Kavli Institute for Theoretical Sciences
University of Chinese Academy of Sciences (UCAS)

Two-stage superconductivity in Hatsugai-Kohmoto-BCS model

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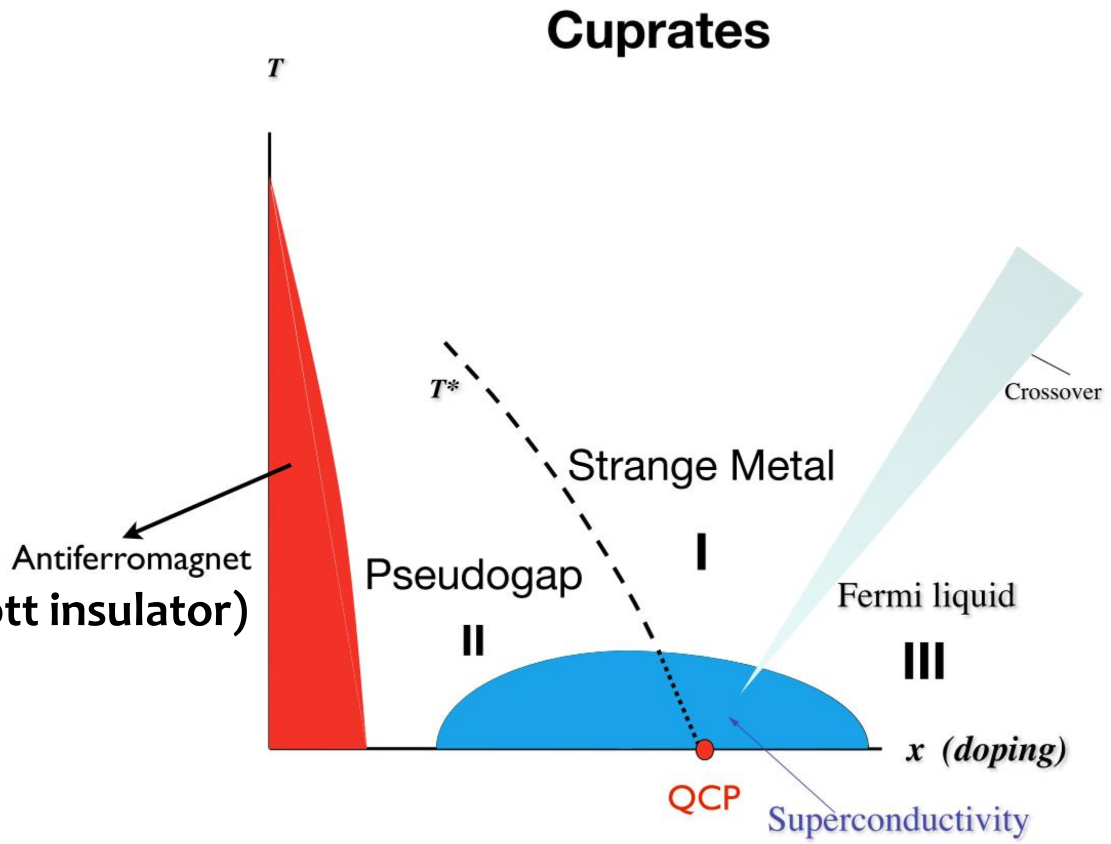
2022-12-23

Ref: Y. Li, V. Mishra, Y. Zhou, and F. -C. Zhang, New J. Phys. **24**, 103019 (2022).

Outline

- Introduction to HK model
- Revisit the Cooper instability
- Two-stage superconductivity
- Ginzburg-Landau analysis
- Summary & Outlook

Where do we approaching to the unconventional SC?

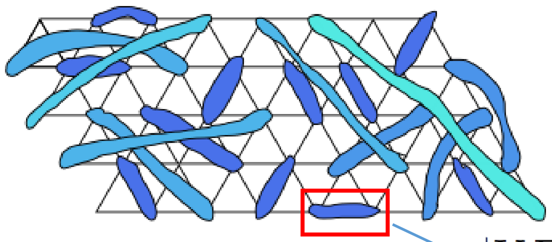


(Mott insulator)

C. Varma, RMP **92**, 031001 (2020)

- Spin-fluctuation mechanism: FL ⇨ SC.
- Interplay of SC & NFL near QCP: NFL ⇨ SC.
- RVB scenario (no pairing glue): QSL ⇨ SC.
- Hubbard model/t-J model: Mott insulator ⇨ Competing orders (AFM/CDW/…) ⇨ SC.
- ……

Resonate-valence-bond state (P. W. Anderson, 1976, 1987)

$|\Psi\rangle_{\mathrm{RVB}} =$  $+ \dots$

$|VB\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

[Motivation]: ***What about a doped Mott insulator with simultaneously beyond Landau's Fermi-liquid theory (NFL) have the SC instability?***

Non-Fermi liquids (previous talk)

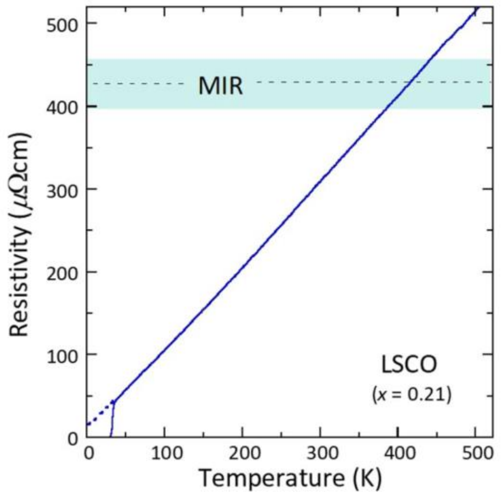
Experiments

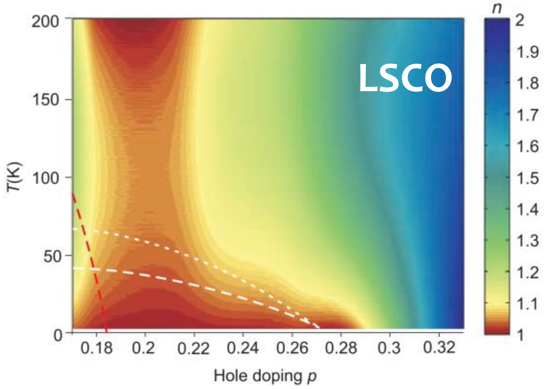
	ρ ∝ T as $T \to \infty$	ρ ∝ T as $T \to 0$	Extended criticality	cot $\Theta_H \propto T^2$ (at low H)	Modified Kohler's (at low H)	H-linear MR (at high H)	Quadrature MR
UD p-cuprates	√ (6)	× (20)	× (21)	√ (22)	√ (23)	—	—
OP p-cuprates	√ (4)	—	—	√ (24)	√ (25)	√ (26)	× (27)
OD p-cuprates	√ (6)	√ (8)	√ (8)	√ (28)	× (29)	√ (29)	√ (29)
$La_{2-x}Ce_xCuO_4$	× (30)	√ (31, 32)	√ (31, 32)	× (33)	× (34)	√ (35)	× (35)
Sr_2RuO_4	√ (36)	× (37)	× (38)	× (39)	× (37)	× (37)	× (37)
$Sr_3Ru_2O_7$	√ (10)	√ (10)	× (10)	×	—	—	—
$FeSe_{1-x}S_x$	× (40)	√ (41)	× (41)	√ (42)	√ (42)	√* (43)	√* (43)
$BaFe_2(As_{1-x}P_x)_2$	× (44)	√ (45)	× (45)	—	√ (46)	√ (47)	√ (47)
$Ba(Fe_{1/3}Co_{1/3}Ni_{1/3})_2As_2$	—	√ (48)	× (48)	—	—	√ (48)	√ (48)
$YbRh_2Si_2$	× (49)	√ (50)	√ (51)	√ (52)	—	—	—
$YbBAl_4$	× (53)	√** (53)	√** (53)	—	—	—	—
$CeCoIn_5$	× (54)	√ (55, 56)	× (55, 56)	√ (54)	√ (54)	—	—
$CeRh_6Ge_4$	× (57)	√ (57)	× (57)	—	—	—	—
$(TMTSF)_2PF_6$	—	√ (58)	√ (58)	—	—	—	—
MATBG	√ (59)	√ (60)	√ (60)	√ (61)	—	—	—

Theories

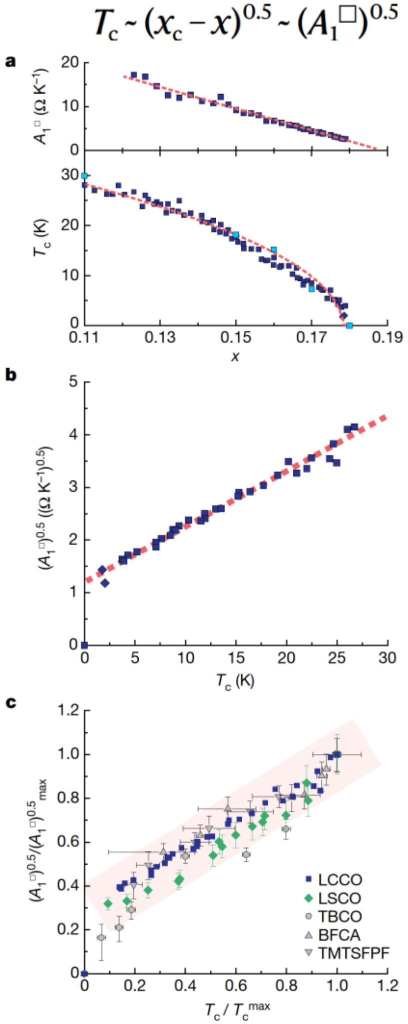
	ρ ∝ T as $T \to 0$	ρ ∝ T as $T \to \infty$	$\sigma \propto \omega^{-2/3}$	Quadrature MR	Extended criticality	Experimental prediction
Phenomenological						
MFL	√ (67)	× (67)	×	×	×	Loop currents (107)
EFL	—*	—	—	×	×	Loop currents (108)
Numerical						
ECFL	×	(109)	—	—	×	×
HM (QMC/ED/CA)	— (110)	√ (110–114)	×	—	—	—
DMFT/EDMFT	√ (115)	√ (116, 117)	×	—	√ (117)	—
QCP	(118)	—	—	—	×	—
Gravity-based						
SYK	√ (119, 120)	√** (120)	×	√*** (121)	—	×
AdS/CFT	√ (122)	√ (122)	√**** (90, 126)	×	×	×
AD/EMD	√ (127–129)	√ (90, 126, 127, 129, 130)	√ (90, 126, 130)	×	√ (126)	Fractional A-B (129)

P. W. Phillips, N. E. Hussey, P. Abbamonte, Science **377**, 169 (2022)





R. A. Cooper et al., Science **323**, 603 (2009)



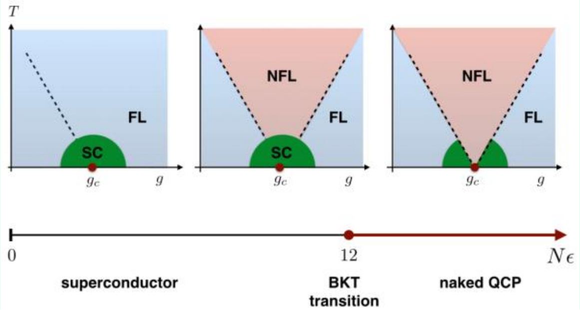
J. Yuan, …, K. Jin, Nature **602**, 431 (2022)

NFL acts as a base state for the Cooper pairing
⟹ **NFL superconductivity**

Scenarios of pairing from NFL (previous talk)

Coupling of fermions and critical bosons

- **Spin-fermion model, hot-spots, patch theory, large-N expansion, ε-expansion, Bosonic quantum criticality, ……**
- Rise and fall of hot spots (E. Berg, R. M. Fernandes, 2017)
- Competition of SC & NFL (M. A. Metlitski, 2015;)
- Quantum fluctuations v.s. thermal fluctuations (H. Wang, G. Torroba, 2017)
- ……

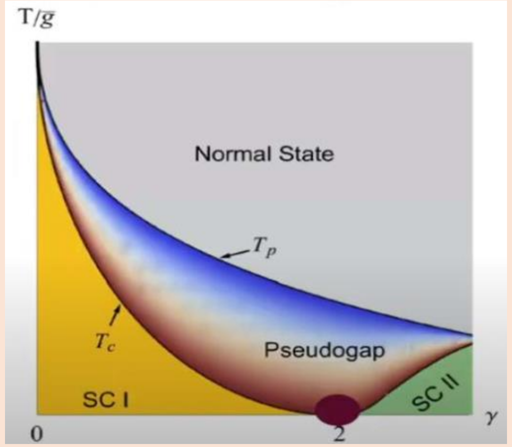


Quantum-critical pairing with anomalous retardations

- **γ-model** (A. V. Chubukov, A. Abanov, Y. Wang)

$$V(\Omega_m) = (\bar{g}/|\Omega_m|)^\gamma$$

- Special role of 1^{st}-Matsubara frequency
- Gap fluctuations due to the sign-changing $\Delta(i\omega_n)$
- Pseudogap behavior
- ……



Fermions with random interactions

- **Spinful SYK, SYK+Hubbard, Yukawa-SYK, …**
- Coherent SC from incoherent metals with large Δ/T_c ratio (D. Chowdhury, E. Berg, 2020)
- 1^{st}-order SC transition (M. Franz, 2021; S. Sachdev, 2022)
- Odd-ω SC (N.V. Gnezdilov, 2019)
- Kosterlitz-Thouless quantum-critical behavior (Y. Wang, 2020);
- Holographic SC (J. Schmalian, 2019, 2020, 2022)
- ……

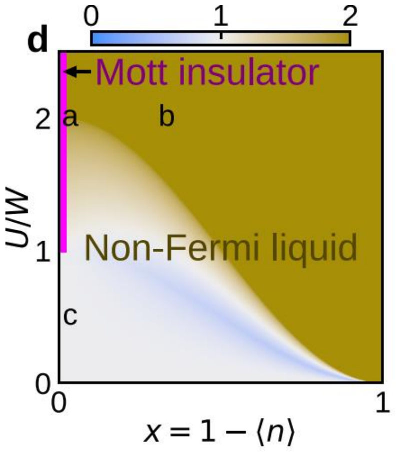
……

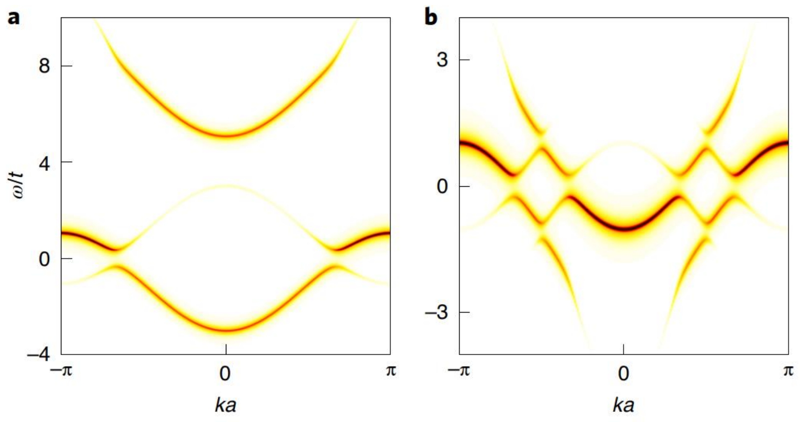
Motivation: Non-Fermi liquid superconductivity

nature physics LETTERS
https://doi.org/10.1038/s41567-020-0988-4
Check for updates

Exact theory for superconductivity in a doped Mott insulator

Philip W. Phillips, Luke Yeo and Edwin W. Huang





$$H_{\rm HK} = \sum_k H_k = \sum_k \left(\xi_k (n_{k\uparrow} + n_{k\downarrow}) + U n_{k\uparrow}\ n_{k\downarrow}\right)$$

- Luttinger count ≠ filling: NFL (beyond Landau's FL theory).
  $2\sum_k \theta(\mathrm{Re}G(k,0))$
- Green's function $G_\sigma(\mathbf{k}, i\omega_n) = \dfrac{1}{i\omega_n - \xi_\mathbf{k} - \langle n_{k\bar\sigma}\rangle U - \frac{\langle n_{k\bar\sigma}\rangle(1-\langle n_{k\bar\sigma}\rangle)U^2}{i\omega_n - \xi_\mathbf{k} - (1-\langle n_{k\bar\sigma}\rangle)U}}$

Green's function of HK model ≈ YRZ's ansatz for cuprates

PHYSICAL REVIEW B **101**, 184506 (2020)

Pairing instability on a Luttinger surface: A non-Fermi liquid to superconductor transition and its Sachdev-Ye-Kitaev dual

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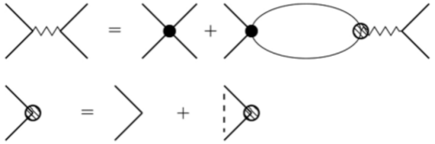
(Received 6 September 2019; revised manuscript received 15 April 2020; accepted 22 April 2020; published 13 May 2020)

$$G(\mathbf{p}, \epsilon_n)^{-1} = i\epsilon_n - \xi(\mathbf{p}) - \Sigma(\mathbf{p}, \epsilon_n), \qquad \Sigma(\mathbf{p}, i\epsilon_n) = \frac{u^2}{i\epsilon_n + \xi(\mathbf{p})}.$$



$$L^{-1}(\mathbf{q}, \Omega) = -g^{-1} + \Pi(\mathbf{q}, \Omega),$$

PHYSICAL REVIEW B 73, 174501 (2006)

Phenomenological theory of the pseudogap state

Kai-Yu Yang,[1] T. M. Rice,[1,2] and Fu-Chun Zhang[1]
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[2]*Institut für Theoretische Physik, ETH Zurich, CH-8093 Zürich, Switzerland*

(Received 18 January 2006; published 3 May 2006)

$$G_a^{RPA}(\mathbf{k},\omega) = \frac{z_a}{\omega - \epsilon_a(k) - t_\perp(k_\perp) - \Delta^2/[\omega + \epsilon_a(k)]}.$$

- Through the similarity of the Green's function, any relation between HK model and the cuprates?

Phenomenological theory of the pseudogap state

Renormalized t-J model (doped valence bond insulator)

$$H_{eff} = g_t T + g_s J \sum_{\langle i,j \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j \qquad g_t = \frac{2x}{1+x}, \quad g_s = \frac{4}{(1+x)^2}$$

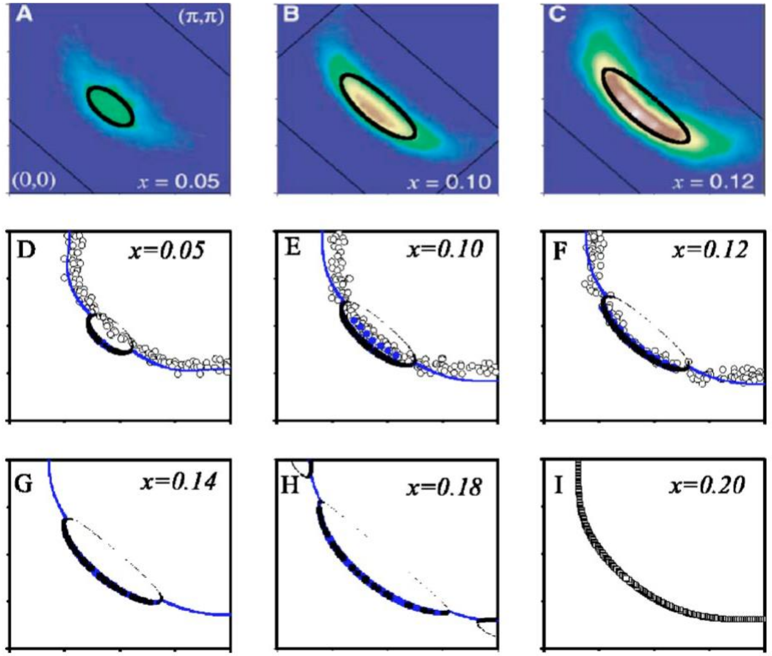
- RVB ansatz $\chi_{i,j} = \langle c_{i,\sigma}^\dagger c_{j,\sigma} \rangle \quad \Delta_{i,j} = \langle c_{i,\sigma} c_{j,\sigma} \rangle$
- A phenomenological Green's function

Two-leg Hubbard ladders (doped spin liquid) [R. M. Konik et al., 2000, 2001, 2006]

$$G^{RVB}(\boldsymbol{k},\omega) = \frac{g_t}{\omega - \xi(\boldsymbol{k}) - \Delta_R^2/[\omega + \xi_0(\boldsymbol{k})]} + G_{inc}, \quad \Sigma_R(\boldsymbol{k},\omega) = |\Delta_R(\boldsymbol{k})|^2/[\omega + \xi_0(\boldsymbol{k})] \text{ (RVB self-energy)}$$

- Green's function for hole-doped d-wave SC

$$G_{coh}^S(\boldsymbol{k},\omega) = \frac{g_t}{\omega - \xi(\boldsymbol{k}) - \Sigma_R(\boldsymbol{k},\omega) - |\Delta_S(\boldsymbol{k})|^2/[\omega + \xi(\boldsymbol{k}) + \Sigma_R(\boldsymbol{k},-\omega)]},$$



K.-Y. Yang, T. M. Rice, F.-C. Zhang, PRB **73**, 174501 (2006).

- The ansatz shows good agreement with ARPES data.
- The transition in the normal state from the doped RVB (underdoped) to a standard Landau Fermi liquid (overdoped) is characterized by the change of the form of Green's function: at the QCP, RVB state gapped the spectrum, leading a Luttinger surface coinciding with the Umklapp surface.
- In the d-wave SC state, another Luttinger surface emerging in the nodal direction, which is converted from the normal-state FS.

Revisit: Hatsugai-Kohmoto model (1992)

Hatsugai-Kohmoto model (1992)

$$H_{\mathrm{HK}} = \sum_{\mathbf{k},\sigma} (\varepsilon_{\mathbf{k}} - \mu)\, c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + U \sum_{\mathbf{k}} n_{\mathbf{k}\uparrow} n_{\mathbf{k}\downarrow}$$

(Y. Hatsugai, M. Kohomoto, JPSJ **61**, 2056 (1992))

- There are 4 states at each **k** point:

$$|0\rangle,\ c^{\dagger}_{\mathbf{k},\uparrow}|0\rangle,\ c^{\dagger}_{\mathbf{k},\downarrow}|0\rangle,\ c^{\dagger}_{\mathbf{k},\uparrow}c^{\dagger}_{\mathbf{k},\downarrow}|0\rangle$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

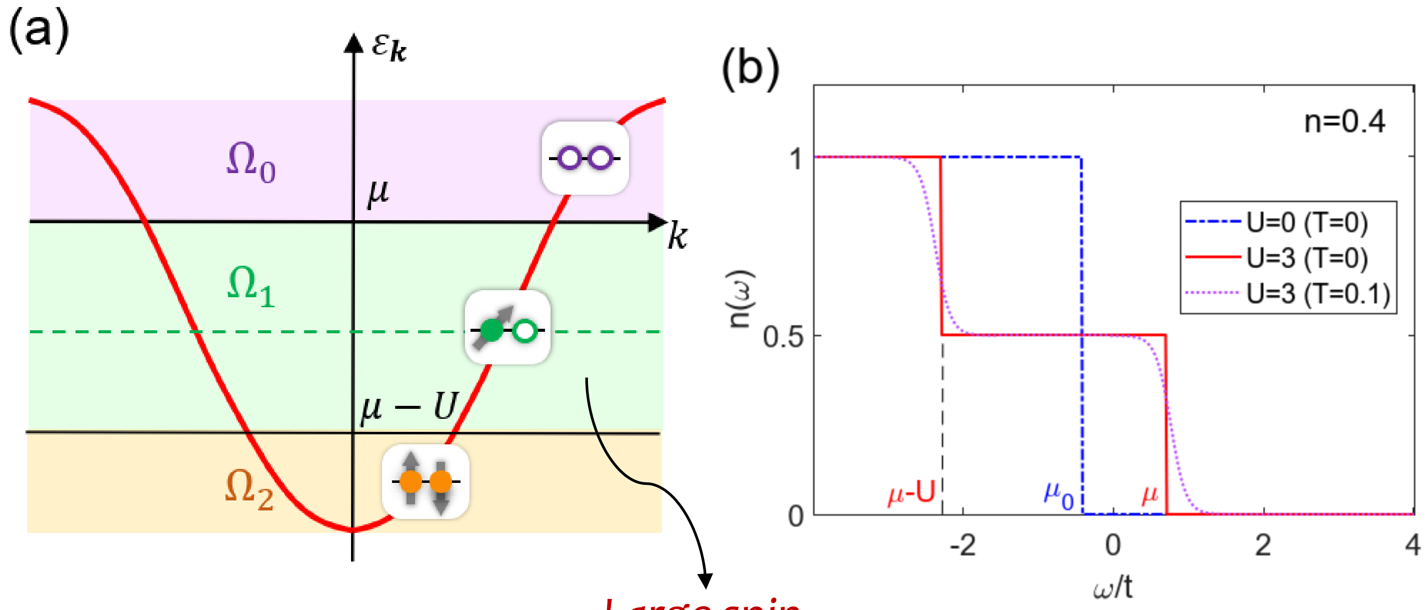
$$0 \qquad \xi_{\mathbf{k}} \qquad \xi_{\mathbf{k}} \qquad 2\xi_{\mathbf{k}} + U$$

- **Density distribution**

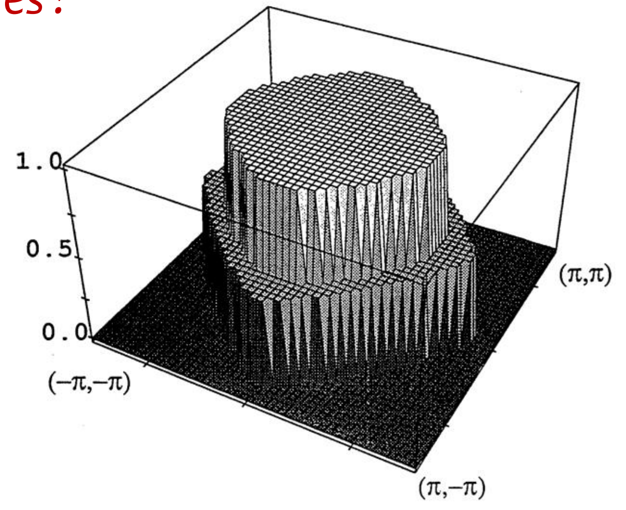
$$\langle n_{\mathbf{k}\sigma}(T)\rangle = \frac{1}{Z_{\mathrm{HK}}} \mathrm{Tr}\left(n_{\mathbf{k}\sigma} e^{-\beta H_{\mathrm{HK}}}\right) = \frac{e^{-\beta\xi_{\mathbf{k}}} + e^{-\beta(2\xi_{\mathbf{k}}+U)}}{1 + 2e^{-\beta\xi_{\mathbf{k}}} + e^{-\beta(2\xi_{\mathbf{k}}+U)}}$$

At T=0, $$n_{\mathbf{k}\sigma} = \frac{1}{2}\left[\Theta(\mu - \varepsilon_{\mathbf{k}}) + \Theta(\mu - \varepsilon_{\mathbf{k}} - U)\right] = \begin{cases} 0, & \xi_{\mathbf{k}} > 0 \quad (\Omega_0) \\ \frac{1}{2}, & -U < \xi_{\mathbf{k}} < 0 \quad (\Omega_1) \\ 1 & \xi_{\mathbf{k}} < -U \quad (\Omega_2) \end{cases}$$

- **Ground state** $$|g\rangle = \prod_{\mathbf{k}_2 \epsilon \Omega_2} c^{\dagger}_{\mathbf{k}_2\uparrow} c^{\dagger}_{\mathbf{k}_2\downarrow} \prod_{\mathbf{k}_1 \epsilon \Omega_1} \left(\frac{c^{\dagger}_{\mathbf{k}_1\uparrow} + e^{i\phi_{\mathbf{k}_1}} c^{\dagger}_{\mathbf{k}_1\downarrow}}{\sqrt{2}} \right) |0\rangle,$$



Large spin degeneracies !



Exact solution: equation of motion

- Define the retarded Green's function $G_\sigma(\mathbf{k},\tau) = -\Theta(\tau)\left\langle \hat{T}\left\{ c_{\mathbf{k}\sigma}(\tau), c^\dagger_{\mathbf{k}\sigma}(0)\right\}\right\rangle$
- Using the equation-of-motion method on G, $\partial_\tau G_\sigma(\mathbf{k},\tau) = -\delta(\tau) - \xi_\mathbf{k} G_\sigma(\mathbf{k},\tau) - UQ_\sigma(\mathbf{k},\tau)$,

  where, $Q_\sigma(\mathbf{k},\tau) \equiv -\Theta(\tau)\left\langle \hat{T}\left\{ n_{\mathbf{k}\bar{\sigma}}(\tau) c_{\mathbf{k}\sigma}(\tau), c^\dagger_{\mathbf{k}\sigma}(0)\right\}\right\rangle$.
- Similar for Q, we have $\partial_\tau Q_\sigma(\mathbf{k},\tau) = -\delta(\tau)\langle n_{\mathbf{k}\bar{\sigma}}\rangle - (\xi_\mathbf{k}+U)Q_\sigma(\mathbf{k},\tau)$.

$$\Rightarrow \quad Q_\sigma(\mathbf{k}, i\omega_n) = \frac{\langle n_{\mathbf{k}\bar{\sigma}}\rangle}{i\omega_n - \xi_\mathbf{k} - U},$$

- Then, the **Green's function** has the expression

$$G_\sigma(\mathbf{k}, i\omega_n) = \frac{1-\langle n_{\mathbf{k}\bar{\sigma}}\rangle}{i\omega_n - \xi_\mathbf{k}} + \frac{\langle n_{\mathbf{k}\bar{\sigma}}\rangle}{i\omega_n - \xi_\mathbf{k} - U}.$$

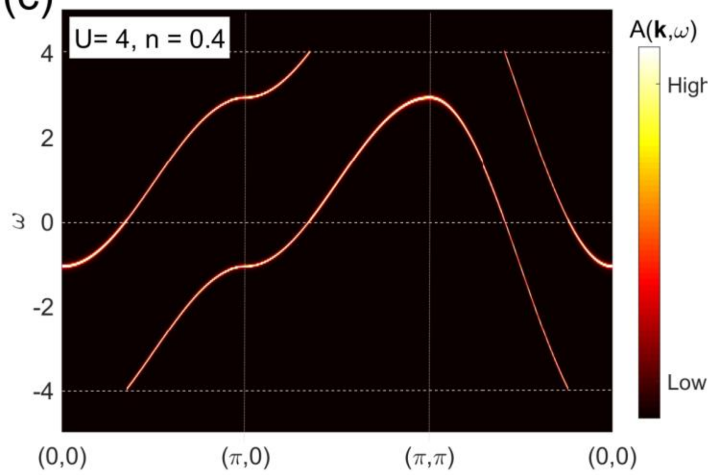
- **Spectral function**

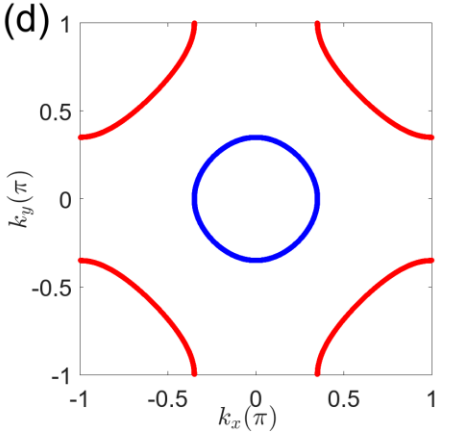
$$A_\sigma(\mathbf{k},\omega) = -\frac{1}{\pi}\operatorname{Im} G_\sigma(\mathbf{k},\omega) = (1-\langle n_{\mathbf{k}\bar{\sigma}}\rangle)\,\delta(\omega-\xi_\mathbf{k}) + \langle n_{\mathbf{k}\bar{\sigma}}\rangle\,\delta(\omega-\xi_\mathbf{k}-U)$$

- **An emergent "two-band" description** $\Rightarrow$ **"two Fermi surfaces"**

Holon $S^\dagger_{\mathbf{k}\sigma} \equiv (1-n_{\mathbf{k}\bar{\sigma}})\,c^\dagger_{\mathbf{k}\sigma}$ (empty $\Rightarrow$ singly-occupied)

Doublon $D^\dagger_{\mathbf{k}\sigma} \equiv n_{\mathbf{k}\bar{\sigma}}\,c^\dagger_{\mathbf{k}\sigma}$ (singly-occupied $\Rightarrow$ doubly-occupied)

(c)


(d)


Related works on the HK model

Fermi arcs for sign-changing HK interaction

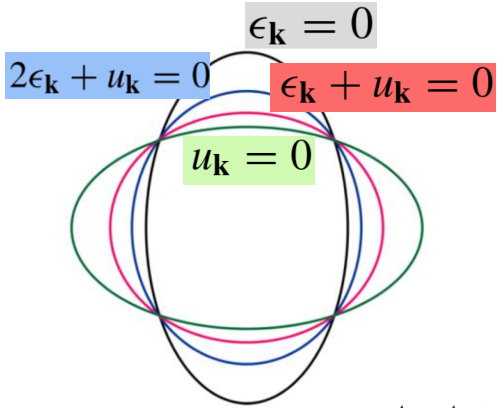
K. Yang, PRB **103**, 024529 (2021)

$$H = \sum_{\mathbf{k}} [\epsilon_{\mathbf{k}}(n_{\mathbf{k}\uparrow} + n_{\mathbf{k}\downarrow}) + u_{\mathbf{k}} n_{\mathbf{k}\uparrow} n_{\mathbf{k}\downarrow}],$$

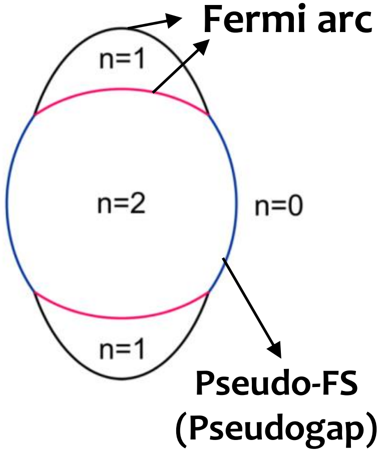
$$\epsilon_{\mathbf{k}} = -2t_x \cos(k_x) - 2t_y \cos(k_y) - \mu,$$

$$u_{\mathbf{k}} = -2T_x \cos(k_x) - 2T_y \cos(k_y) + U,$$

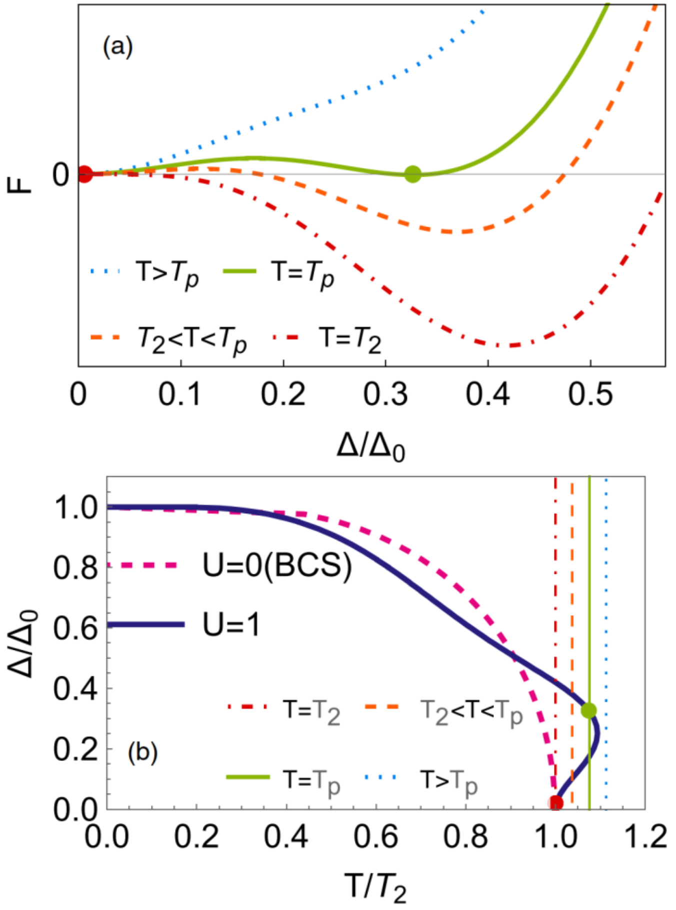
$$n_{\mathbf{k}} = \begin{cases} 0, & \epsilon_{\mathbf{k}} > 0 \text{ and } 2\epsilon_{\mathbf{k}} + u_{\mathbf{k}} > 0, \\ 1, & \epsilon_{\mathbf{k}} < 0 \text{ and } \epsilon_{\mathbf{k}} + u_{\mathbf{k}} > 0, \\ 2, & \epsilon_{\mathbf{k}} + u_{\mathbf{k}} < 0 \text{ and } 2\epsilon_{\mathbf{k}} + u_{\mathbf{k}} < 0. \end{cases}$$



$\Delta = \Delta_0 + \frac{\pi}{8}U,$



1st-order SC transition

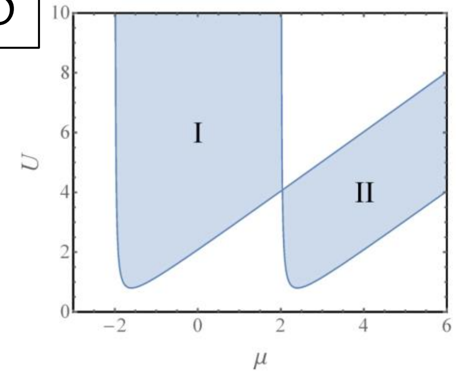


(Ginzburg criterion, pair susceptibility, GL analysis, Heat capacity, NMR $1/TT_1$, ultrasonic attenuation, superfluid stiffness)

J. Zhao,…, P. W. Phillips, PRB **105**, 184509 (2022)

Topological s-wave SC

1D

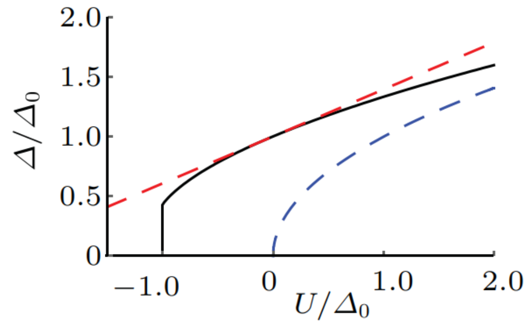


$$\hat{P} = (-1)^{\sum_{\mathbf{k}\sigma} \hat{n}_{\mathbf{k}\sigma}}$$

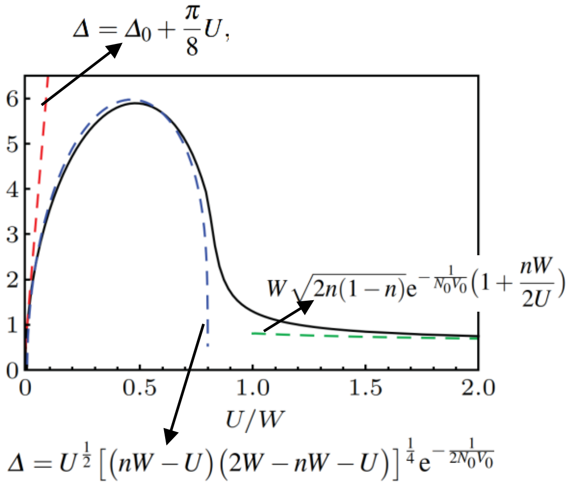
H.-S. Zhu, et al., PRB 103, 024514 (2012); CPB **30**, 107401 (2021)

Correlation effects on SC

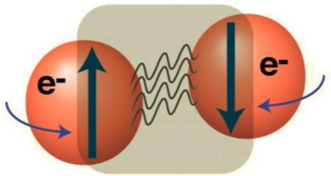
2D



U<0 suppress SC;
U>0 enhanced SC



Recall: Cooper instability (Cooper, 1956)



- Hamiltonian $H = \sum_{\boldsymbol{k},\sigma} \xi_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} - V \sum_{\boldsymbol{k},\boldsymbol{k}'} c^{\dagger}_{\boldsymbol{k}\uparrow} c^{\dagger}_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow},$
- Construct a wavefunction with creating a pair electrons from |G>, $|\phi\rangle = \sum_{\boldsymbol{k}\in\{0<\xi_{\boldsymbol{k}}<\omega_D\}} u_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\uparrow} c^{\dagger}_{-\boldsymbol{k}\downarrow} |G\rangle$
- Evaluate the energy change with adding a pair of electrons

$$E = \langle\phi| H |\phi\rangle - \langle G| H |G\rangle = \sum_{\boldsymbol{k}\in\{0<\xi_{\boldsymbol{k}}<\omega_D\}} 2\xi_{\boldsymbol{k}} |u_{\boldsymbol{k}}|^2 - V \sum_{\boldsymbol{k},\boldsymbol{k}'\in\{0<\xi_{\boldsymbol{k}}<\omega_D\}} u^*_{\boldsymbol{k}'} u_{\boldsymbol{k}}.$$

- Introduce a Lagrange multiplier λ via the normalization condition $\sum_{\boldsymbol{k}\in\{0<\xi_{\boldsymbol{k}}<\omega_D\}} |u_{\boldsymbol{k}}|^2 = 1,$

$$E' = E + \lambda \left(\sum_{\boldsymbol{k}\in\{0<\xi_{\boldsymbol{k}}<\omega_D\}} |u_{\boldsymbol{k}}|^2 - 1 \right),$$

Variational condition →

$$(2\xi_{\boldsymbol{k}} - \lambda) u_{\boldsymbol{k}} = V \sum_{\boldsymbol{k}'\in\{0<\xi_{\boldsymbol{k}}<\omega_D\}} u_{\boldsymbol{k}'},$$

- Equation for the Cooper-pair bound state $1 = V \sum_{\boldsymbol{k}\in\{0<\xi_{\boldsymbol{k}}<\omega_D\}} \frac{1}{2\xi_{\boldsymbol{k}} - E} \approx N(0) V \int_0^{\omega_D} d\xi \frac{1}{2\xi - E} = \frac{N_F V}{2} \ln \left| \frac{2\omega_D - E}{-E} \right|$

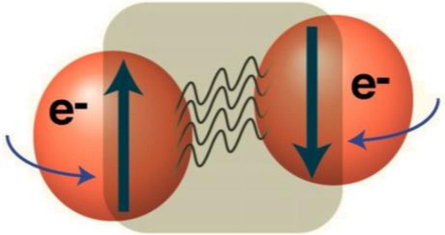
$$\longrightarrow E = -\frac{2\omega_D}{e^{\frac{2}{N(0)V}} - 1} \approx -2\omega_D e^{-\frac{2}{N(0)V}} \quad < 0$$

The electrons near the Fermi surfaces can always favoring Cooper pairing even when the attractive pairing interaction is infinitesimally small.

L. N. Cooper, Phys. Rev. **15,** 1189 (1956)

Cooper instability in HK model (1)

Cooper instability (1956): The electrons near the Fermi surfaces can always favoring Cooper pairing even when the attractive pairing interaction is infinitesimally small.



- Consider an additional pairing interaction, $H_{\text{pairing}} = -V \sum_{\mathbf{k},\mathbf{k}'} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}$,
- According to Cooper's approach, in HK model, we construct a wavefunction which creating a Cooper pair from |g>,

$$|\psi\rangle = \sum_{\mathbf{k}\epsilon\Omega_0} \alpha_{\mathbf{k}} b^\dagger_{\mathbf{k}} |g\rangle + \sum_{\mathbf{k}\epsilon\Omega_1} \beta_{\mathbf{k}} b^\dagger_{\mathbf{k}} |g\rangle \,,$$

$$b^\dagger_{\mathbf{k}} = c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}$$

- Normalization condition $\langle\psi|\psi\rangle = \sum_{\mathbf{k}\epsilon\Omega_0} |\alpha_{\mathbf{k}}|^2 + \frac{1}{4}\sum_{\mathbf{k}\epsilon\Omega_1} |\beta_{\mathbf{k}}|^2 = 1.$
- Evaluate the change of energy with adding a Cooper pair,

$$\begin{aligned} E_{\mathrm{C}} &= \langle\psi| H |\psi\rangle - \langle g| H |g\rangle \\ &= \sum_{\mathbf{k}\epsilon\Omega_0} 2\xi_{\mathbf{k}} |\alpha_{\mathbf{k}}|^2 + \frac{1}{4}\sum_{\mathbf{k}\epsilon\Omega_1} (\xi_{\mathbf{k}} + U) |\beta_{\mathbf{k}}|^2 - V \sum_{\mathbf{k},\mathbf{k}'\epsilon\Omega_0} \alpha^*_{\mathbf{k}}\alpha_{\mathbf{k}'} - \frac{V}{16}\sum_{\mathbf{k},\mathbf{k}'\epsilon\Omega_0} \beta^*_{\mathbf{k}}\beta_{\mathbf{k}'} - \frac{V}{4}\sum_{\mathbf{k}\epsilon\Omega_0,\mathbf{k}'\epsilon\Omega_1} (\alpha^*_{\mathbf{k}}\beta_{\mathbf{k}'} + \alpha_{\mathbf{k}}\beta^*_{\mathbf{k}'}) \end{aligned}$$

- Introduce a Lagrange multiplier λ $\quad Q = E_{\mathrm{C}} - \lambda\left(\langle\psi|\psi\rangle - 1\right),$
- Variational conditions $\quad \frac{\partial Q}{\partial \alpha^*_{\mathbf{k}}} = 0 \text{ and } \frac{\partial Q}{\partial \beta^*_{\mathbf{k}}} = 0$

Cooper instability in HK model (2)

- The variational conditions give the 2 equations

$$\alpha_{\mathbf{k}} = \frac{V}{2\xi_{\mathbf{k}} - E_C}\left(\sum_{\mathbf{k}'\epsilon\Omega_0} \alpha_{\mathbf{k}'} + \frac{1}{4}\sum_{\mathbf{k}'\epsilon\Omega_1} \beta_{\mathbf{k}'}\right),$$

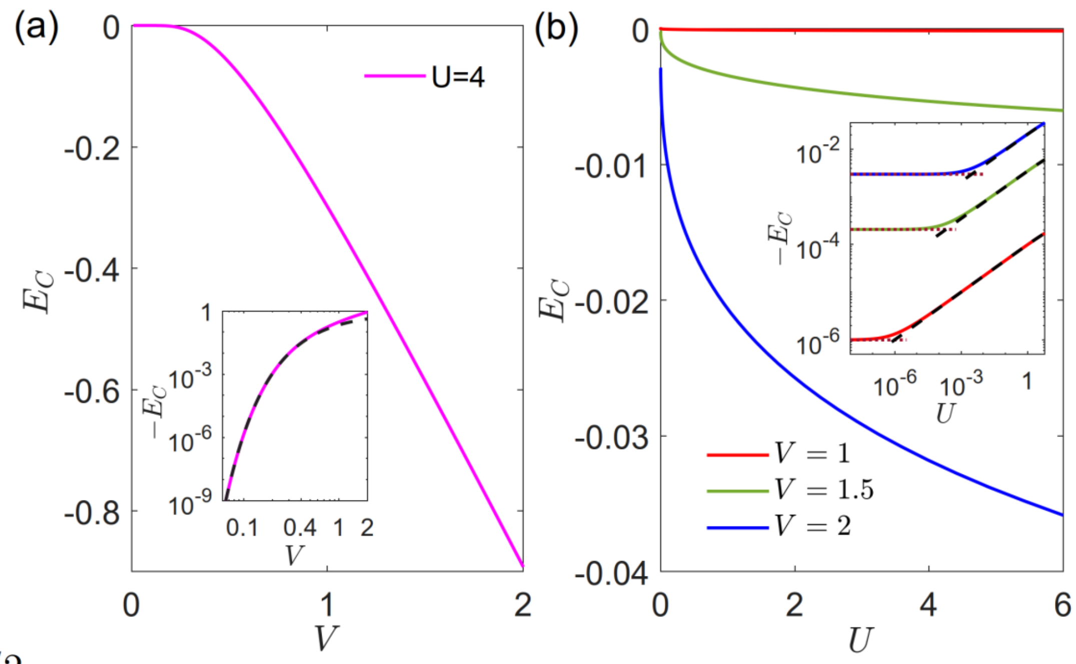
$$\beta_{\mathbf{k}} = \frac{V}{\xi_{\mathbf{k}} + U - E_C}\left(\sum_{\mathbf{k}'\epsilon\Omega_0} \alpha_{\mathbf{k}'} + \frac{1}{4}\sum_{\mathbf{k}'\epsilon\Omega_1} \beta_{\mathbf{k}'}\right).$$

- Summing $\alpha_{\mathbf{k}}$, $\beta_{\mathbf{k}}$ over **k** in Ω_0, Ω_1 regions, respectively, one can obtain the equation

$$1 = \sum_{\mathbf{k}\epsilon\Omega_0} \frac{V}{2\xi_{\mathbf{k}} - E_C} + \frac{1}{4}\sum_{\mathbf{k}\epsilon\Omega_1} \frac{V}{\xi_{\mathbf{k}} + U - E_C}.$$

- With taking $\rho(\omega) = \sum_{\mathbf{k}} \delta(\omega - \varepsilon_{\mathbf{k}}) = \frac{1}{W}$ for $-W/2 < \omega < W/2$

$$\Rightarrow \quad 1 = \frac{V}{4W} \ln\left|\frac{(W - 2\mu - E_C)^2 (U - E_C)}{E_C^3}\right|.$$



(1) U is finite, $E_C \approx -(W - 2\mu)^{2/3} U^{1/3} e^{-\frac{4W}{3V}}$

(2) U→0, reproduce the BCS results:

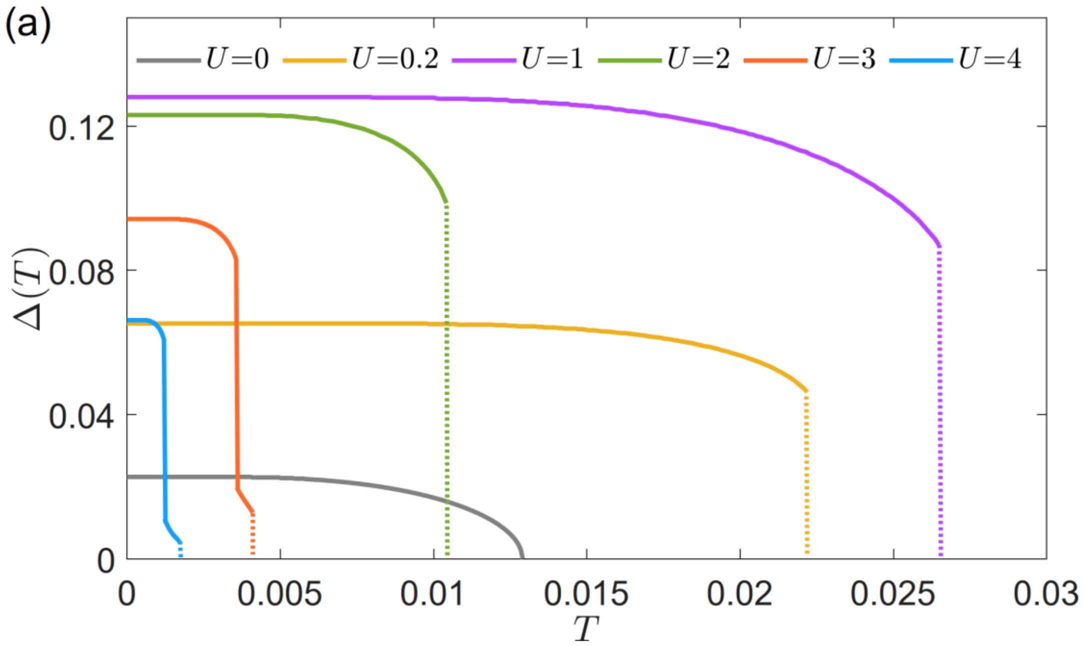
$$E_C \approx -(W - 2\mu) e^{-\frac{2W}{V}}$$

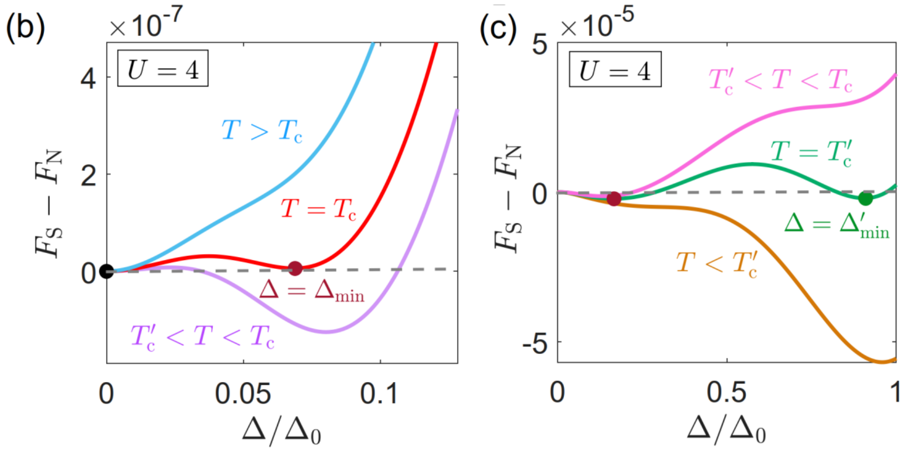
- Distinct from the BCS case, except for the empty region (Ω_0), the emergent singly-occupied region (Ω_1) also contribute to Cooper pairs. [In Phillips's paper, the contribution from Ω_1 is neglected.]
- The Cooper instability is smoothly inherited from the BCS theory as turning on U from U=0.

Two-stage superconductivity in HK-BCS model

HK-BCS model

$$H=\sum_{\mathbf{k},\sigma}(\varepsilon_{\mathbf{k}}-\mu)c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}+U\sum_{\mathbf{k}}n_{\mathbf{k}\uparrow}n_{\mathbf{k}\downarrow}+\sum_{\mathbf{k}}\left(\Delta c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow}+\mathrm{H.c.}\right)+\frac{\Delta^2}{V},$$





- SC transition is of first order as U≠0.
- The two-stage superconductivity happens when U is large enough.
- The transition at T_c and changes at T_c' are manifested as switching of the global minimal of free energy.

Method

- Folding the BZ,

$$H=\sum_{\mathbf{k}\in\frac{1}{2}\mathrm{BZ}}\left[\xi_{\mathbf{k}}\left(n_{\mathbf{k},\uparrow}+n_{-\mathbf{k},\uparrow}+n_{\mathbf{k},\downarrow}+n_{-\mathbf{k},\downarrow}\right)\right.$$
$$+U\left(n_{\mathbf{k}\uparrow}n_{\mathbf{k}\downarrow}+n_{-\mathbf{k}\uparrow}n_{-\mathbf{k}\downarrow}\right)$$
$$\left.+\left(\Delta c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow}+\Delta c^{\dagger}_{-\mathbf{k}\uparrow}c^{\dagger}_{\mathbf{k}\downarrow}+\mathrm{H.c.}\right)\right],$$

where $Z_{\mathbf{k}}=\sum_n e^{-E_{n,\mathbf{k}}/T}$ and $\Delta\equiv -V\sum_{\mathbf{k}}\langle c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}\rangle$

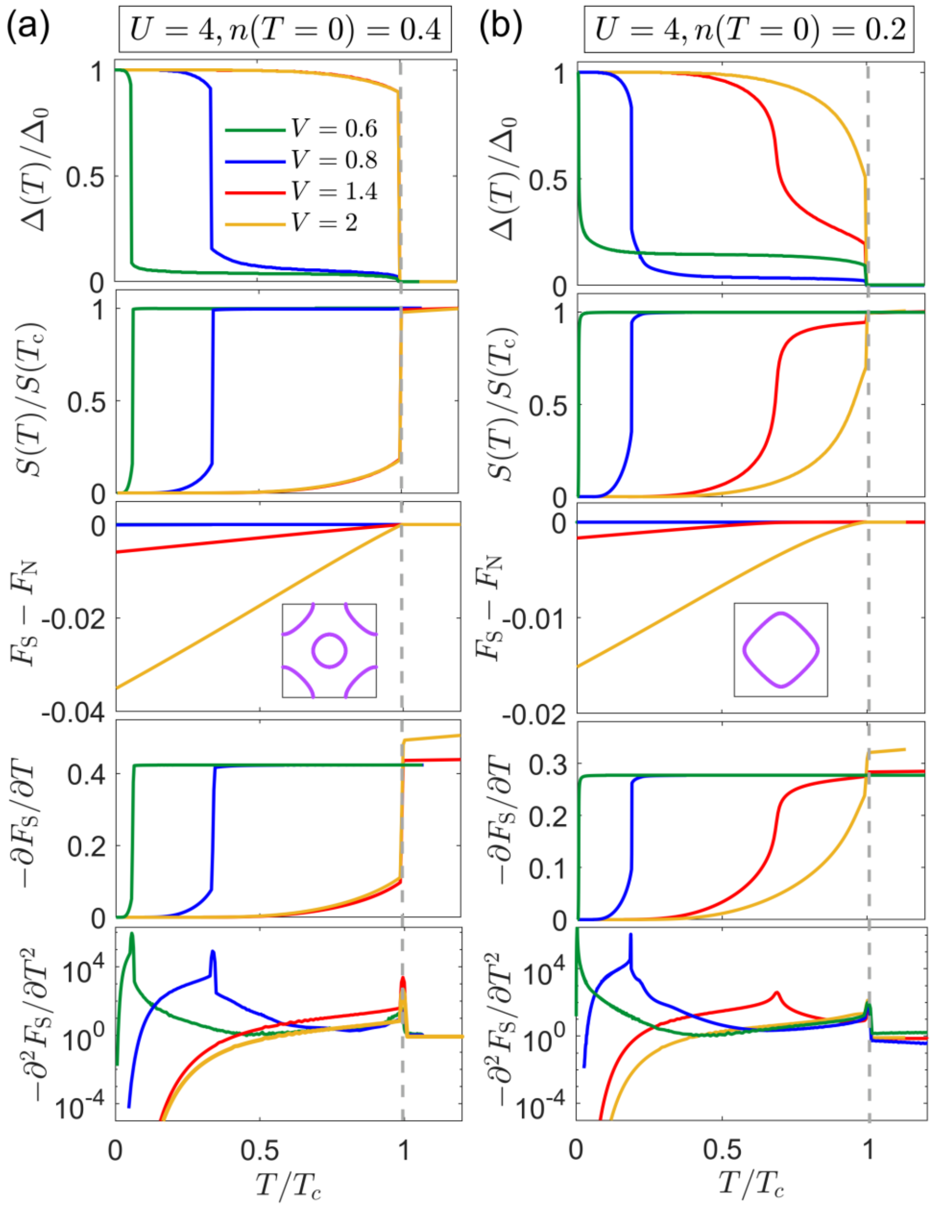
- Evaluate the Free energy

$$F_S[\Delta]=-T\ln Z=-T\sum_{\mathbf{k}\in\frac{1}{2}\mathrm{BZ}}\ln Z_{\mathbf{k}},$$

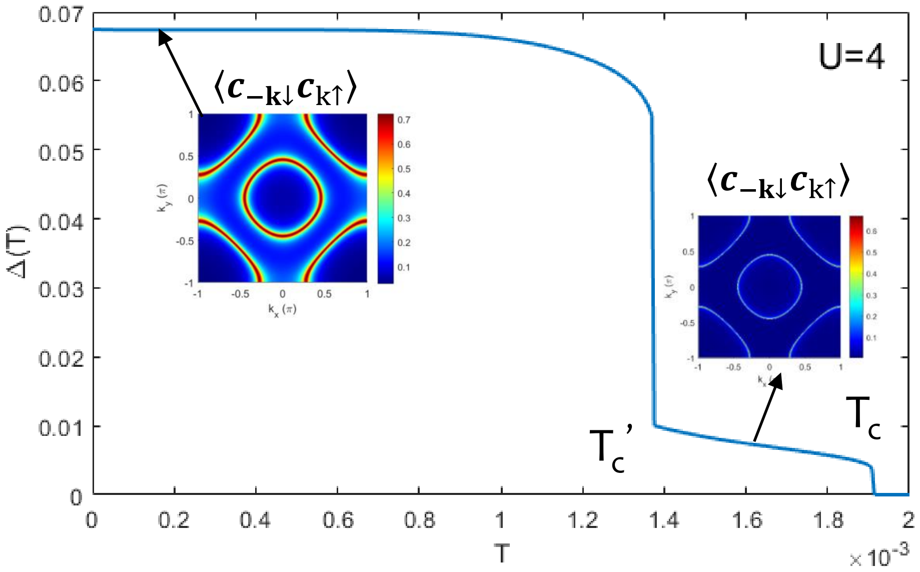
- Variational condition

$$\frac{\partial F_S[\Delta]}{\partial\Delta}=0.$$ (& Find out the global minimal of F_S.)

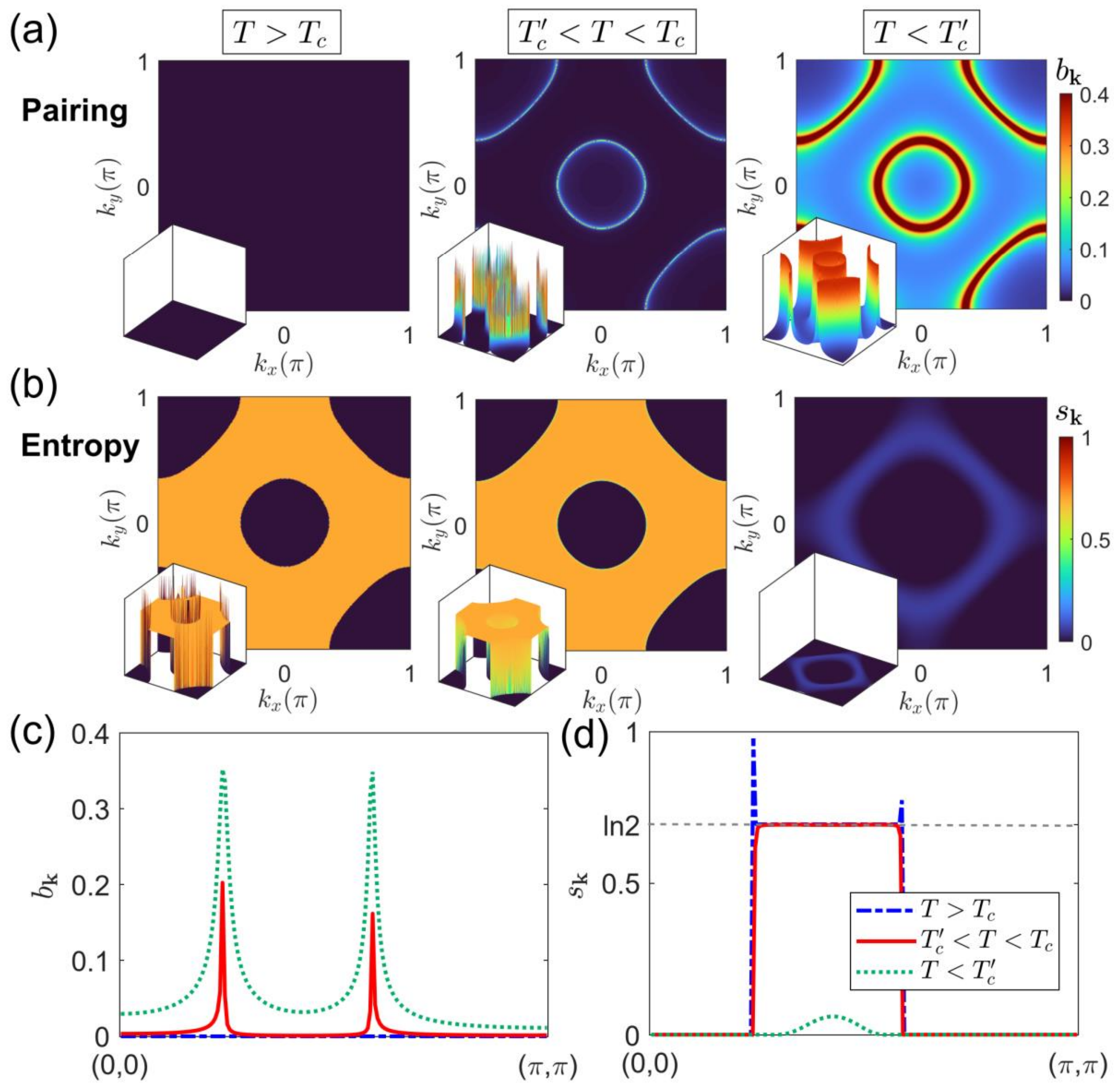
Thermodynamic evolutions



- The 1st-order transition are also manifested in S(T) and derivatives of $F_S(T)$.
- Below T_c, Δ(T) and thermodynamics shows a 1st-order-like changes at T_c'.
- Two-gap SC is excluded since SC on both FSs occur simultaneously at T_c.



Two-stage SC: Cooper-pair distribution & entropy distribution



Cooper-pair amplitude $b_{\mathbf{k}} = \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle$

Entropy distribution $s_{\mathbf{k}} = -\frac{1}{2}\sum_n \rho_{n,\mathbf{k}} \ln \rho_{n,\mathbf{k}}$

Two-stage superconductivity

(1) $T > T_c$: no SC, entropy mainly distributed on the Fermi surfaces & within the singly-occupied region.

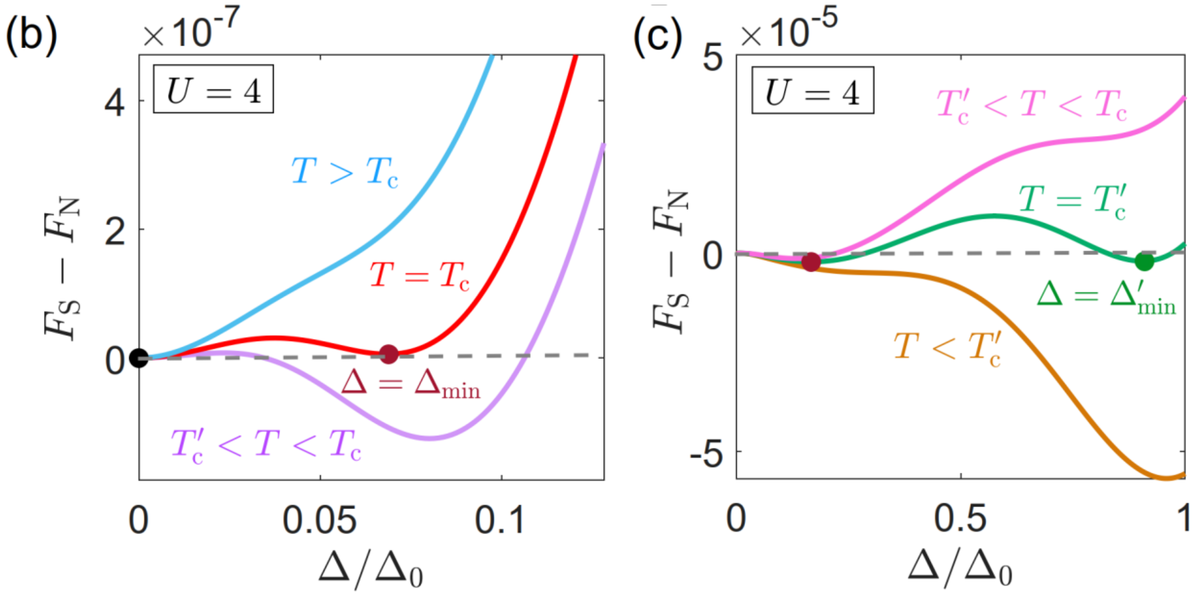
(2) $T_c' < T < T_c$: SC happens only near the Fermi surfaces, and where the entropy is released simultaneously.

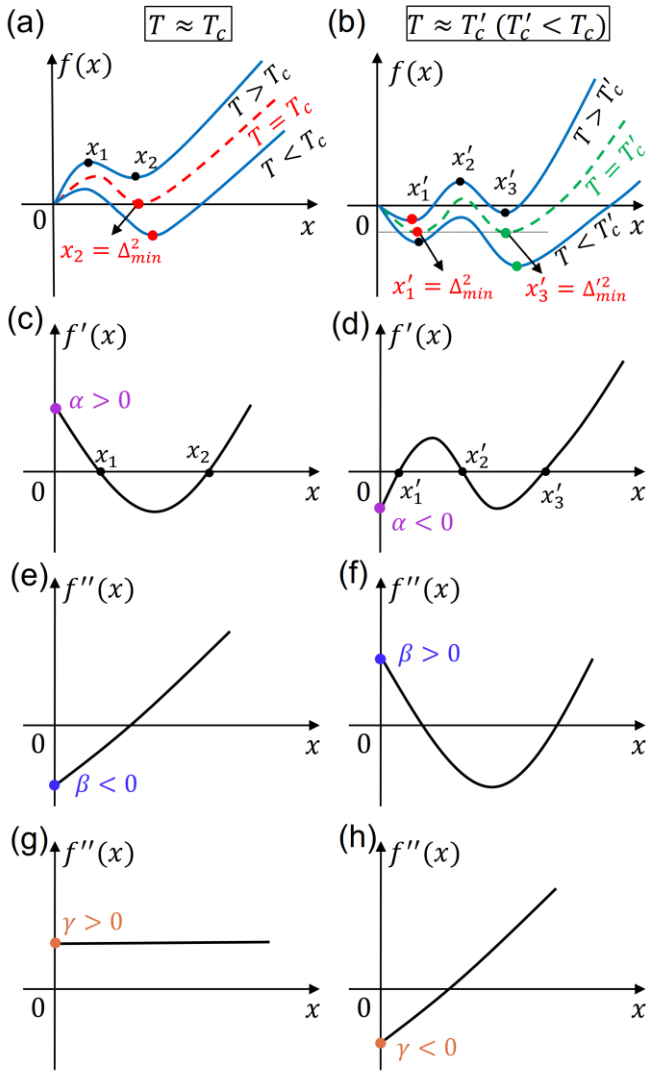
(3) $T < T_c'$: the electrons within the singly-occupied region pairs due to the proximity effect.

Ginzburg-Landau analysis

$$F_S\left[\Delta\right] = -T\ln Z = -T\sum_{\mathbf{k}\in\frac{1}{2}\mathrm{BZ}} \ln Z_{\mathbf{k}},$$

$$f\left(x\right) = \alpha x + \frac{\beta}{2}x^2 + \frac{\gamma}{3!}x^3 + \frac{\eta}{4!}x^4.$$





- From the free-energy evolution, the two-stage SC are manifested as the jumps of minima near $T \approx T_c$ and $T \approx T_c^{'}$, respectively.
- It can be explained from the Ginzburg-Landau analysis by counting up to the eighth-order terms.

$$\delta\mathcal{F}\left[\Delta\right] = \alpha\Delta^2 + \frac{\beta}{2}\Delta^4 + \frac{\gamma}{3!}\Delta^6 + \frac{\eta}{4!}\Delta^8 + O\left(\Delta^8\right),$$

GL analysis at T ≈ T_c & T ≈ T_c'

$$\delta\mathcal{F}[\Delta] = \alpha\Delta^2 + \frac{\beta}{2}\Delta^4 + \frac{\gamma}{3!}\Delta^6 + \frac{\eta}{4!}\Delta^8 + O\left(\Delta^8\right), \quad (7)$$

where α, β, γ and η are the expansion coefficients and depend on temperature T, and $\eta > 0$, or $\eta = 0$ and $\gamma > 0$, ensures the stability of the system. To study phase transitions for the HK-BCS model, we consider critical regions: $T \approx T_c$ and $T \approx T_c'$.

(1) $T \approx T_c$: It turns out that the occurrence of a first-order transition at $T = T_c$ impose constraints for expansion coefficients at this critical point as follows [53],

$$\alpha > 0, \quad (8a)$$

$$\eta \geq 0, \quad (8b)$$

$$\frac{9\alpha\eta - 2\beta\gamma}{4\gamma^2 - 9\beta\eta} = \frac{4\left(\beta^2 - 2\alpha\gamma\right)}{9\alpha\eta - 2\beta\gamma} > 0. \quad (8c)$$

And the superconducting gap at T_c reads

$$\Delta(T = T_c) = \sqrt{\frac{3\left(9\alpha\eta - 2\beta\gamma\right)}{4\gamma^2 - 9\beta\eta}} = \sqrt{\frac{12\left(\beta^2 - 2\alpha\gamma\right)}{9\alpha\eta - 2\beta\gamma}}. \quad (9)$$

In the limit of $\eta = 0$, it becomes

$$\Delta(T = T_c) = \sqrt{\frac{-3\beta}{2\gamma}} = \sqrt{\frac{6\left(2\alpha\gamma - \beta^2\right)}{\beta\gamma}}, \quad (10)$$

which restores the result in Ref. [51].

(2) $T \approx T_c'$: In the presence of the first-order-like jump at $T = T_c'$, the sign of expansion coefficients can be determined in the critical region as follows [53],

$$\alpha < 0, \ \beta > 0, \ \gamma < 0 \ \text{and} \ \eta > 0. \quad (11)$$

The critical condition at $T = T_c'$ is given by

$$\alpha = \gamma\left(\frac{\beta}{\eta} - \frac{\gamma^2}{3\eta^2}\right), \quad (12)$$

and the temperature regions $T > (<) T_c'$ are separated from each other in accordance with the inequality as follows,

$$\alpha < (>) \gamma\left(\frac{\beta}{\eta} - \frac{\gamma^2}{3\eta^2}\right), \ \text{for} \ T > (<) T_c'. \quad (13)$$

The superconducting order parameters at $T_c'^{\pm}$ read

$$\Delta_{\text{min}} \equiv \Delta(T = T_c'^{+}) = \sqrt{-\frac{\gamma}{\eta} - \sqrt{\frac{\gamma^2}{\eta^2} - \frac{6\alpha}{\gamma}}}, \quad (14a)$$

$$\Delta'_{\text{min}} \equiv \Delta(T = T_c'^{-}) = \sqrt{-\frac{\gamma}{\eta} + \sqrt{\frac{\gamma^2}{\eta^2} - \frac{6\alpha}{\gamma}}}. \quad (14b)$$

To study the temperature dependence $\Delta(T)$ around T_c', we introduce the dimensionless parameter $t' = \left(T - T_c'\right)/T_c'$, and find for small t',

$$\Delta(T) \approx \Delta(T_c'^{\pm})\left(1 - b_{\pm}t'\right) \ \text{at} \ T \gtrless T_c', \quad (15)$$

where $b_{\pm} > 0$ are two positive parameters that can be determined from experimental data or microscopic theory [53].

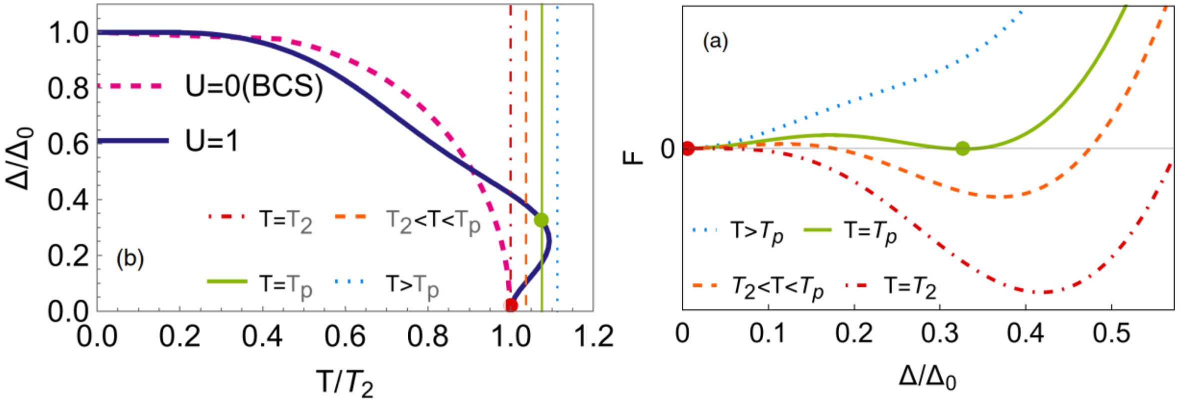
HK-BCS model: Phillips's works v.s. our works

Phillips's works

- Cooper instability: **only for electrons in Ω_0.**

$$1 = -\frac{g}{L^d}\sum_{k\in\Omega_0}\frac{\langle 1 - n_{k\uparrow} + n_{-k\downarrow}\rangle}{E - 2\xi_k - U\langle n_{k\downarrow} + n_{-k\uparrow}\rangle}$$

- Finite-T mean-field in *strong-pairing regime*:
  - ➢ **1^{st}-order SC transition.**
  - ➢ **Switch of the minima of free energy at T_c (or T_p)**



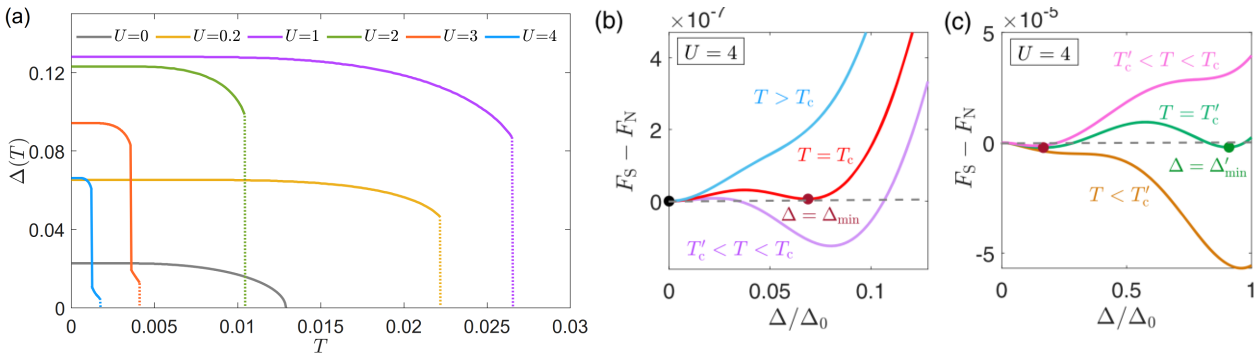
P. W. Phillips et al., Nat. Phys. **16**, 1175 (2020)
J. Zhao,…, P. W. Phillips, PRB **105**, 184509 (2022)

Our works

- Cooper instability: **not only for electrons in Ω_0, but also for Ω_1.**

$$1 = \sum_{\mathbf{k}\epsilon\Omega_0}\frac{V}{2\xi_{\mathbf{k}} - E_C} + \frac{1}{4}\sum_{\mathbf{k}\epsilon\Omega_1}\frac{V}{\xi_{\mathbf{k}} + U - E_C}.$$

- Finite-T mean-field in *weak and intermediate-pairing regime*:
  - ➢ **1^{st}-order SC transition & two-stage SC.**
  - ➢ **Switch of the minima of free energy at T_c & T_c'.**



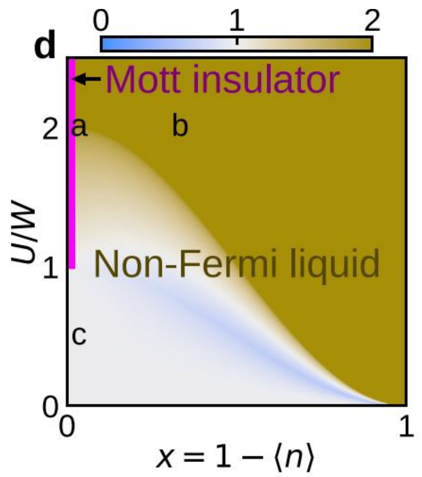
Y. Li, V. Mishra, Y. Zhou, F. -C. Zhang, New J. Phys. **24**, 103019 (2022).

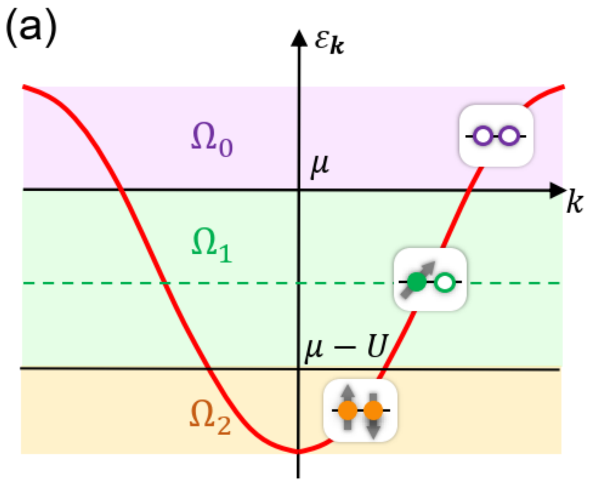
Our works unveiled a hidden superconductivity contributed from the singly-occupied electrons (Ω_1), which manifested in $\Delta(T)$ as a two-stage SC in weak-pairing regime, and hidden in a crossover in strong-pairing regime.

Interesting physics from HK model

- An **exactly-solvable** strongly-correlated model which can generate **Mott physics** at strong U.
- The existence of **zeros in Green's function** is **beyond the conventional Luttinger's theorem**, and the existence of **Luttinger surfaces** can also show a SC instability, universal to the strongly-correlated SC?
- A **sign-changing U** can generate **Fermi arcs** and **pesudogap state**.
- A **Green's function** is similar to the phenomenological form of **Yang-Rice-Zhang's ansatz for cuprates**, any significance?
- For the entropy, not like the 'area-law' for the Fermi surfaces, the **large spin-degeneracy in singly-occupied states** suggest a '**volume-law**'.
- ……

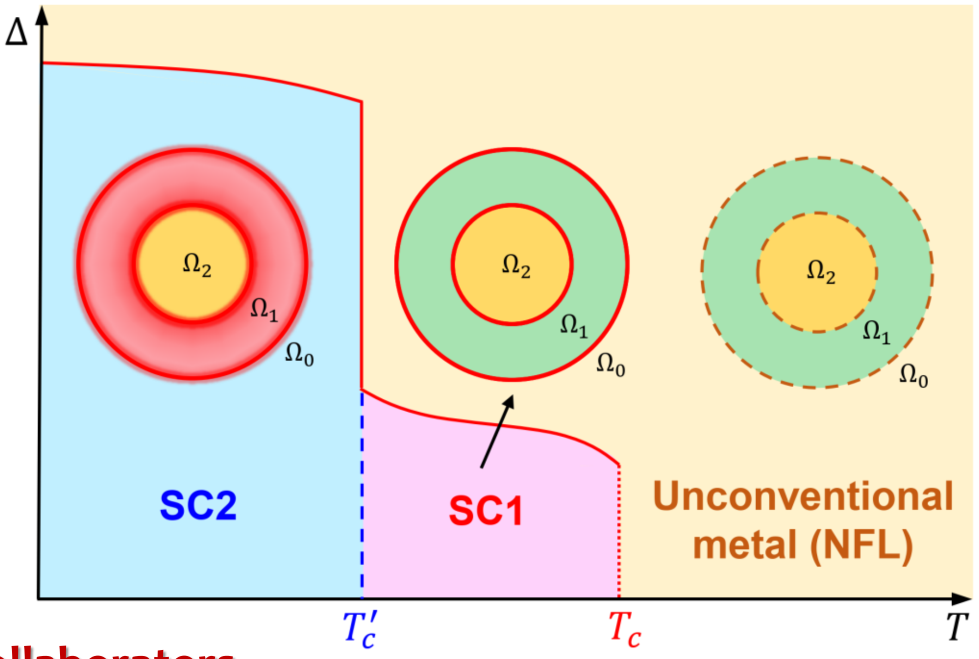
$$H_{\mathrm{HK}} = \sum_{\mathbf{k},\sigma} (\varepsilon_{\mathbf{k}} - \mu)\, c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + U \sum_{\mathbf{k}} n_{\mathbf{k}\uparrow} n_{\mathbf{k}\downarrow}$$





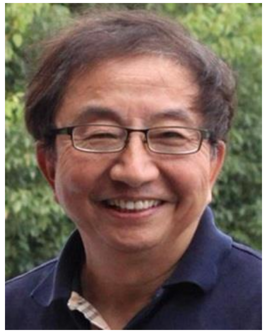
Summary

Two-stage superconductivity



- HK-BCS model exhibit a novel two-stage SC from a NFL state.
- As T lowering, at the first stage, the SC happens only near the Fermi surfaces below T_c; while at the second stage, the SC within the singly-occupied region suddenly pairs also below T_c' ($T_c' < T_c$).
- More extensions are needed to account for the many unsolved experiments in unconventional SC, e.g., **k-**dependent U+ unconventional pairing? HK+el-ph interaction?

Collaborators



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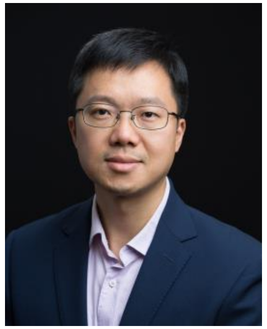

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Thanks for your attention !