbf{k}-\mathbf{k}',i\omega_n-i\omega_m),$$

$$\phi_\mu(\mathbf{k},i\omega_n) = -\pi T\sum_{\nu,m}\int_{\mathrm{FS}_\nu}\frac{d\mathbf{k}'_\parallel}{(2\pi)^3 v_{\mathbf{k}'_\mathrm{F}}}\frac{V^{\mu\nu}(\mathbf{k}-\mathbf{k}',i\omega_n-i\omega_m)}{|\omega_m Z_\nu(\mathbf{k}',i\omega_m)|}\phi_\nu(\mathbf{k}',i\omega_m)$$

Y. Li, and Y.-F. Yang, Chin. Sci. Bull. **62**, 4068 (2017)   李宇, 盛玉韬, 杨义峰, 物理学报 **70**, 017402 (2021)

Phenomenological pairing interactions

PHYSICAL REVIEW B VOLUME 42, NUMBER 1 1 JULY 1990

Phenomenological model of nuclear relaxation in the normal state of $YBa_2Cu_3O_7$

A. J. Millis
AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, New Jersey 07974

Hartmut Monien and David Pines
Physics Department, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801
(Received 27 November 1989; revised manuscript received 7 March 1990)

A. J. Millis, H. Monien, and D. Pines, Phys. Rev. B **42**, 167 (1990)

Static susceptibility

$$\chi\left(\boldsymbol{q},\omega\right)=\frac{\chi_0}{1+\xi^2\left(\boldsymbol{q}-\boldsymbol{Q}\right)^2-i\omega/\omega_{sf}},$$

Magnetic correlation length

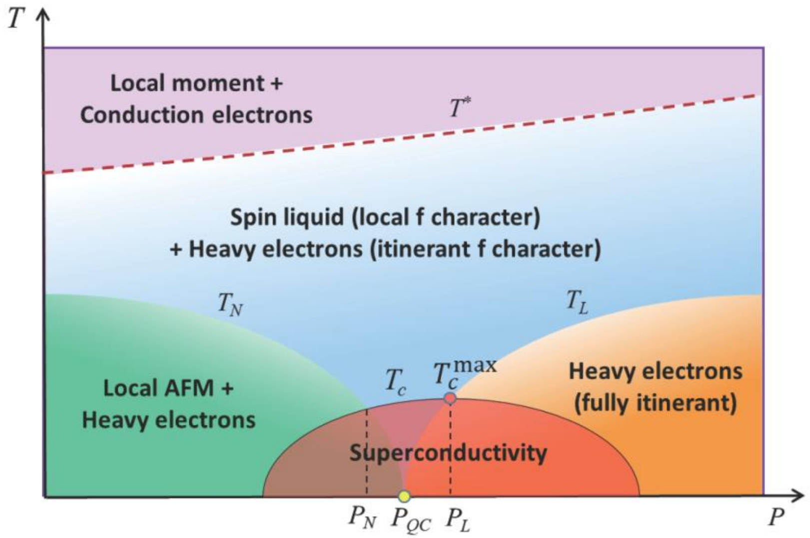
Magnetic characteristic wavevectors

Characteristic frequency for spin fluctuations

- Beyond the RPA approach, which could fit the experimental spin fluctuation behaviors.
- The parameters can be extracted from neutron scattering and NMR experiments.
- The phenomenological approach avoided the need for the criterion of Migdal-like theorem.

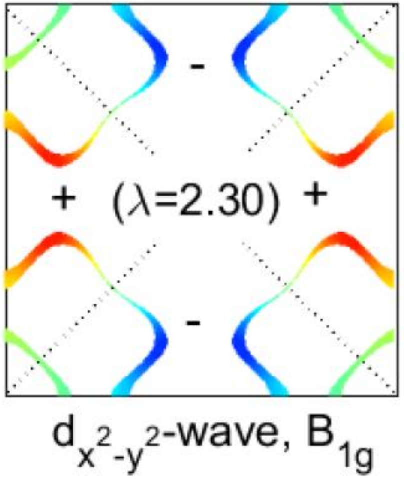
$CeCoIn_5$: d-wave pairing & T_c-scaling behavior

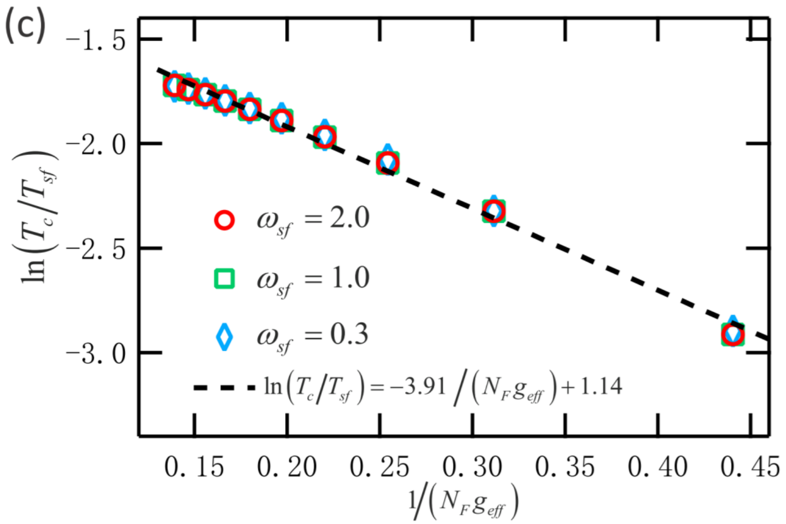
Two-fluid picture for heavy fermions



$$T_c(p) = 0.14T_m^* exp\left[-\frac{1}{N_F(p,T_c)V_{eff}(p)}\right]$$

What's the microscopic correspondence of this T_c-formula?





$$T_c = AT_{sf}exp\left(-\frac{1}{BN_Fg_{eff}}\right),$$

- The pairing symmetry is d_{x2-y2}-wave, as expected.
- The T_c can be captured by a simple formula similar to two-fluid model.
- Both T_{sf} and T^* come from the nearest-neighbor magnetic-exchange coupling J_{ex}. which provide a microscopic support for two-fluid model.

Y. -F. Yang *et al*, nature **454**, 611 (2008)
Y. -F. Yang and D. Pines, Proc, Natl. Acad. Sci. USA, **109**, E3060 (2012)
Y. -F. Yang and D. Pines. Proc. Natl. Acad. Sci. USA, **111**, 18178 (2014)
Y. -F. Yang, Rep. Prog. Phys. **79**, 074501 (2016)

Y. Li, and Y.-F. Yang, Chin. Sci. Bull. **62**, 4068 (2017)

$CeCu_2Si_2$: nodeless SC from experiments

Before 2014

Specific heat $C/T \propto T$

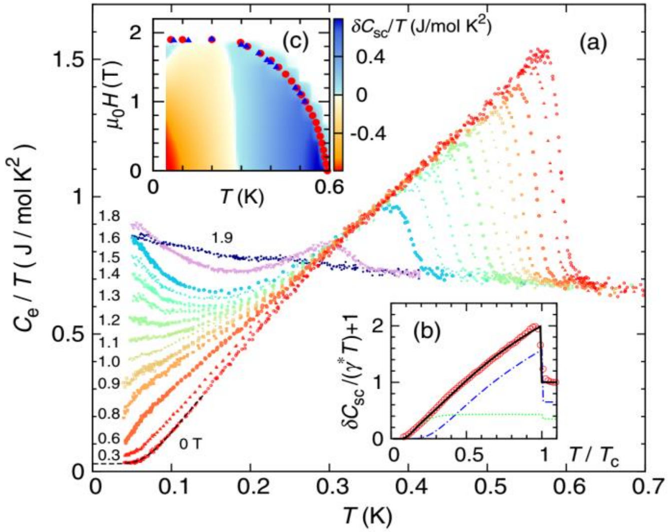
NMR $1/T_1 \propto T^3$

➡ **line-nodal SC**

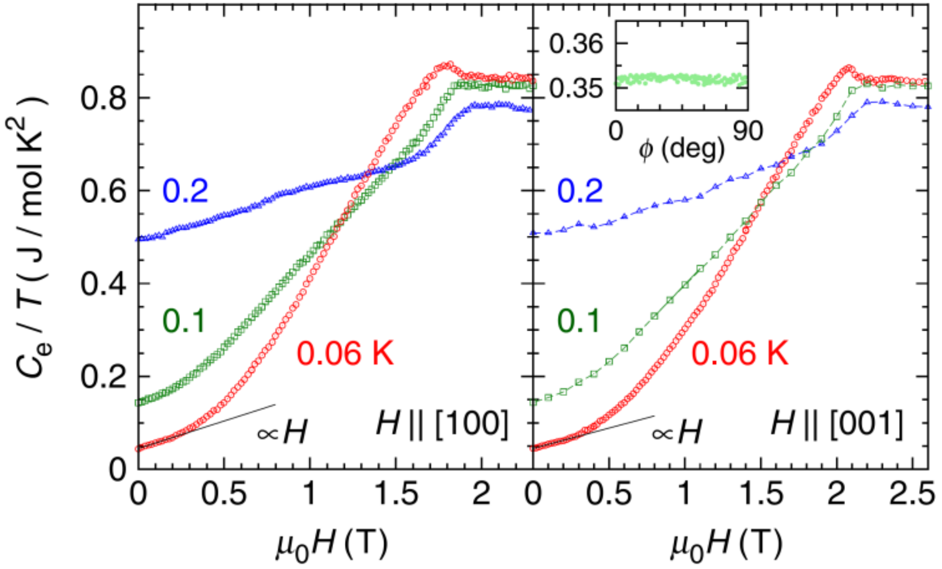
Specific heat (2014)

Fully gapped SC behaviors !

S. Kittaka, *et al*, Phys. Rev. Lett. **112**, 067002 (2014)



$$C(T) = A\exp(-\Delta_0/T) + \gamma_0 T$$



$$C(H)/T \propto H$$

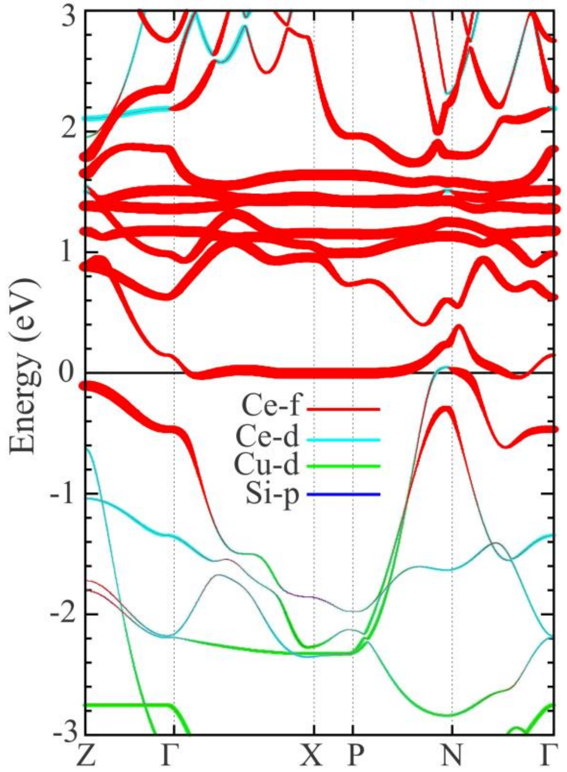
Later on， ……

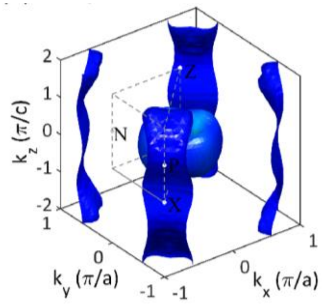
(1) STM: two-gap behaviors	M. Enayat, *et al*, Phys. Rev. B **93**, 045123 (2016)
(2) Angle-resolved specific heat: no nodes	S. Kittaka, *et al*, Phys. Rev. B **94**, 054514 (2016)
(3) Penetration depth } Fully gapped	G.-M. Pang *et al*, Proc. Natl. Acad. Sci. **115**, 5343 (2018)
(4) Thermal conductivity } Fully gapped	T. Yamashita *et al*, Sci. Adv. **3**, e1601667 (2017)
(5) NMR: no coherence peak	S. Kitagawa, *et al*, Phys. Rev. B **96**, 134506 (2017)

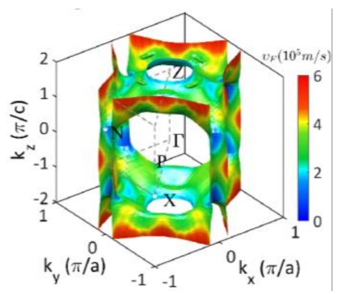
Experimental fittings: $s^{\pm}$? s^{++}? 'd+d'? ➡ **Theoretically ?**

$CeCu_2Si_2$: nodeless $s^{\pm}$-wave pairing from calculations

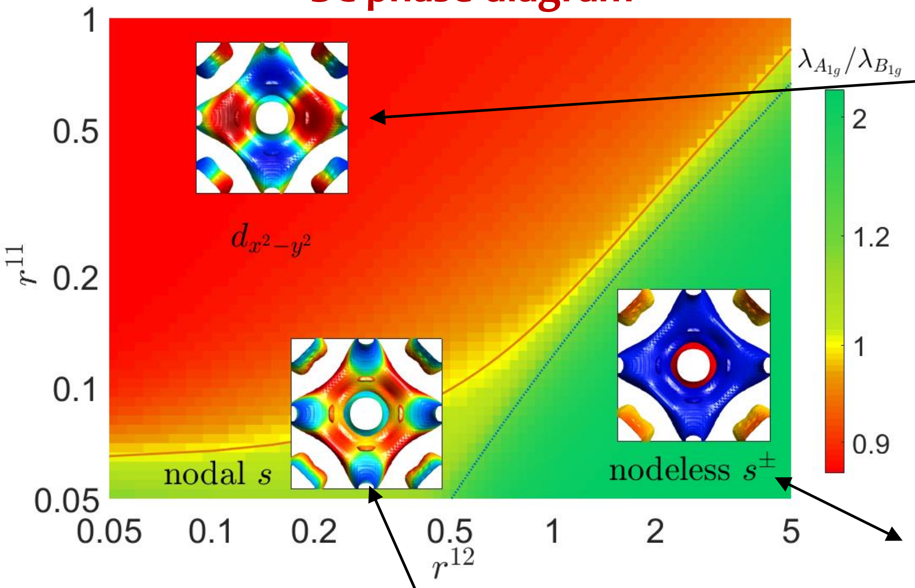
LDA+U



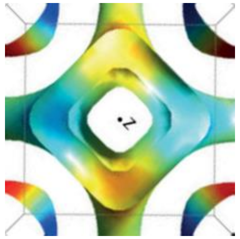




SC phase diagram



(RPA)



H. Ikeda *et al*, Phys. Rev. Lett. **114**, 147003 (2015)

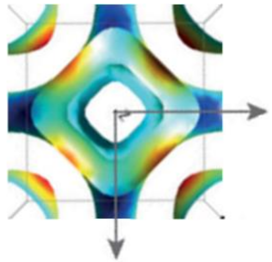
I. Eremin *et al*, Phys. Rev. Lett. **101**, 187001 (2008)

d_{x2-y2} (✘)

(2nd-order perturbation theory)

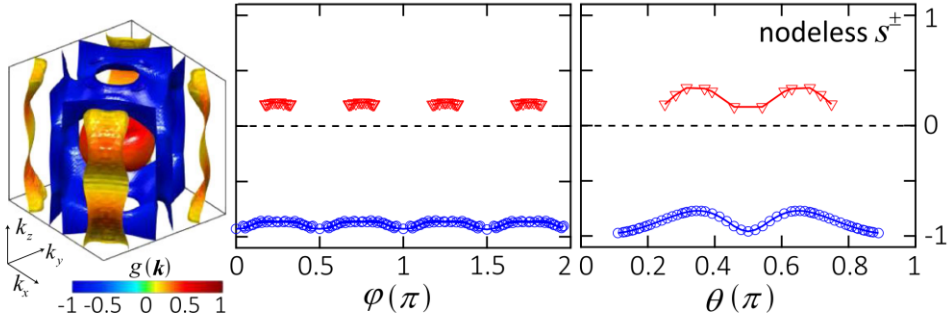
loop-nodal $s^{\pm}$

(✘)



H. Ikeda *et al*, Phys. Rev. Lett. **114**, 147003 (2015)

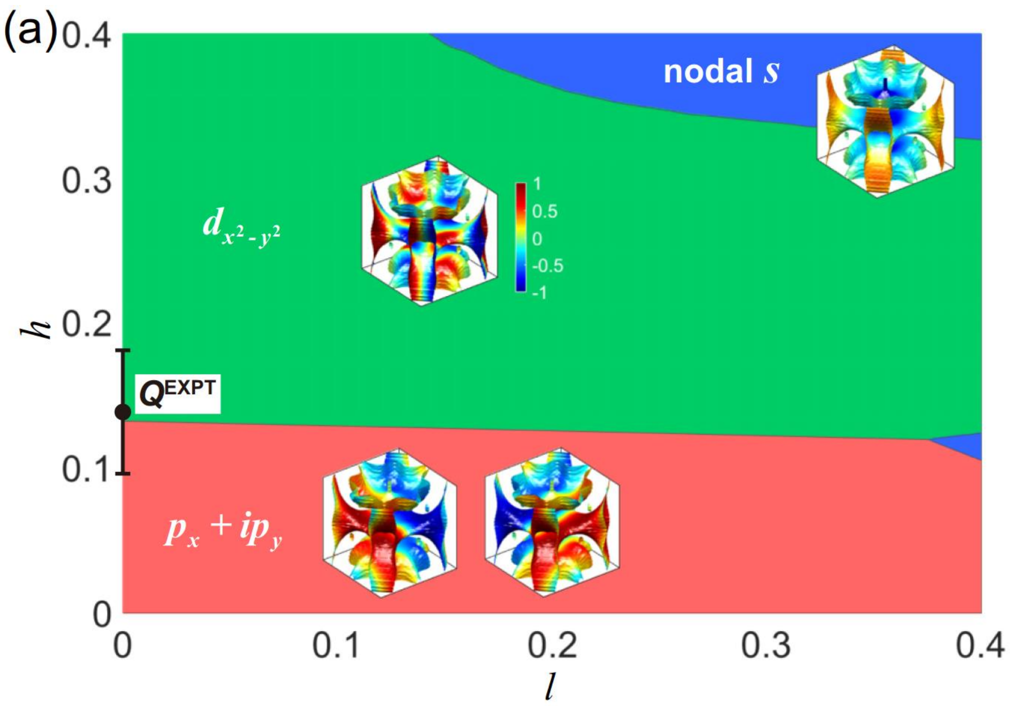
Strong interband pair interaction-induced $s^{\pm}$-wave.

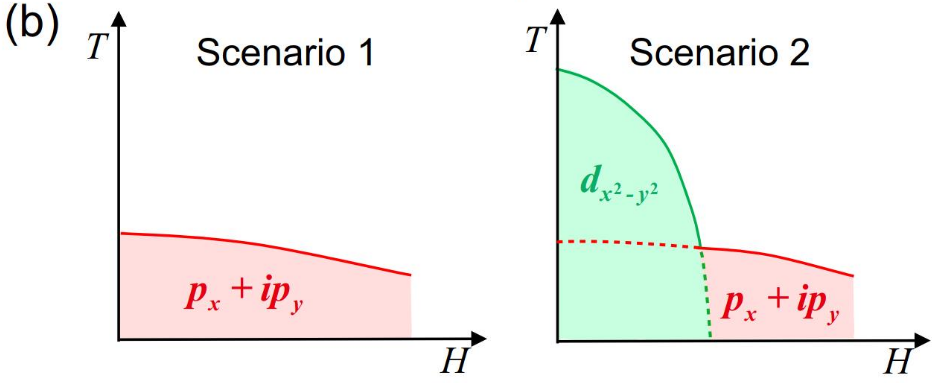


($\left|\frac{\Delta_2^{max}}{\Delta_1^{max}}\right|$ & $\left|\frac{\bar{\Delta}_2}{\bar{\Delta}_1}\right| \approx 2.6\sim3$, consistent with experimental fittings.)

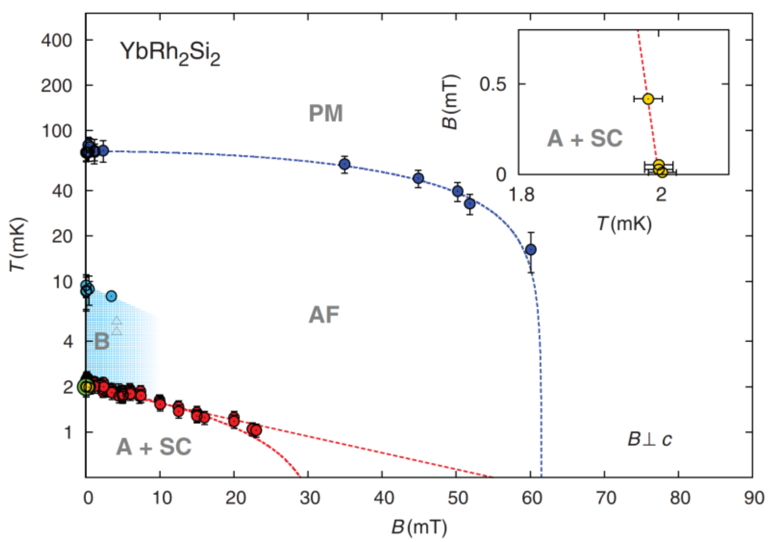
Y. Li *et al*, Phys. Rev. Lett. **120**, 217001 (2018)

$YbRh_2Si_2$: nearly degenerate p_x+ip_y and d_{x2-y2}-wave

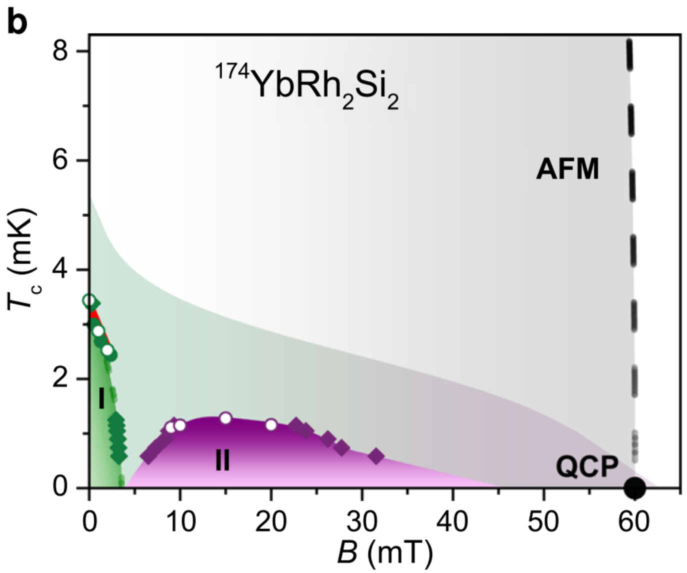




Y. Li *et al*, Phys. Rev. B **100**, 085132 (2019)



E. Schuberth et al. Science **351**, 485 (2016).



D. H. Nguyen et al., Nat. Comm. **12**, 4341 (2021).

- A nearly degenerate p_x+ip_y and d_{x2-y2}-wave is obtained.
- We predict a T-H scenario with two SC phases, which has been observed on experiments recently.

TBG ($s^{\pm}$) & Sr_2RuO_4 ($d_{x2-y2}+ig$)

Chin. Phys. B Vol. 28, No. 7 (2019) 077103

RAPID COMMUNICATION

Possible nodeless $s^{\pm}$-wave superconductivity in twisted bilayer graphene*

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PHYSICAL REVIEW B **106**, 054516 (2022)

Multipole-fluctuation pairing mechanism of $d_{x^2-y^2} + ig$ superconductivity in Sr_2RuO_4

Yutao Sheng,[1,2] Yu Li,[3] and Yi-feng Yang[1,2,4,*]

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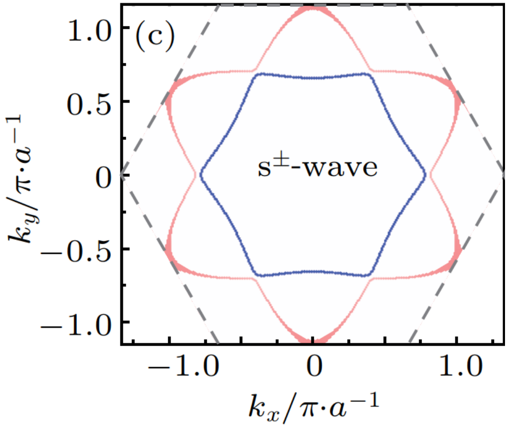
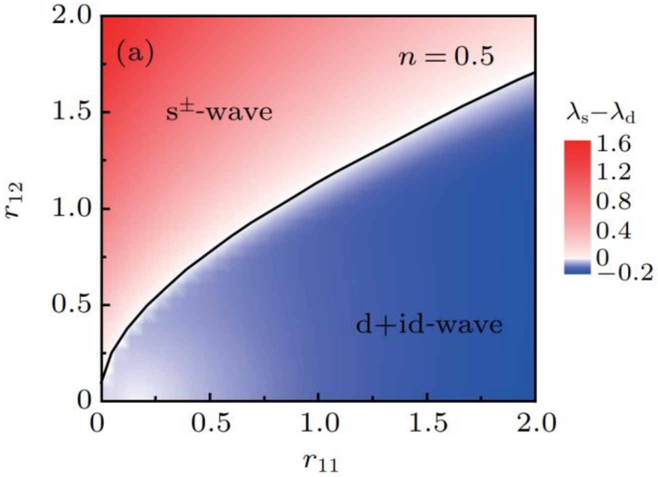
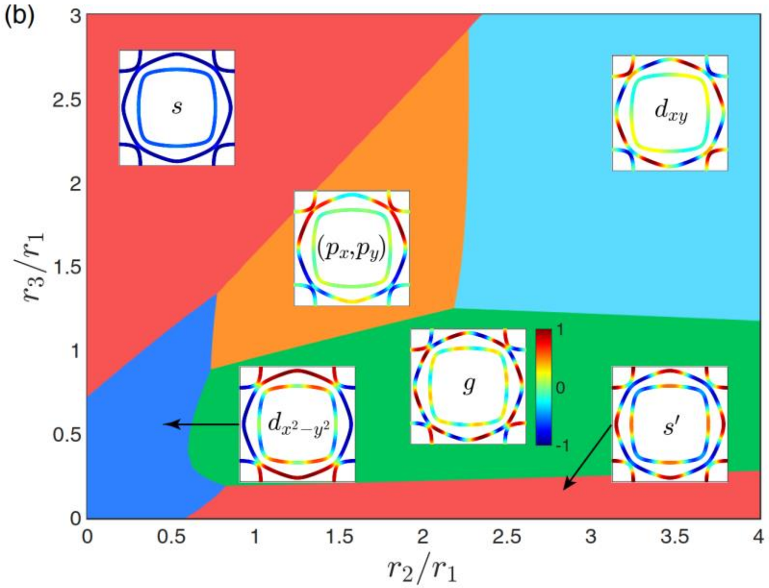






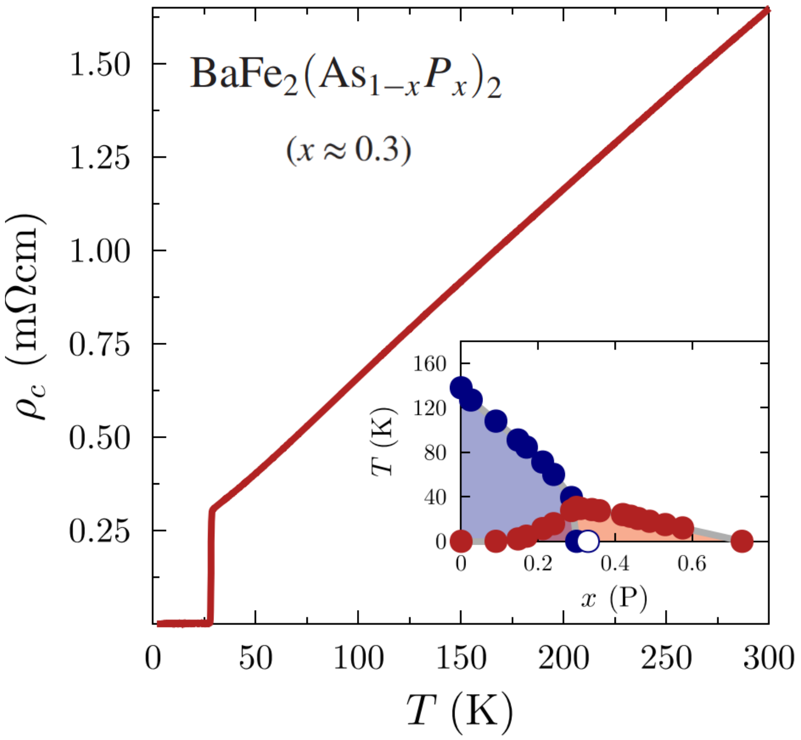
Z. Liu, Y. Li, Y.-f. Yang, Chin. Rev. B **28**, 077103 (2019)

Y. Sheng, Y. Li, Y.-f. Yang, Phys. Rev. B **106**, 054516 (2022)

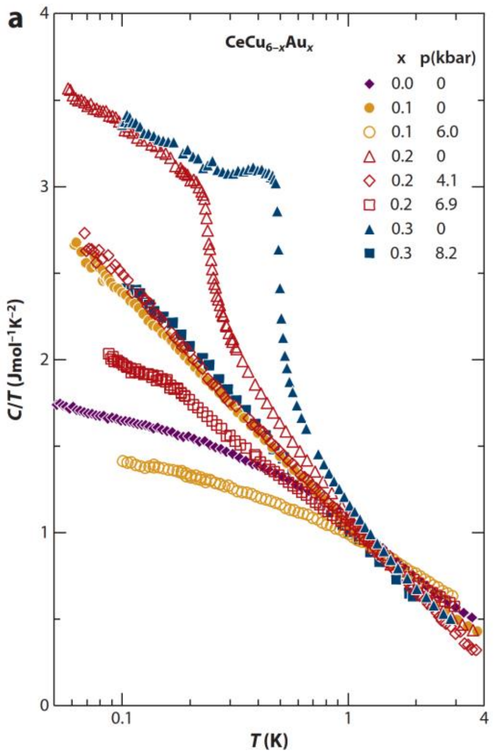
III. Non-Fermi-liquid SC

- Non-Fermi liquid phenomenology
- Advance in pairing from NFLs

Non-Fermi liquids

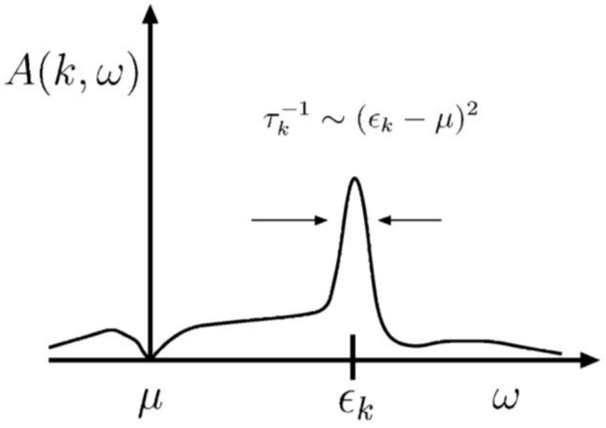


J. Chu et al., Phys. Rev. B **79**, 014506 (2009).



O. Stockert, F. Steglich, Ann. Rev. Condens. Matter Phys. **2**, 79 (2011)

Physical quantities	FL	NFL
Resistivity $\rho(T)$	T^2	T^α (α≠2)
Specific heat $C(T)$	T	e.g., $T \log T$
QP damping rate $\mathrm{Im}\Sigma(\omega)$	ω^2	ω^α (α<2)



In NFL, the coherent quasiparticle peak is broadened into incoherent states.

Non-Fermi liquid (NFL)

- Anomalous metallic transport & thermodynamic properties deviating from the Landau's Fermi liquid theory.
- Beyond Landau's quasiparticle description.

Non-Fermi liquids

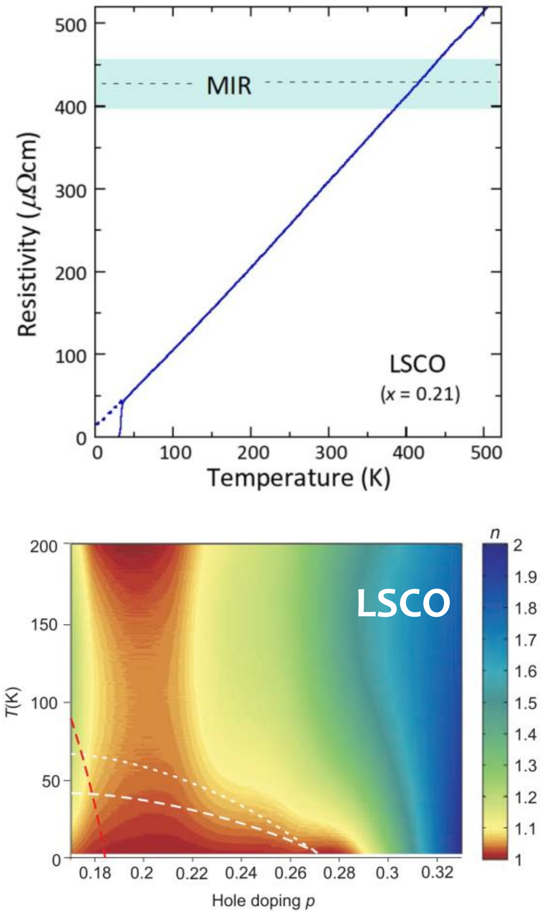
Experiments

	$\rho \propto T$ as $T \to \infty$	$\rho \propto T$ as $T \to 0$	Extended criticality	$\cot \Theta_H \propto T^2$ (at low H)	Modified Kohler's (at low H)	H-linear MR (at high H)	Quadrature MR
UD p-cuprates	√ (6)	× (20)	× (21)	√ (22)	√ (23)	—	—
OP p-cuprates	√ (4)	—	—	√ (24)	√ (25)	√ (26)	× (27)
OD p-cuprates	√ (6)	√ (8)	√ (8)	√ (28)	× (29)	√ (29)	√ (29)
$La_{2-x}Ce_xCuO_4$	× (30)	√ (31, 32)	√ (31, 32)	× (33)	× (34)	√ (35)	× (35)
Sr_2RuO_4	√ (36)	× (37)	× (38)	× (39)	× (37)	× (37)	× (37)
$Sr_3Ru_2O_7$	√ (10)	√ (10)	× (10)	×	—	—	—
$FeSe_{1-x}S_x$	× (40)	√ (41)	× (41)	√ (42)	√ (42)	√* (43)	√* (43)
$BaFe_2(As_{1-x}P_x)_2$	× (44)	√ (45)	× (45)	—	√ (46)	√ (47)	√ (47)
$Ba(Fe_{1/3}Co_{1/3}Ni_{1/3})_2As_2$	—	√ (48)	× (48)	—	—	√ (48)	√ (48)
$YbRh_2Si_2$	× (49)	√ (50)	√ (51)	√ (52)	—	—	—
$YbBAl_4$	× (53)	√** (53)	√** (53)	—	—	—	—
$CeCoIn_5$	× (54)	√ (55, 56)	× (55, 56)	√ (54)	√ (54)	—	—
$CeRh_6Ge_4$	× (57)	√ (57)	× (57)	—	—	—	—
$(TMTSF)_2PF_6$	—	√ (58)	√ (58)	—	—	—	—
MATBG	√ (59)	√ (60)	√ (60)	√ (61)	—	—	—

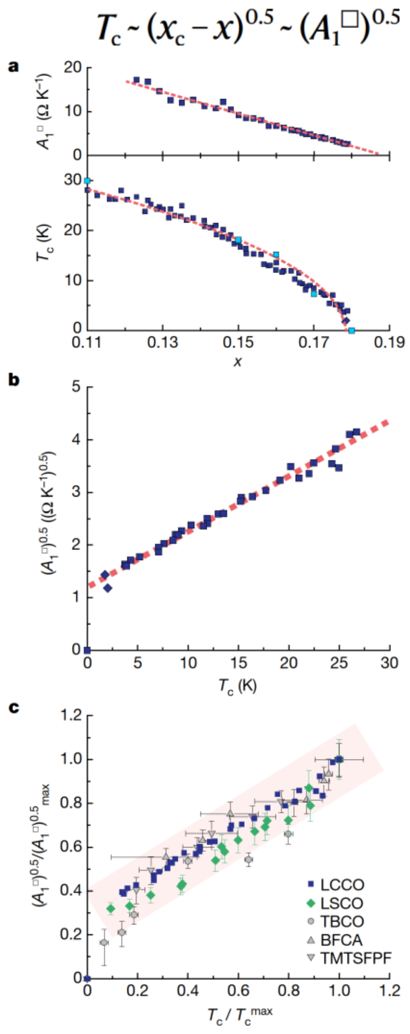
Theories

	$\rho \propto T$ as $T \to 0$	$\rho \propto T$ as $T \to \infty$	$\sigma \propto \omega^{-2/3}$	Quadrature MR	Extended criticality	Experimental prediction
Phenomenological						
MFL	√ (67)	× (67)	×	×	×	Loop currents (107)
EFL	— *	—	—	×	×	Loop currents (108)
Numerical						
ECFL	×	(109)	—	—	×	×
HM (QMC/ED/CA)	— (110)	√ (110–114)	×	—	—	—
DMFT/EDMFT	√ (115)	√ (116, 117)	×	—	√ (117)	—
QCP	(118)	—	—	—	×	—
Gravity-based						
SYK	√ (119, 120)	√** (120)	×	√*** (121)	—	×
AdS/CFT	√ (122)	√ (122)	√**** (90, 126)	×	×	×
AD/EMD	√ (127–129)	√ (90, 126, 127, 129, 130)	√ (90, 126, 130)	×	√ (126)	Fractional A-B (129)

P. W. Phillips, N. E. Hussey, P. Abbamonte, Science **377**, 169 (2022)



R. A. Cooper et al., Science **323**, 603 (2009)



J. Yuan, …, K. Jin, Nature **602**, 431 (2022)

NFL acts as a base state for the Cooper pairing

⟹ **NFL superconductivity**

Quantum critical metals (NFL): Monte Carlo simulations

PRL **117**, 097002 (2016) PHYSICAL REVIEW LETTERS week ending 26 AUGUST 2016

Competing Orders in a Nearly Antiferromagnetic Metal

Yoni Schattner,[1] Max H. Gerlach,[2] Simon Trebst,[2] and Erez Berg[1]

PHYSICAL REVIEW B **95**, 035124 (2017)

Quantum critical properties of a metallic spin-density-wave transition

Max H. Gerlach,[1] Yoni Schattner,[2] Erez Berg,[2] and Simon Trebst[1]

PHYSICAL REVIEW X **6**, 031028 (2016)

Ising Nematic Quantum Critical Point in a Metal: A Monte Carlo Study

Yoni Schattner,[1] Samuel Lederer,[3] Steven A. Kivelson,[2] and Erez Berg[1]

PNAS

Superconductivity and non-Fermi liquid behavior near a nematic quantum critical point

Samuel Lederer[a,1], Yoni Schattner[b,1], Erez Berg[b,c], and Steven A. Kivelson[d,2]

Annual Review of Condensed Matter Physics

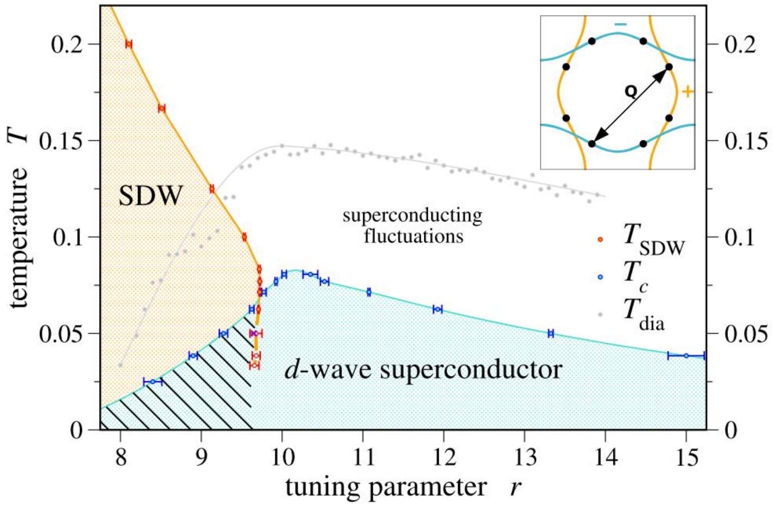
Monte Carlo Studies of Quantum Critical Metals

Erez Berg[1], Samuel Lederer[2], Yoni Schattner[3,4], and Simon Trebst[5]

E. Berg *et al.*, Ann. Rev. Condens. Matter **10**,63 (2019).

SDW model

$$\chi_{\text{fit}}^{-1} = a|\mathbf{q} - \mathbf{Q}|^2 + b|\omega_n| + c(r - r_c).$$



Y. Schattner *et al*, Phys. Rev. Lett. **117**, 097002 (2016).
M. H. Gerlach *et al*, Phys. Rev. B **95**, 035124 (2017).

$$H_{nem} = -h \sum_{\langle \mathbf{r},\mathbf{r}'\rangle} \mu^x_{\mathbf{r},\mathbf{r}'} + V \sum_{\{\mathbf{r},\mathbf{r}'\},\{\mathbf{r}',\mathbf{r}''\}} \mu^z_{\mathbf{r},\mathbf{r}'}\mu^z_{\mathbf{r}',\mathbf{r}''},$$

$$H_{int} = \alpha t \sum_{\langle \mathbf{r},\mathbf{r}'\rangle,\sigma} \mu^z_{\mathbf{r},\mathbf{r}'} \left(c^\dagger_{\mathbf{r}\sigma} c_{\mathbf{r}'\sigma} + h.c.\right).$$

Fermionic spectral weight

- weak coupling: renormalized FL behavior;
- strong coupling: NFL scaling above T_c.
- ImΣ is nearly independent on ω and T.

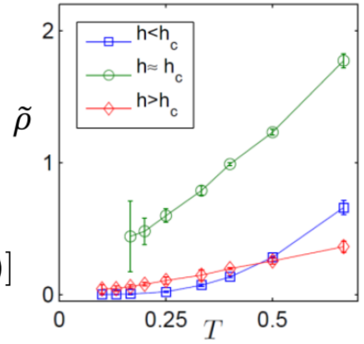
Transport

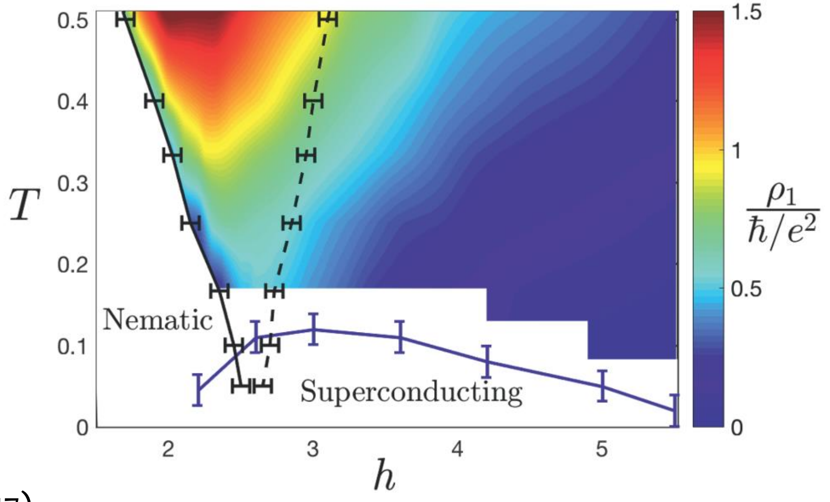
- T-linear resistivity near the QCP.

Y. Schattner *et al*, Phys. Rev. X **6**, 031028 (2016).
S. Lederer *et al*, Proc. Natl. Acad. Sci. USA. **114**, 4905 (2017).

Nematic model

$$\tilde{D}^{-1} = A[T + b|\mathbf{q}|^2 + c(h - h_c)]$$





Rise and fall of hot spots in SC mediated from AFM QCP

Hamiltonian: two-band model

$$\mathcal{H}_0 = \sum_{\mathbf{k}\alpha} \varepsilon_{c,\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} + \sum_{\mathbf{k},\alpha} \varepsilon_{d,\mathbf{k}} d^\dagger_{\mathbf{k}\alpha} d_{\mathbf{k}\alpha}.$$

with

$$\varepsilon_{c,\mathbf{k}} = \mu - 2(t+\delta)\cos k_x - 2(t-\delta)\cos k_y,$$

$$\varepsilon_{d,\mathbf{k}+\mathbf{Q}} = -\mu + 2(t-\delta)\cos k_x + 2(t+\delta)\cos k_y,$$

(Sign-problem free for DQMC calculation.)

Spin-fermion model

$$S_{\text{int}} = \lambda \sum_j \int_\tau \mathbf{M}_j e^{i\mathbf{Q}\cdot\mathbf{x}_j} \cdot (c^\dagger_{j,\alpha} \boldsymbol{\sigma}_{\alpha\beta} d_{j,\beta} + \text{H.c.}).$$

$$S_{\text{mag}} = \frac{1}{2}\int_{\mathbf{x},\tau}\left[\frac{1}{v_s^2}(\partial_\tau \mathbf{M})^2 + (\nabla\mathbf{M})^2 + rM^2 + \frac{u}{2}M^4\right].$$

Susceptibility (one-loop approximation)

$$\chi^{-1}(\mathbf{q}, i\Omega_n) = \tilde{r} + \mathbf{q}^2 a^2 + \frac{|\Omega_n|}{\gamma},$$

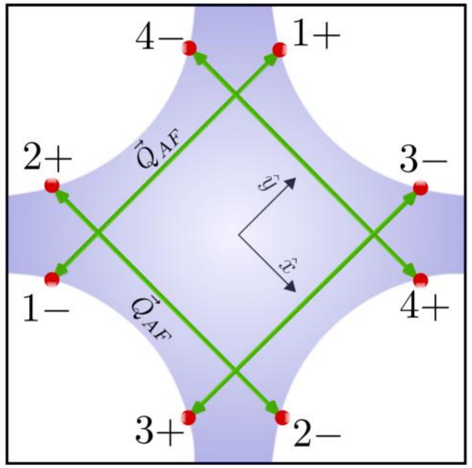
$$\gamma = \frac{\pi v_F^2 \sin\theta_{\text{hs}}}{\lambda^2 N}.$$

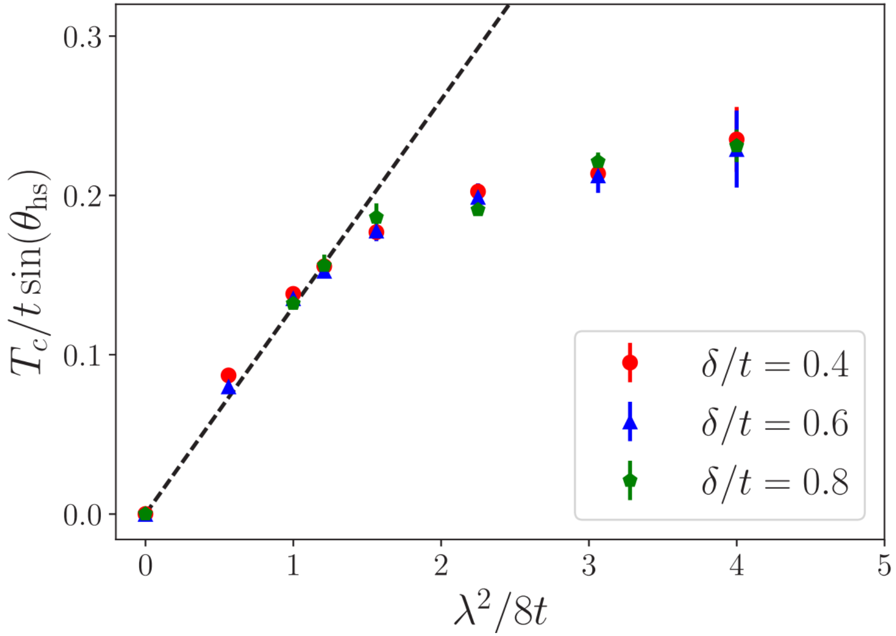
Eliashberg equations

$$(1 - Z_{1,k}) i\omega_n = -\frac{n_b\lambda^2}{\beta V}\sum_{\omega_m,\mathbf{p}} \chi(\mathbf{k}-\mathbf{p}, i\omega_n - i\omega_m)\left(\frac{Z_{2,p} i\omega_m}{Z_{2,p}^2\omega_m^2 + \varepsilon_{2,\mathbf{p}}^2}\right),$$

$$\phi_{1,k} = -\frac{n_b\lambda^2}{\beta V}\sum_{\omega_m,\mathbf{p}} \chi(\mathbf{k}-\mathbf{p}, i\omega_n - i\omega_m)\left(\frac{\phi_{2,p}}{Z_{2,p}^2\omega_m^2 + \varepsilon_{2,\mathbf{p}}^2}\right).$$

hot spots





Hot-spots-dominated pairing (small λ²)	→	Fermi surface-dominated pairing (large λ²)
$T_c = A_c\lambda^2 \sin\theta_{\text{hs}}$		$T_c \propto p_0 v_F \sin\theta_{\text{hs}}$

X. Wang, Y. Schattner, E. Berg, and R. Fernandes, PRB **95**, 174520 (2017)

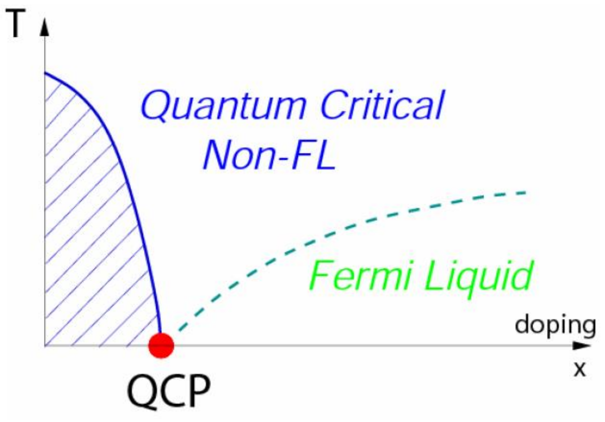
γ-model (A. Chubukov, et al.)

Eliashberg equations

$$\begin{cases} Z(\omega_n) = 1 + \dfrac{\pi T}{\omega_n} \displaystyle\sum_{\omega_m} \lambda(\omega_m - \omega_n) \dfrac{\omega_m}{\sqrt{\omega_m^2 + \Delta^2(\omega_m)}}, \\ Z(\omega_n)\Delta(\omega_n) = \pi T \displaystyle\sum_{\omega_m} \lambda(\omega_m - \omega_n) \dfrac{\Delta(\omega_m)}{\sqrt{\omega_m^2 + \Delta^2(\omega_m)}}. \end{cases}$$

Near QCP ($\xi^{-2} \to 0$),

$$\Sigma(\omega) = \omega^{1-\gamma} (\omega_0)^\gamma$$



<u>γ-model</u> $\lambda(\Omega) = \left(\dfrac{\Omega_0}{|\Omega|}\right)^\gamma$

- The same interaction responsible for NFL and SC pairing simultaneously.
- Singular fermionic self-energy (NFL) kills the Cooper logarithm.
- Pairing gaps out low-energy states and restores a FL.

$\gamma = 1/2$	2D antiferromagnetic QCP
$\gamma = 1/3$	2D ferromagnetic QCP/interaction with gauge field/nematic
$\gamma = 1/4$	2D $2k_F$ QCP
$\gamma = 2$	QCP of Einstein phonons
$\gamma = 1$	2D QCP of fermions interacting with undamped bosons
$\gamma = +0$ (log ω)	3D QCP, Color superconductivity
$\gamma = 0(\varepsilon)$	QCP in 3-ε dimension

(from A. Chubukov's talk)

Special role of the first Matsubara frequency

PRL **117**, 157001 (2016) PHYSICAL REVIEW LETTERS week ending 7 OCTOBER 2016

Superconductivity near a Quantum-Critical Point: The Special Role of the First Matsubara Frequency

Yuxuan Wang,[1] Artem Abanov,[2] Boris L. Altshuler,[3] Emil A. Yuzbashyan,[4] and Andrey V. Chubukov[5]

Eliashberg equations

$$\begin{cases} \Sigma(\omega_n) = \pi T \sum_{\omega_m} \lambda(\omega_m - \omega_n) \dfrac{\omega_m + \Sigma(\omega_m)}{\sqrt{[\omega_m + \Sigma(\omega_m)]^2 + \Phi^2(\omega_m)}}, \\ \Phi(\omega_n) = \pi T \sum_{\omega_m} \lambda(\omega_m - \omega_n) \dfrac{\Phi(\omega_m)}{\sqrt{[\omega_m + \Sigma(\omega_m)]^2 + \Phi^2(\omega_m)}}, \end{cases}$$

where $\lambda(\Omega) = \left(\frac{\Omega_0}{|\Omega|}\right)^\gamma$,

- Recast the gap equation,

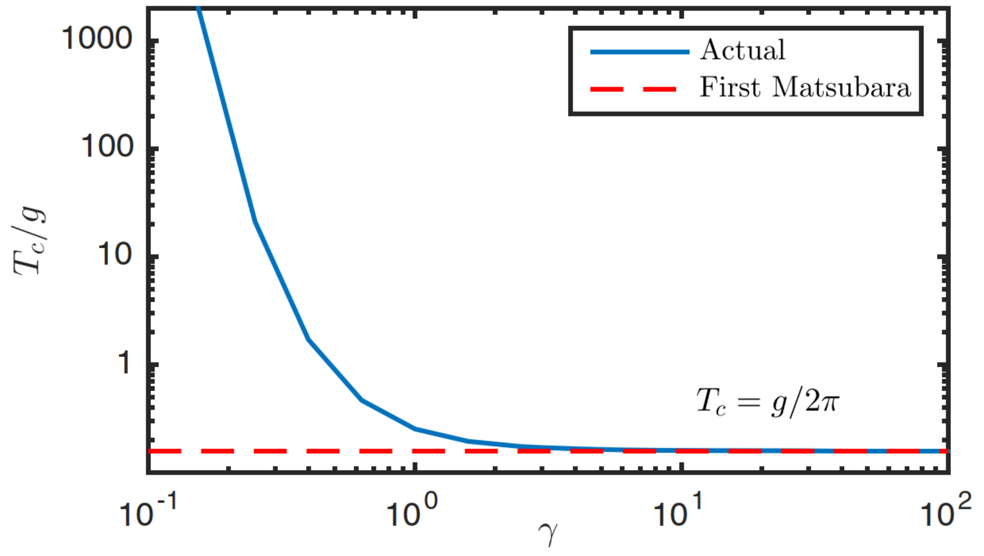
$$\Delta(\omega_n) = \pi T \sum_{\omega_m} \frac{\lambda(\omega_m - \omega_n)}{\sqrt{\omega_m^2 + \Delta^2(\omega_m)}} \left(\Delta(\omega_m) - \Delta(\omega_n) \frac{\omega_m}{\omega_n} \right).$$

- Self-energy at first Matsubara frequency

$$\Sigma(\pi T) = [g/(2\pi T)]^\gamma \pi T \textstyle\sum_{m' \neq 0} \ \mathrm{sgn}(2m'+1)/|m'|^\gamma,$$

$$T_c \sim \omega_0 (\gamma N)^{-1/\gamma} \sim \frac{g}{2\pi N^{1/\gamma}} e^{\log(b/\gamma)/\gamma} \gg \frac{g}{2\pi N^{1/\gamma}},$$

Y. Wang *et al*, Phys. Rev. Lett. **117**, 157001 (2016)

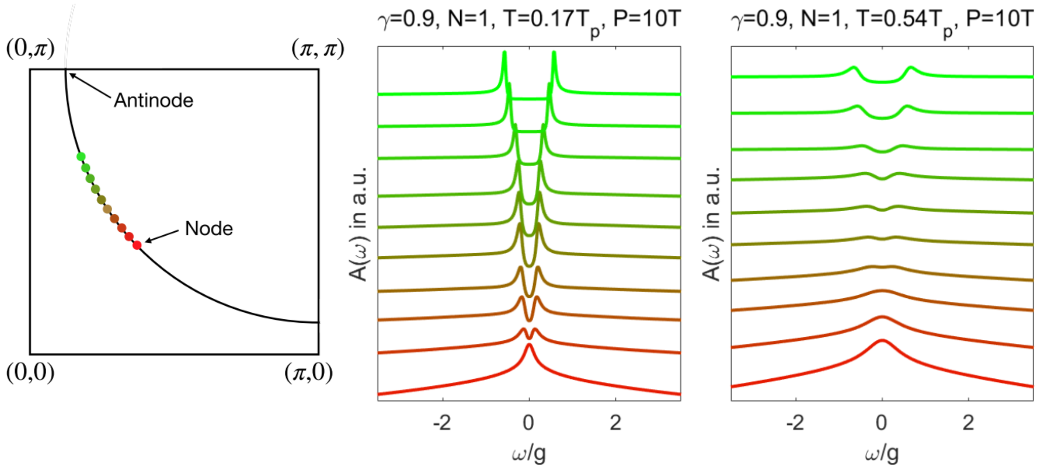


- In $\omega_n = \omega_m$: there is an exact cancellation of the singular critical fluctuations in gap equation. ⇨ **QCP will not destroy SC.**
- $\Sigma(\pm\pi T) = 0$ suggest it is irrelevant to pairing.
- A non-zero T_c can always be obtained due to $\Sigma(\pm\pi T) = 0$ ⇨ **SC always wins out NFL.**

SC can always emerging out near a QCP, no matter what the interplay between pairing interaction and fermionic incoherence.

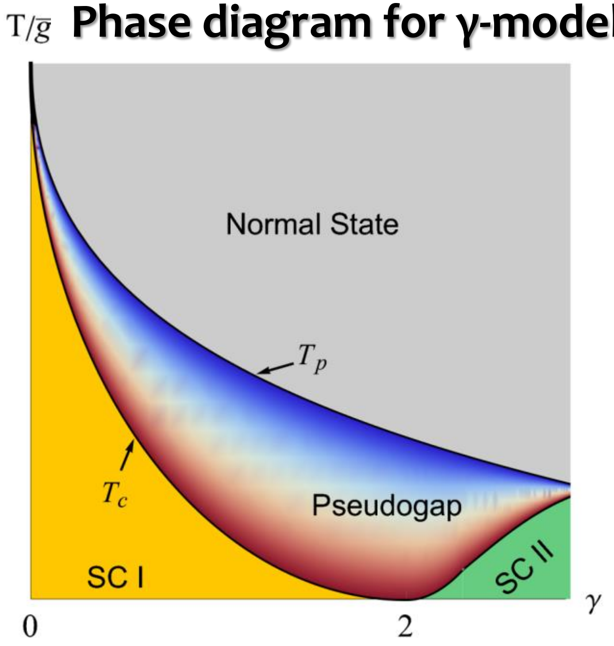
Interplay of SC and NFL near QCP (A. Chubukov, et al.)

"gap-filling" behavior -- Pseudogap



Abanov. *et al*, PRB **99**, 180506(R) (2019); Y.-M. Wu. *et al*, PRB **99**, 144512 (2019)

Phase diagram for γ-model

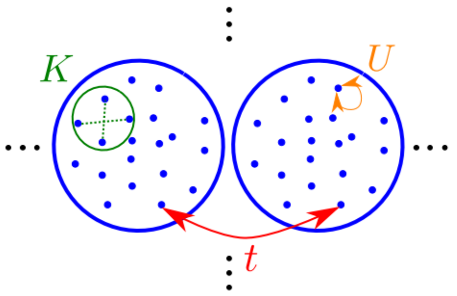


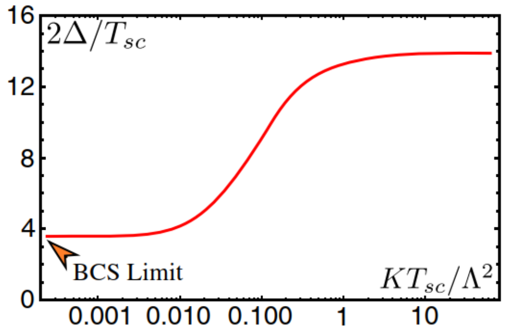
A. Abanov et al, PRB **102**, 024524 (2020).
Y.-M. Wu. *et al*, PRB **102**, 024525 (2020).
V. Chubukov. *et al*, Ann. Phys. **417**, 168142 (2020)
Y.-M. Wu. *et al*, PRB **103**, 024522 (2021)
Y.-M. Wu. *et al*, PRB **103**, 184508 (2021)
S.-S. Zhang. *et al*, PRB **104**, 144509 (2021)

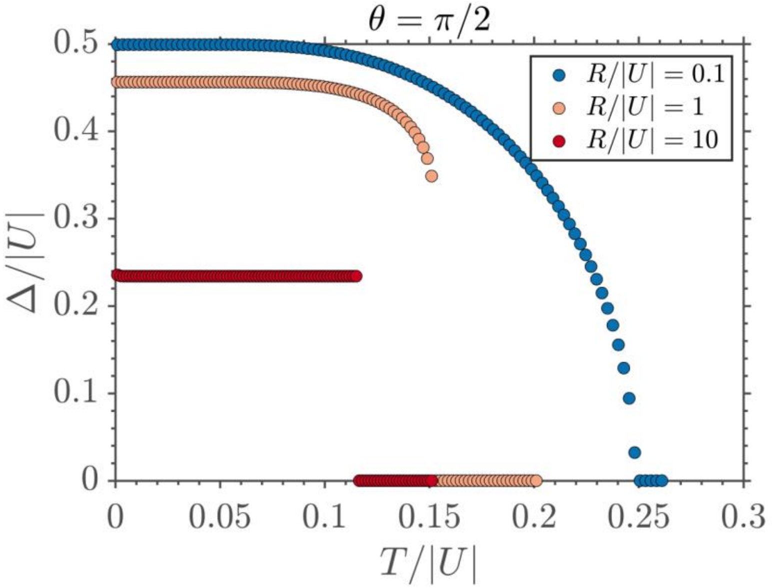
- $T_c<T<T_p$: incoherence of fermions induce a gap-filling behavior; $T<T_c$: fermions acquire coherence and develop a SC gap.
- Gap fluctuations due to the sign-changing $\Delta(i\omega_n)$
- ……

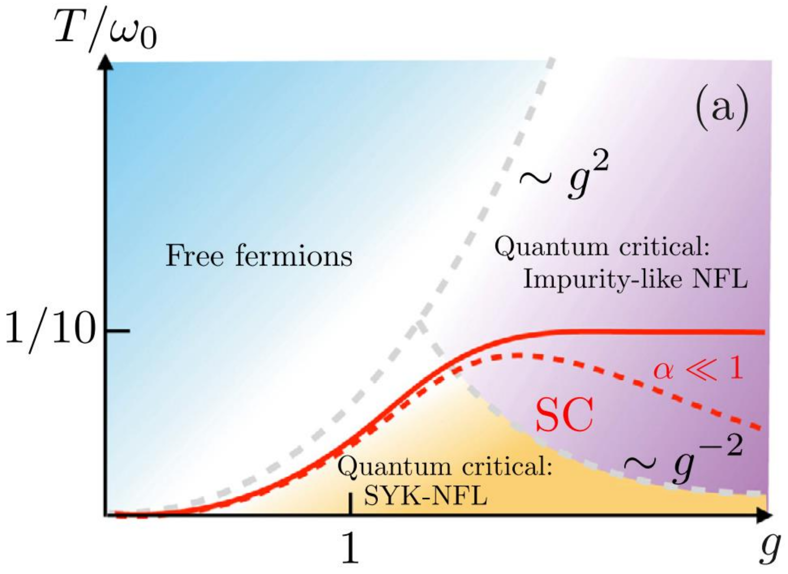
SC from incoherent metals (SYK model)

$$H_1 = \sum_m \sum_{i_1,\ldots,i_4=1}^{N} [K^{am}_{i_1,\ldots,i_4} a^\dagger_{i_1 m} a^\dagger_{i_2 m} a_{i_3 m} a_{i_4 m} + (a \leftrightarrow b)]$$
$$- t \sum_{\langle mn \rangle} \sum_{i=1}^{N} [a^\dagger_{im} a_{in} + (a \leftrightarrow b) + \text{H.c.}]$$
$$- \frac{U}{N} \sum_m \sum_{i,j=1}^{N} b^\dagger_{im} a^\dagger_{im} a_{jm} b_{jm},$$









- **Spinful SYK, SYK+Hubbard, Yukawa-SYK, …**
- Coherent SC from incoherent metals with large Δ/T_c ratio (D. Chowdhury, E. Berg, 2020)
- 1st-order SC transition (M. Franz, 2021; S. Sachdev, 2022)
- Odd-ω SC (N.V. Gnezdilov, 2019)
- Kosterlitz-Thouless quantum-critical behavior (Y. Wang, 2020);
- Holographic SC (J. Schmalian, 2019, 2020, 2022)
- ……

Summary: Effectiveness & Failure of Eliashberg theories

- Combining with a phenomenological pairing interaction, Eliashberg theory provide a simple way to tackle with real materials and compare with experiments.
- The effectiveness of the Eliashberg theory is preserved when the perturbation theory become exact, which relate to the Migdal's theorem in el-ph systems, and the assumptions $v_{\mathrm{boson}} \ll v_{\mathrm{fermion}}$ in spin-fermion model, and the large-N expansion in SYK models.
- The intertwining of SC and NFL or other orders, made the validity of Eliashberg calculations become a more subtle issue.
- Comparisons with DQMC simulations show a threshold for the validity of Migdal-Eliashberg theory, due to the phonon softening at strong coupling.

PHYSICAL REVIEW B **97**, 140501(R) (2018)

Rapid Communications

Breakdown of the Migdal-Eliashberg theory: A determinant quantum Monte Carlo study

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Eliashberg theory of phonon-mediated superconductivity — When it is valid and how it breaks down☆

Check for updates

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Thanks for your attention !