Lattice models of fracton topological orders

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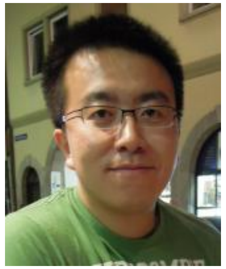
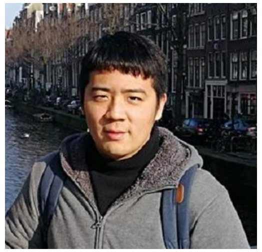




Publications:

[1] **MYL** and Peng Ye, Fracton physics of spatially extended excitations, Phys. Rev. B 101, 245134 (2020).

[2] **MYL** and Peng Ye, “Fracton physics of spatially extended excitations. II. Polynomial ground state degeneracy of exactly solvable models,” Phys. Rev. B 104, 235127 (2021).

[3] Chengkang Zhou, **MYL**, Zheng Yan, Peng Ye, and Zi Yang Meng, “Evolution of dynamical signature in the Xcube fracton topological order,” Physical Review Research 4, 033111(2022)

[4] Chengkang Zhou, **MYL**, Zheng Yan, Peng Ye, Zi Yang Meng, Detecting Subsystem Symmetry Protected Topological Order Through Strange Correlators, Phys.Rev.B 106.214428(2022)

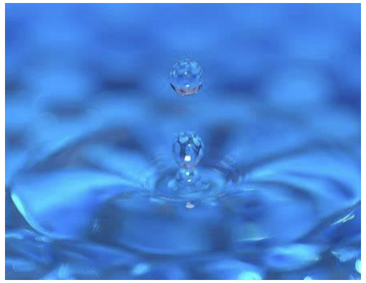
[5] **MYL**, Peng Ye, Hierarchy of Entanglement Renormalization and Long-Range Entangled States, arXiv:2211.14136 [quant-ph].

Outline

- 1. Topological orders and toric code model
  - 1.1 From symmetry breaking to topological order
  - 1.2 2D toric code model
  - 1.3 3D toric code model
- 2. Fracton (topological) orders and X-cube model
  - 2.1 From (pure) topological order to fracton order
  - 2.2 X-cube model
- 3. Foliated fracton order theory of Type-I fracton orders
- 4. Summary

Symmetry breaking in Ising model

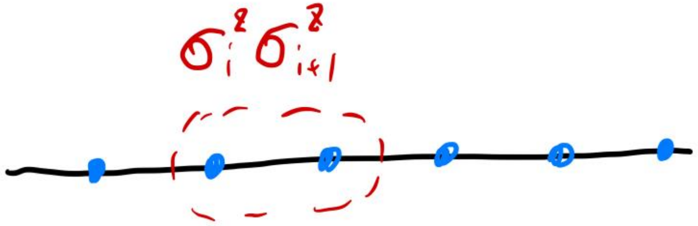
- Phases classified by symmetries



Continuous

Discrete

- 1D Ising model



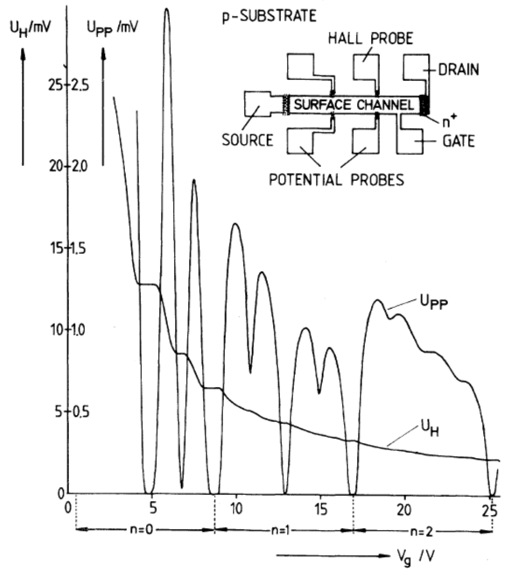
  - One spin-$1/2$ on each site
  - $H = -\sum_i \sigma_i^z \sigma_{i+1}^z$, with global symmetry generated by $S = \prod_i \sigma_i^x$
  - At $T = 0$, symmetry spontaneously break, two degenerate ground states $|000 \ldots 0\rangle$ and $|111 \ldots 1\rangle$, local order parameter $\langle \sigma^z \rangle = \pm 1$.
  - At $T > 0$, symmetry restored, $\langle \sigma^z \rangle = 0$.

[6] Strocchi, F. (2008). Symmetry Breaking in the Ising Model. In: Symmetry Breaking. Lecture Notes in Physics, vol 732. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-540-73593-9_21

Beyond symmetry breaking paradigm

- Quantum Hall effect

Fig from Ref.[7]

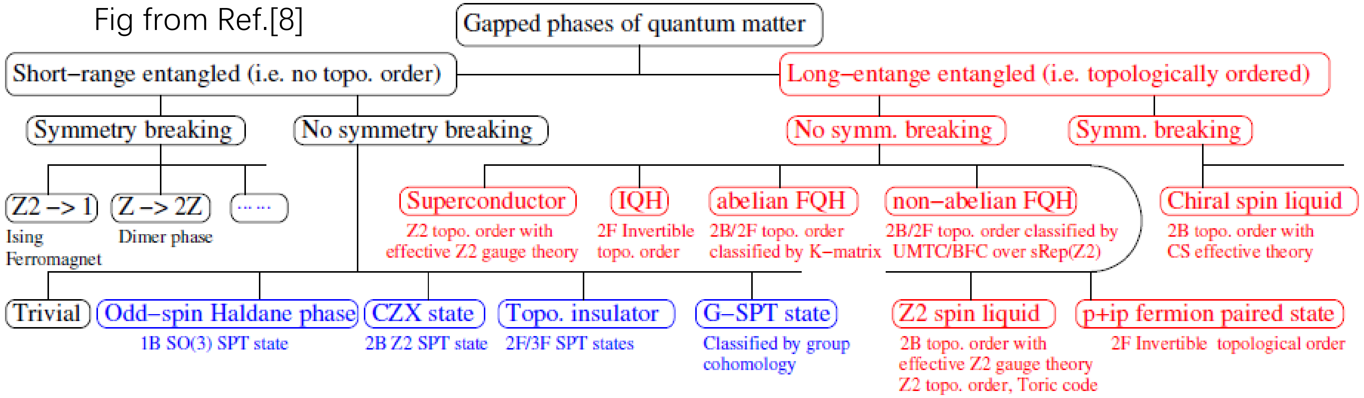


QHE shows transitions **without** any change of symmetries

- Beyond symmetry breaking
  - Short range entangled: symmetry protected topological/trivial (**SPT**) order
  - **Long range entangled: topological order**

- Gapped phases of quantum matter

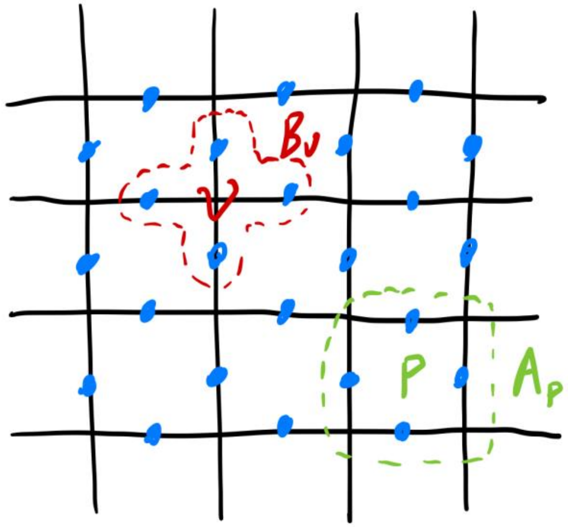
Fig from Ref.[8]



[7] K. v. Klitzing, G. Dorda, and M. Pepper. New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance. Phys. Rev. Lett. 45, 494 (1980
[8] Xiao-Gang Wen. Choreographed entanglement dances: Topological states of quantum matter. Science 363, 6429 (2019).

Topologically ordered 2D toric code model

- 2D toric code model (**TCM**)



$$H = -\sum_p A_p - \sum_v B_v,$$

$$A_p = \prod_{i\in \mathrm{p}} \sigma_i^x,$$
$$B_v = \prod_{i\in v} \sigma_i^z.$$

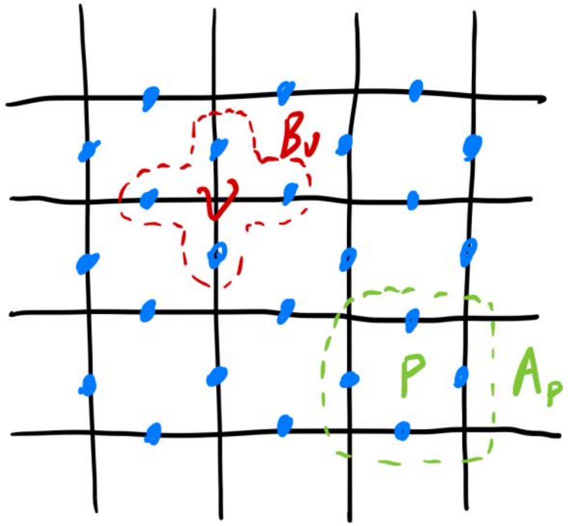
Though 2D TCM looks complicated , it is **exactly solvable**, as all terms commute with each other (notice that $\{\sigma_i^x, \sigma_i^z\} = 0$).

- Some properties of topological orders
  - **Ground states**: topological degeneracy
  - **Excited states**: fusion and braiding statistics
  - **Entanglement**: topological entanglement entropy
  - …

[9] A. Y. Kitaev, Fault-tolerant quantum computation by anyons, Ann. Phys. (N.Y.) 303, 2 (2003).

String-net picture of 2D TCM ground states

- 2D toric code model



$$H = -\sum_p A_p - \sum_v B_v,$$

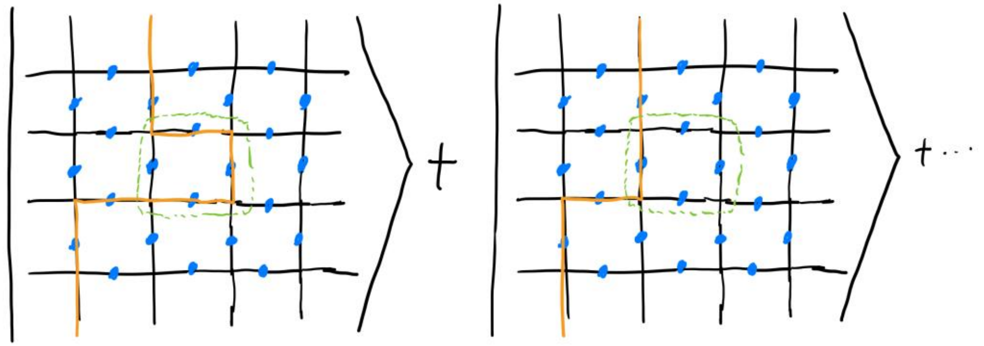
$$A_p = \prod_{i\in \mathrm{p}} \sigma_i^x,$$
$$B_v = \prod_{i\in v} \sigma_i^z.$$

Due to the exactly solvability, we can obtain a ground state $|\psi\rangle$ by solving the following equations:

$$B_v|\psi\rangle = |\psi\rangle, \forall\, v,$$
$$A_p|\psi\rangle = |\psi\rangle, \forall\, p,$$

- String-net picture of 2D TCM ground states
  - We use **Ising configurations**, where each spin is of an eigenstate of σ^z with eigenvalue ± 1, as a complete basis of the Hilbert space.

$|\psi\rangle =$

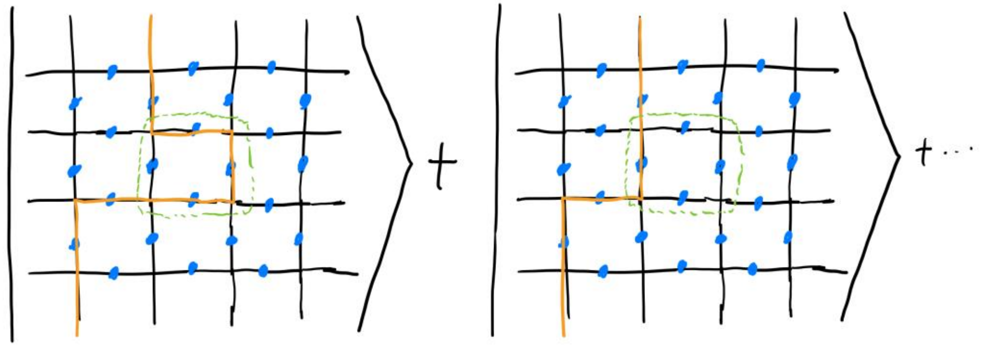


Here down spins (i.e. $\sigma^z = -1$) are regarded as forming strings. Then, we can see that $B_v|\psi\rangle = |\psi\rangle, \forall\, v$ requires all strings to be closed; $A_p|\psi\rangle = |\psi\rangle, \forall\, p$ requires configurations contractible closed strings to be equally superpositioned.

[10] Michael A. Levin and Xiao-Gang Wen, String-net condensation: A physical mechanism for topological phases, Phys. Rev. B 71, 045110 (2005).

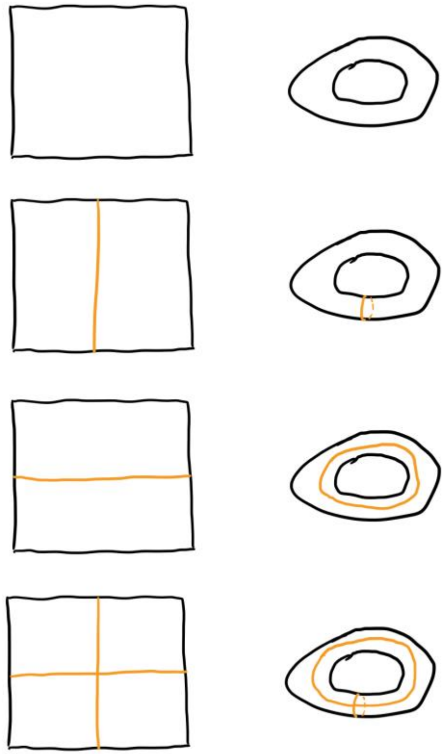
Topological degeneracy of 2D TCM

Then, we can see that:
1) $B_v|\psi\rangle = |\psi\rangle, \forall\, v$ requires all strings to be **closed**;
2) $A_p|\psi\rangle = |\psi\rangle, \forall\, p$ requires configurations differed by contractible closed strings to be **equally superpositioned**.

$|\psi\rangle =$ 

Thus a **loop condensed** ground state $|\psi\rangle$ can be largely determined by the above conditions (dubbed as **A and B constraints** for convenience).
The only ambiguity is originate from the existence of **non-contractible closed strings**. Locally $|\psi\rangle$ is featureless.

On 2-torus (PBC), by counting non-contractible closed strings, we obtain $GSD = 2^2 = 4$.

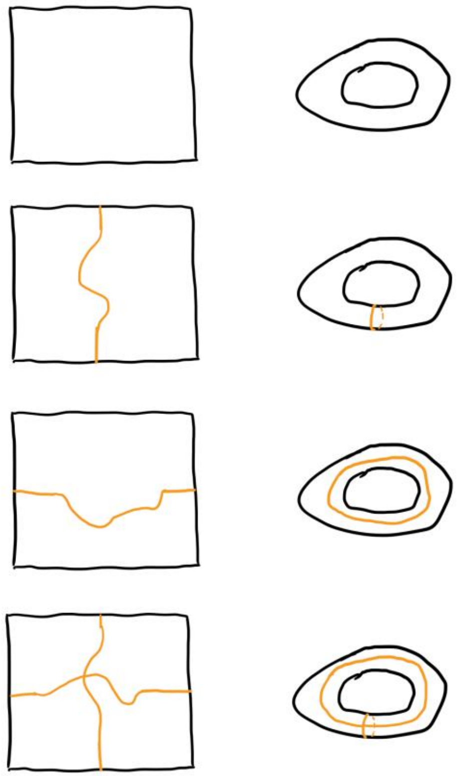


Mathematically, the number of independent non-contractible closed strings is the first Betti number, that is a **topological invariant** of the base manifold.

[9] A. Y. Kitaev, Fault-tolerant quantum computation by anyons, Ann. Phys. (N.Y.) 303, 2 (2003).
[10] Michael A. Levin and Xiao-Gang Wen, String-net condensation: A physical mechanism for topological phases, Phys. Rev. B 71, 045110 (2005).

2D TCM as a topological quantum memory

From a perspective of quantum information, we can use the ground state subspace as two logical qubits.



$|00\rangle$

$|01\rangle$

$|10\rangle$

$|11\rangle$

To do a bit-flip operation, we need to use apply a non-local string operator (dubbed as X-type logical operator):

$$W(l) = \prod_{i \in l} \sigma_i^x,$$

Here l is a non-contractible closed string (also dubbed as loop).

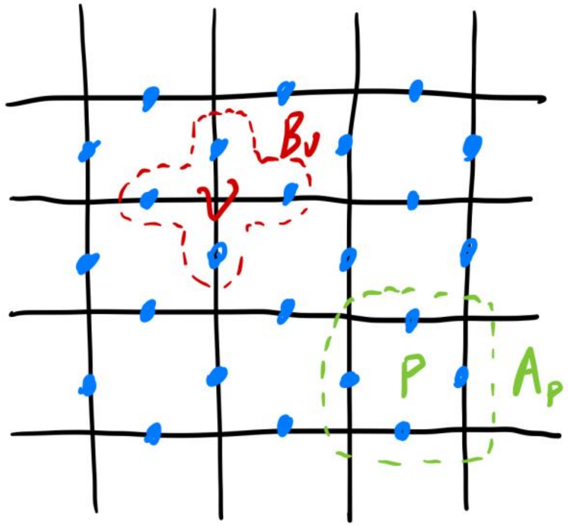
Because such logical operators are non-local, local pertubations **cannot** flip the encoded qubits.

Nevertheless, when $T > 0$, due to the free mobility of excitations (to be introduced), the encoded information can be blurred.

[11] Eric Dennis, Alexei Kitaev, Andrew Landahl and John Preskil, Topological quantum memory, J. Math. Phys. 43, 4452 (2002).

Topological excitations of 2D TCM

- 2D toric code model

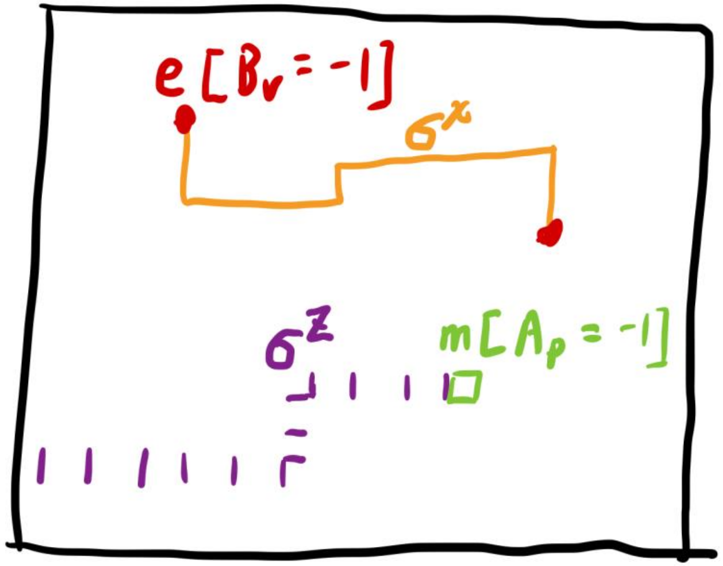


$$H = -\sum_p A_p - \sum_v B_v,$$

$$A_p = \prod_{i\in \mathrm{p}} \sigma_i^x,$$
$$B_v = \prod_{i\in v} \sigma_i^z.$$

Then we can consider open strings, which leads to excited states.

As we can see, Hamilton terms are flipped at the end of open string operators, leads to **excitations** located at such endpoints.



Charge excitation e
-endpoint of $\prod_{i\in l} \sigma_i^x$

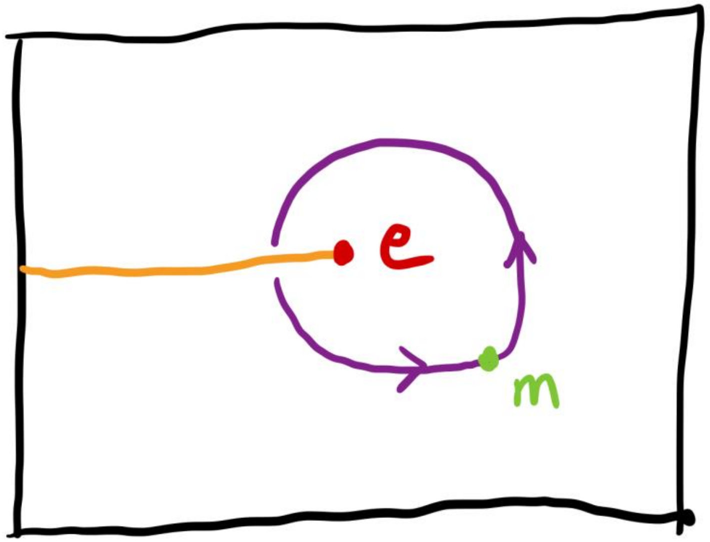
Flux excitation m
-endpoint of $\prod_{i\in l} \sigma_i^z$

These excitations are **topological**, in the sense that a single excitation cannot be generated locally.
At $T > 0$, under thermal perturbation, a pair of topological excitations may **circle the torus and form a logical operator**, which disturbs the information encoded in ground states.

[9] A. Y. Kitaev, Fault-tolerant quantum computation by anyons, Ann. Phys. (N.Y.) 303, 2 (2003).
[10] Michael A. Levin and Xiao-Gang Wen, String-net condensation: A physical mechanism for topological phases, Phys. Rev. B 71, 045110 (2005).

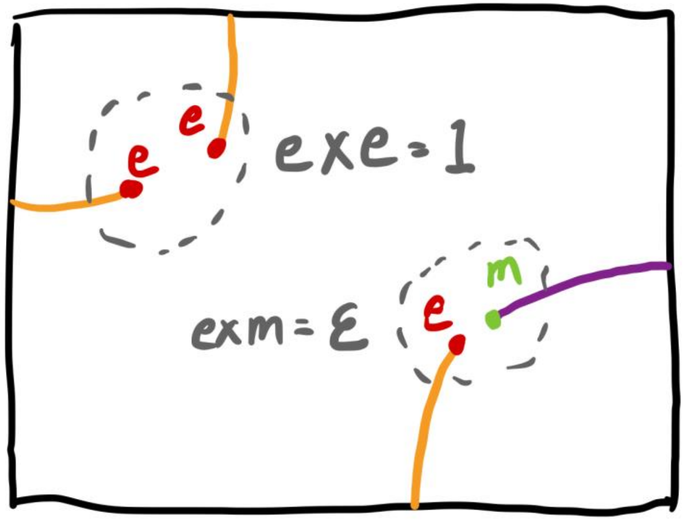
Fusion and braiding in 2D TCM

- Braiding statistics in 2D TCM



As Abeilian anyons, topological excitations have **non-trivial mutual statistics**: the state acquires a -1 phase when e circles m and the contrary (while their self-statistics are trivial).

- Fusion rules in 2D TCM



	1	e	m	ϵ
1	1	e	m	ϵ
e	e	1	ϵ	m
m	m	ϵ	1	e
ϵ	ϵ	m	e	1

When two excitations are very close, the pair behaves like a single new excitation: as a trivial example, a pair of e excitations now can be locally generated, thus it equals to a trivial topological excitation 1. This process is a **fusion** (for abelian excitations, they form an **Abelian group** under fusion).

Non-trivially, the fusion result of e and m dubbed as ϵ is a **fermion**

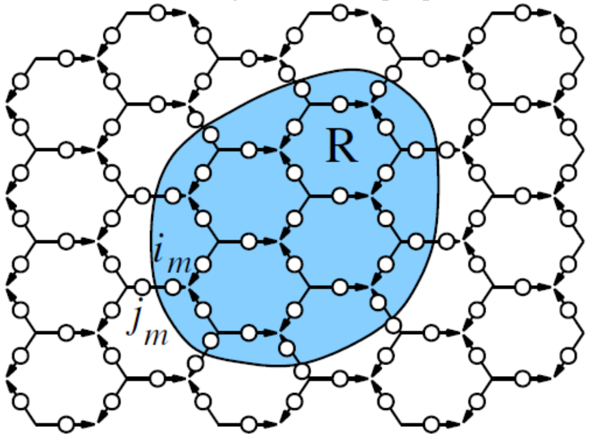
[9] A. Y. Kitaev, Fault-tolerant quantum computation by anyons, Ann. Phys. (N.Y.) 303, 2 (2003).
[10] Michael A. Levin and Xiao-Gang Wen, String-net condensation: A physical mechanism for topological phases, Phys. Rev. B 71, 045110 (2005).

Long range entanglement in 2D TCM

- Topological entanglement entropy

Due to the long range entangled nature of 2D TCM ground states, the entanglement entropy contains **a constant term**, in addition to the area law term of local gapped systems.

Fig from Ref.[13]



$S(\rho_R) = \alpha L - \gamma$, where $-\gamma = -\log 2$ is the **topological entanglement entropy**.

- Entanglement renormalization

Due to the existence of short range entanglement, without appropriate treatment, during a renormalization process the dimension of local Hilbert space would explode.
To avoid this problem, Vidal proposed a renormalization scheme where the **short range entanglement is also renormalized**.

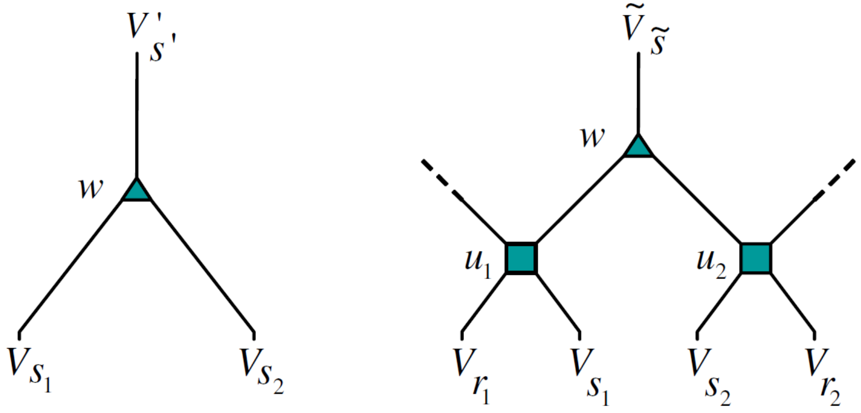


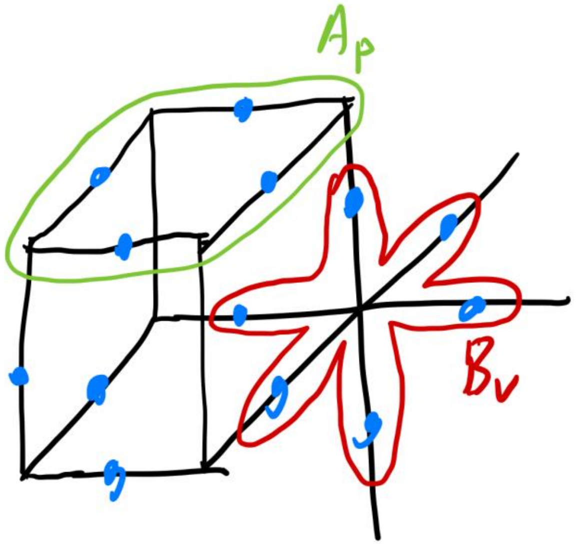
Fig from Ref.[14]

Long range entanglement pattern is preserved in this process, thus it is convenient to study topological orders.

[12] Alexei Kitaev and John Preskill, Topological Entanglement Entropy, Phys. Rev. Lett. 96, 110404 (2006).
[13] Michael A. Levin and Xiao-Gang Wen, Detecting Topological Order in a Ground StateWave Function, 96, 110405 (2006).
[14] G. Vidal, Entanglement Renormalization, Phys. Rev. Lett. 99, 220405 (2007).

Topologically ordered 3D toric code model

- 3D toric code model (**TCM**)



$$H = -\sum_p A_p - \sum_v B_v,$$

$$A_p = \prod_{i\in \mathrm{p}} \sigma_i^x,$$
$$B_v = \prod_{i\in v} \sigma_i^z.$$

The 3D TCM has exactly the same form as 2D TCM, and we can also use the string-net picture to solve all ground states and excited states.

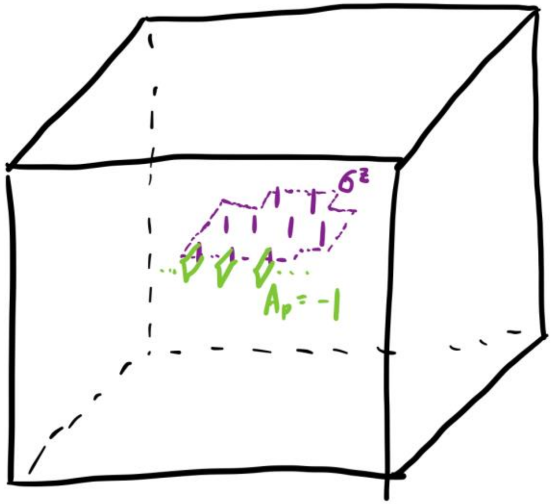
- Some similarities between 2D and 3D TCM:
  - String-net pattern: ground states of 3D TCM also have **loop condensation**.
  - Topological degeneracy: the GSD of 3D GSD can also be obtained by **counting non-contractible loops**.
  - Long range entanglement: 3D TCM also has **constant topological entanglement entropy**.

[15] Alioscia Hamma, Paolo Zanardi and Xiao-Gang Wen, String and membrane condensation on three-dimensional lattices, Phys. Rev. B 72, 035307 (2005).

Topological excitations in 3D TCM

- Loop excitation in 3D TCM

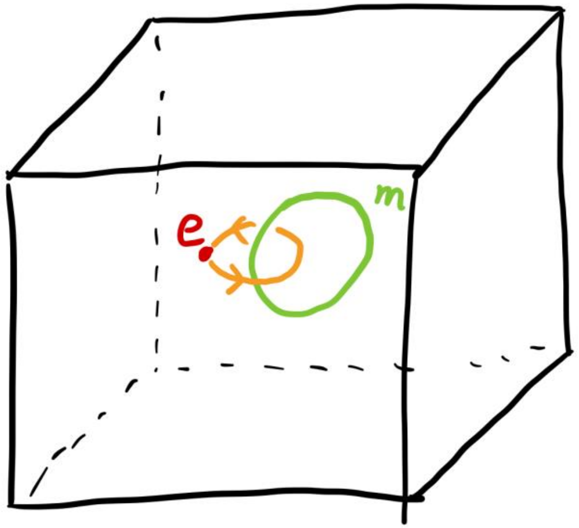
In addition to loop condensation, 3D TCM ground states also have **closed membrane condensation**.



Therefore, we have **flux loop excitations** at the boundary of an open membrane $\prod_{i\in m}\sigma_i^z$.

- Braiding statistics in 3D TCM

In 3D, point excitations are all fermions or bosons. But due to the existence of loop excitations, we can still have non-trivial braiding statistics.



Above is one of the simplest example, that leads to a phase -1.

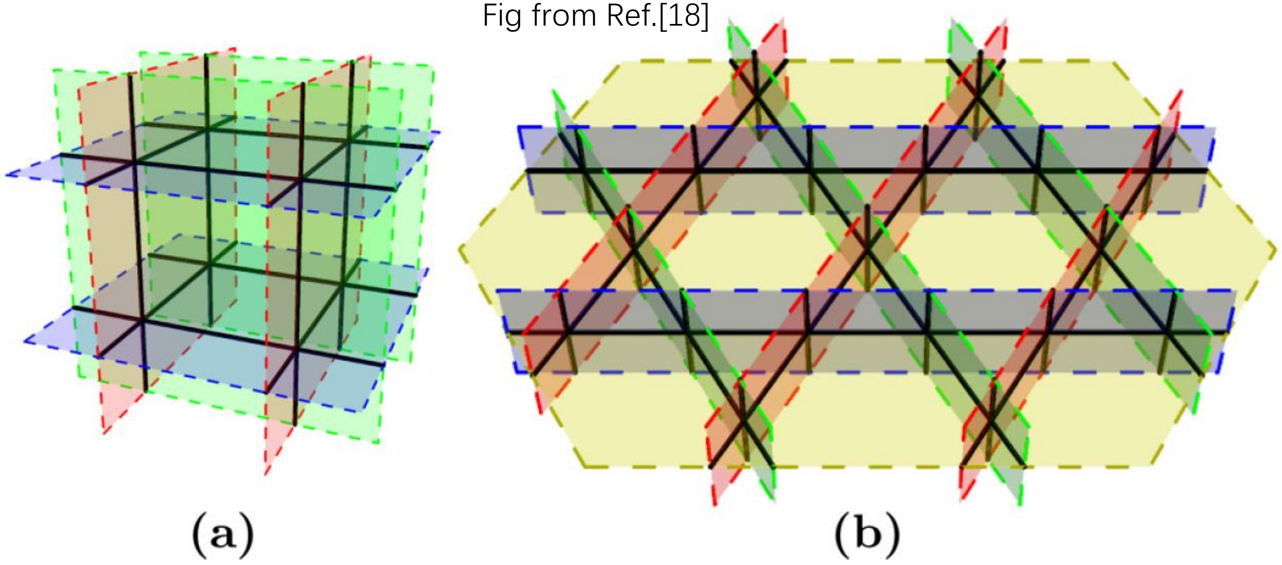
[15] Alioscia Hamma, Paolo Zanardi and Xiao-Gang Wen, String and membrane condensation on three-dimensional lattices, Phys. Rev. B 72, 035307 (2005).
[16] M. G. Alford and Frank Wilczek, Aharonov-Bohm interaction of cosmic strings with matter, Phys. Rev. Lett. 62, 1071 (1989).

More than topology

- Recently, topological excitations with **restricted mobility** are discovered in various topologically ordered systems, that introduces a series of exotic properties.

- As topological excitations are now restricted in certain subspaces, intuitively, we expect the **geometry** of such subspaces to be a part of the ``topological'' order (thus TCM-like orders are dubbed as **pure topological orders**).

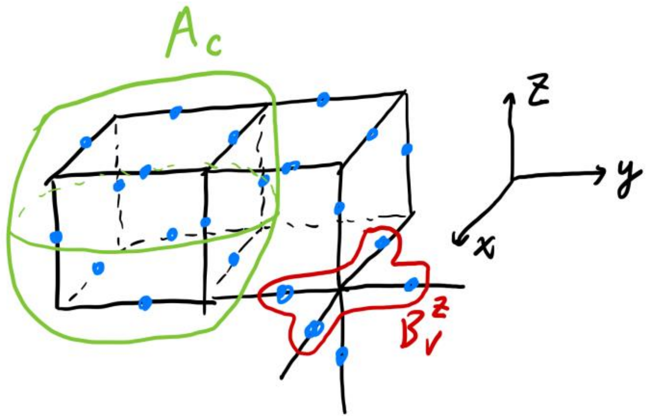
Fig from Ref.[18]



[17] Sagar Vijay, Jeongwan Haah, and Liang Fu. A new kind of topological quantum order: A dimensional hierarchy of quasiparticles built from stationary excitations. Phys. Rev. B 92, 235136 (2015).
[18] Kevin Slagle and Yong Baek Kim, X-cube model on generic lattices: Fracton phases and geometric order, Phys. Rev. B 97, 165106 (2018).

Cage-net picture of X-cube model

- 3D X-cube model

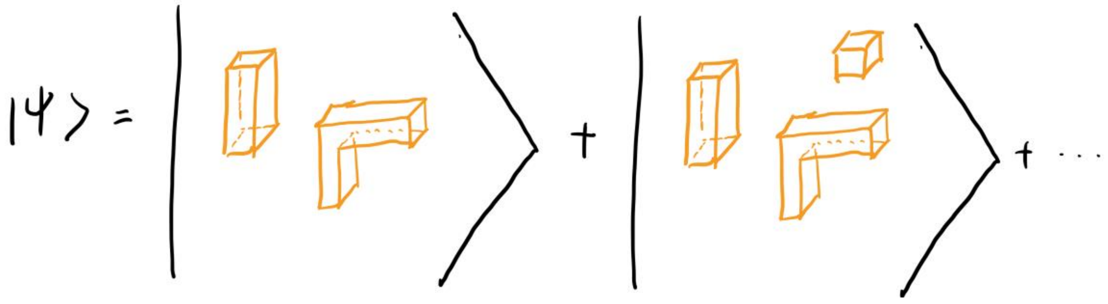


$$H = -\sum_c A_c - \sum_v \sum_l B_v^l,$$

$$A_c = \prod_{i \in c} \sigma_i^x,$$
$$B_v^l = \prod_{i \in v_l} \sigma_i^z.$$

Similar to toric code model, each link is assigned with a spin-$1/2$.
An A_c term is the product of the x-components of the **12 spins** on the links around cube c, a B_v^l term is the product of the z-components of the **4 spins** on the links around vertex v and inside a plane perpendicular to direction l.

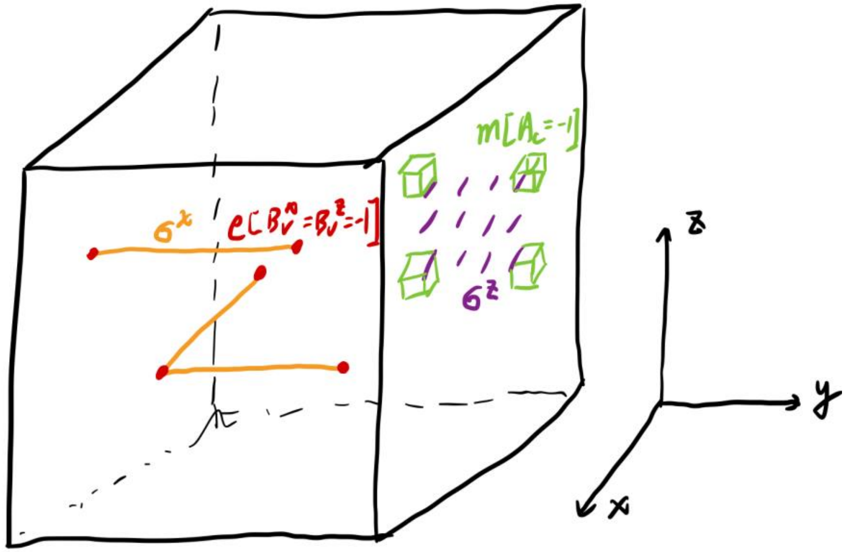
- Cage-net picture of X-cube model
  - X-cube model is also exactly solvable.
  - Similar to toric code model, we can describe ground states and excited states of X-cube model with strings formed of down spins.
  - However, now in a ground state, strings are not simply required to be closed, but to be **``closed'' cages**.



[19] Sagar Vijay, Jeongwan Haah, and Liang Fu, Fracton Topological Order, Generalized Lattice Gauge Theory and Duality, Phys. Rev. B 94, 235157 (2016).
[20] Abhinav Prem, Sheng-Jie Huang, Hao Song, and Michael Hermele, Cage-Net Fracton Models, Phys. Rev. X 9, 021010 (2019).

Topological excitations of X-cube model

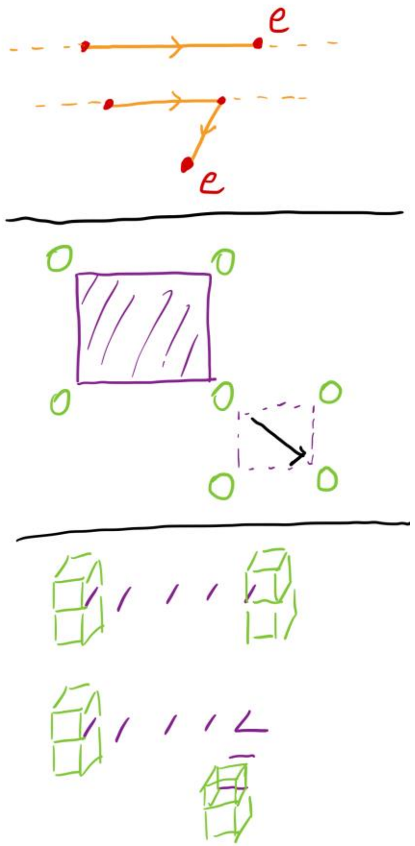
- ``Rigid'' generation operators



Similar to 3D TCM, we have string and membrane generation operators. But now the strings and membranes are ``rigid'':

- Additional e excitations appear at the turning points of strings.
- m excitations are point-like and appear at the corners of membranes.

- Subdimensional particles



Lineon: generated by $\prod_{i\in l} \sigma_i^x$. Turning points of string l leads to **additional energy cost**.

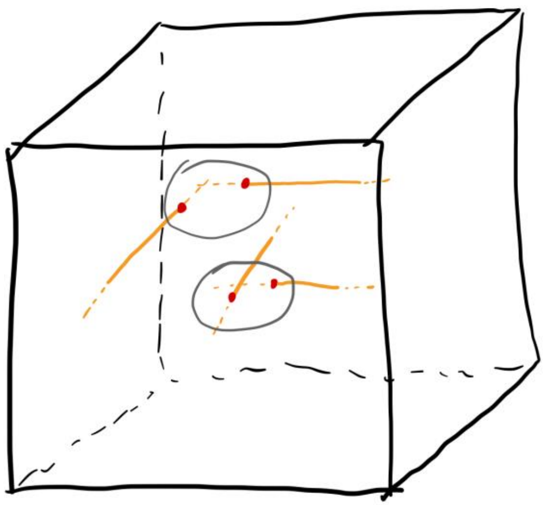
Fracton: generated by $\prod_{i\in m} \sigma_i^z$. Corners of membrane m leads to additional energy cost.

Planon: generated by $\prod_{i\in m} \sigma_i^z$, where membrane m reduced to a string.

[19] Sagar Vijay, Jeongwan Haah, and Liang Fu, Fracton Topological Order, Generalized Lattice Gauge Theory and Duality, Phys. Rev. B 94, 235157 (2016).
[21] Wilbur Shirley, Kevin Slagle and Xie Chen, Foliated fracton order from gauging subsystem symmetries, SciPost Phys. 6, 041 (2019).

Fusion and statistics of X-cube model

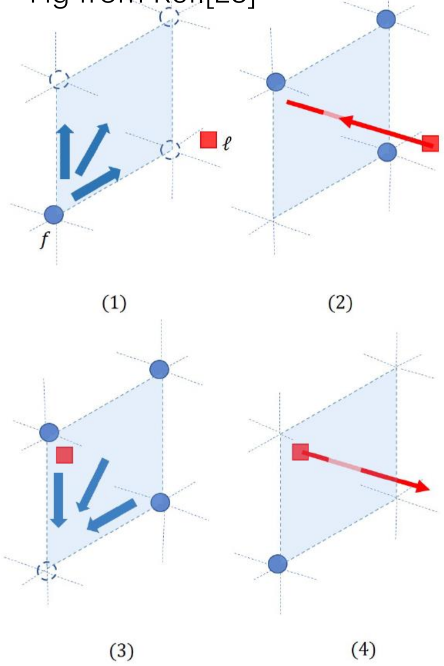
- Location dependent fusion
  - Due to the mobility restrictions, the fusion rules now depend on the location of excitations.
  - Therefore, we can recognize the mobile subspace as a part of the type (i.e. superselection sector) of excitations.



Thus we have **infinite types of excitations** in the thermodynamic limit.

- Another way to realize ``braiding'' in 3D
  - Though X-cube model only has point-like excitations, due to the mobility restriction, we can still have non-trivial statistics.

Fig from Ref.[23]

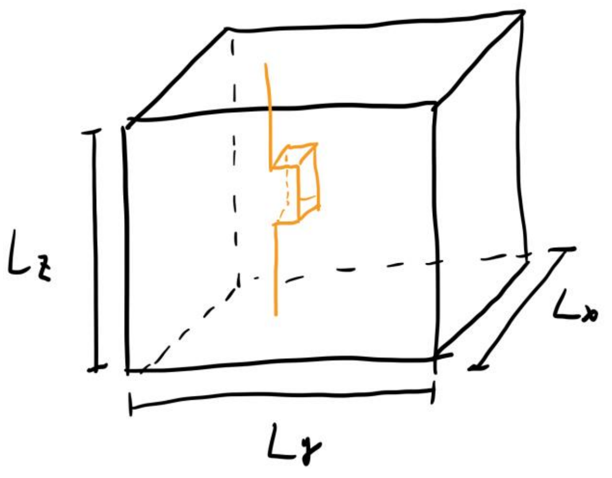


As we can see, if certain virtual processes are allowed, we can have non-trivial processes for a lineon and a fracton, that gives a phase -1.

[22] Shriya Pai and Michael Hermele, Fracton fusion and statistics, Phys. Rev. B 100, 195136 (2019).

Topological degeneracy of X-cube model

- ``Rigid'' logical operators
  - For example, we consider the σ^z configuration basis. **Non-contractible strings** can also change the ground state, but now they are also required to be rigid.

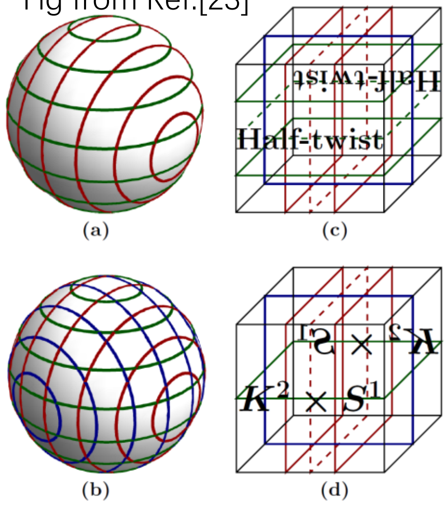


Then, the action of logical operators is location-dependent. On 3-torus, we have $\log_2 GSD = 2L_x + 2L_y + 2L_z - 3$.

- From the perspective of foliation
  - X-cube ground states are also **locally indistinguishable**, thus the degeneracy is also topological.
  - In fact, the degeneracy is found to be related to the topology of 2D subsystems (dubbed as leaves in the language of foliation, to be introduced in next section)

$\log_2 GSD = b_x L_x + b_y L_y + b_z L_z - c$, where b_i is first Betti number.

Fig from Ref.[23]



[19] Sagar Vijay, Jeongwan Haah, and Liang Fu, Fracton Topological Order, Generalized Lattice Gauge Theory and Duality, Phys. Rev. B 94, 235157 (2016).
[23] Wilbur Shirley, Kevin Slagle, Zhenghan Wang, and Xie Chen, Fracton Models on General Three-Dimensional Manifolds, Phys. Rev. X 8, 031051 (2018).

Fracton order as topological quantum memory

As we've demonstrated for toric codes, the mobile excitations can disturb the encoded information. For a fracton order with **only fracton excitations** (dubbed as Type-II fracton orders), such problems may be avoided.

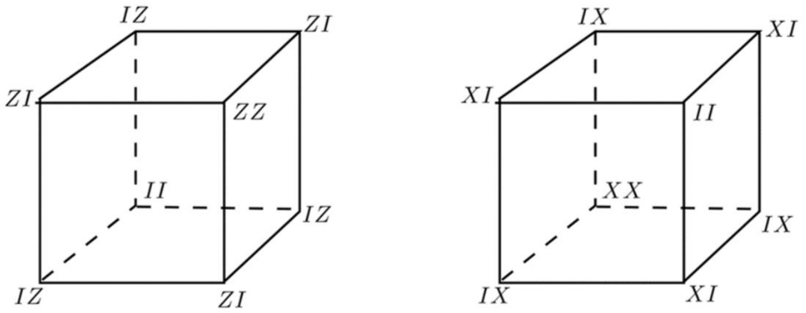
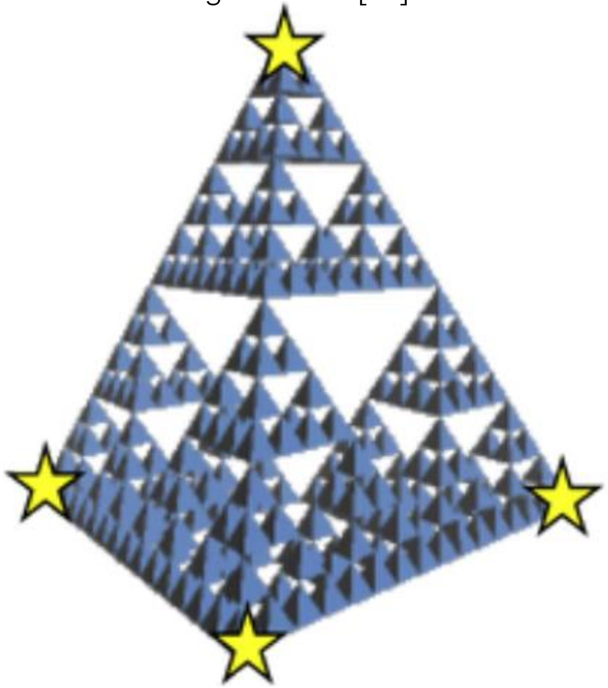


Fig from Ref.[24]

Fig from Ref.[25]



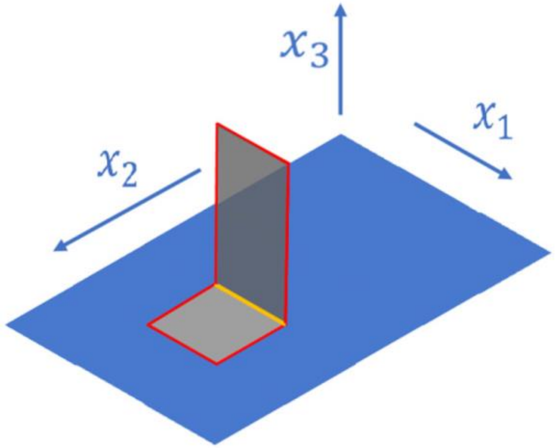
Haah's code is a typical Type-II fracton ordered model, where topological excitations are generated at the corners of a fractal.
Thus Haah's code is expected to be a more robust topological quantum memory.

[24] Jeongwan Haah, Local stabilizer codes in three dimensions without string logical operators, Phys. Rev. A 83, 042330 (2011).
[25] Michael Pretko, Xie Chen and Yizhi You, Fracton phases of matter, International Journal of Modern Physics A Vol. 35, No. 06, 2030003 (2020).

Fracton physics of spatially extended excitations

- Mobility and deformability restriction

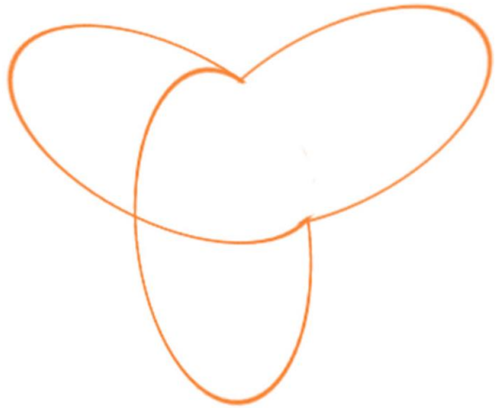
We constructed a series of exactly solvable models labeled by 4 integers. For example, $[1,2,3,4]$ model contains loop excitations with restricted mobility.



As we can see, as a loop cannot escape from the mobile plane by deforming, either, the mobility restriction naturally extends to deformability restriction.

- Complex excitations

Due to the existence of deformability restriction, loops can fuse into string-like excitations with non-manifold shapes:

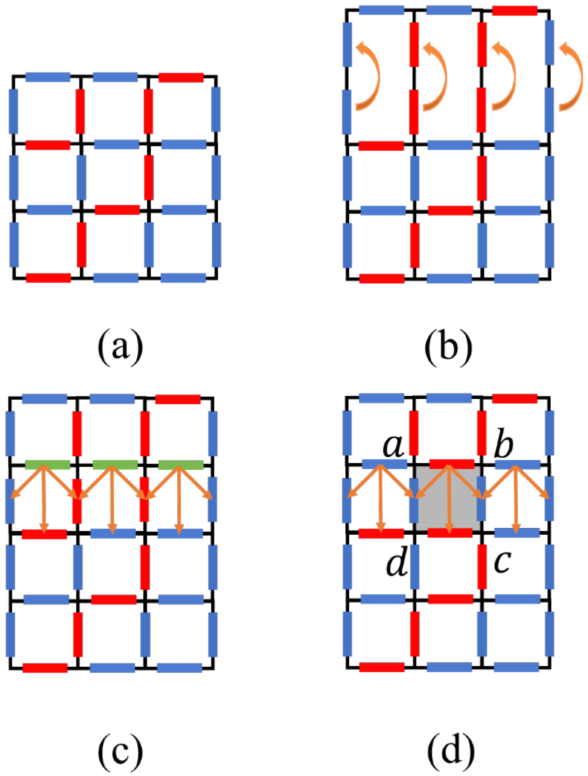


The deformability restriction prevents the loops restricted in perpendicular planes to totally cancel each other, and what remains is such a **complex excitation**.

[1] **MYL** and Peng Ye, Fracton physics of spatially extended excitations, Phys. Rev. B 101, 245134 (2020).

An obstacle to ERG transformation

- Dropping short range entanglement



In a 2D TCM ground state, we can use **local unitary** transformations to decouple some spins, and drop such **unentangled spins**.

- A paradox

If we want to change the size of an X-cube model with ERG, we expect the system size to be changed.

However, $\log_2 GSD = 2L_x + 2L_y + 2L_z - 3$ means that the topological GSD would also be changed, while ERG should not influence long range entanglement pattern.

How to understand that long range entanglement pattern of fracton orders with ERG?

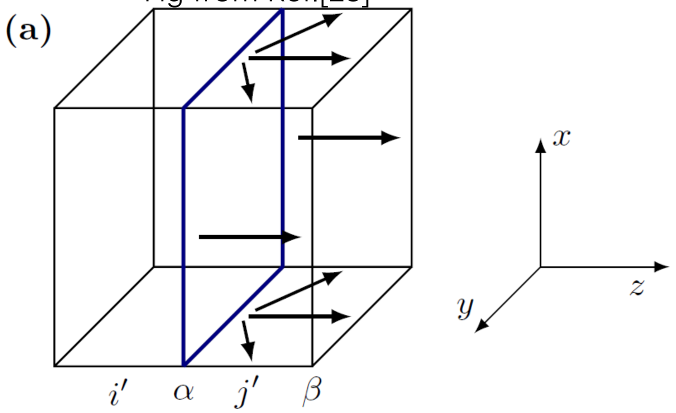
[5] MYL, Peng Ye, Hierarchy of Entanglement Renormalization and Long-Range Entangled States, arXiv:2211.14136 [quant-ph].
[23] Wilbur Shirley, Kevin Slagle, Zhenghan Wang, and Xie Chen, Fracton Models on General Three-Dimensional Manifolds, Phys. Rev. X 8, 031051 (2018).
[26] Miguel Aguado and Guifré Vidal, Entanglement Renormalization and Topological Order, Phys. Rev. Lett. 100, 070404 (2008).

Generalized ERG and foliation

- Source of entanglement

The authors of [23] noticed that, by allowing 2D TCM ground states to be added/removed, the size and GSD of X-cube model can be changed consistently.

Fig from Ref.[23]



- Foliated fracton order theory

That is to say, the fracton order of X-cube model can be understood by considering the ``**total foliation**'' of the base manifold, i.e., how is the base manifold partitioned to a series of 2D subsystems dubbed as leaves.

As a result, we have
$\log_2 GSD = b_x L_x + b_y L_y + b_z L_z - c$.
Where b_i is the first Betti number of leaves stacked along direction i, c depends on the topology of intersection.

[23] Wilbur Shirley, Kevin Slagle, Zhenghan Wang, and Xie Chen, Fracton Models on General Three-Dimensional Manifolds, Phys. Rev. X 8, 031051 (2018).
[26] Miguel Aguado and Guifré Vidal, Entanglement Renormalization and Topological Order, Phys. Rev. Lett. 100, 070404 (2008).
[27] Brian Swingle and John McGreevy, Renormalization group constructions of topological quantum liquids and beyond, Phys. Rev. B 93, 045127 (2016).

3D Type-I fracton phases

- Taming the infinity

In thermodynamic limit, the ``topological'' degeneracy and number of types of ``topological'' excitations all go to infinity, that prevents us to understand fracton orders with such data.

Foliation theory shows how such infinity is originated from concrete topological data.

- 3D Type-I fracton phases

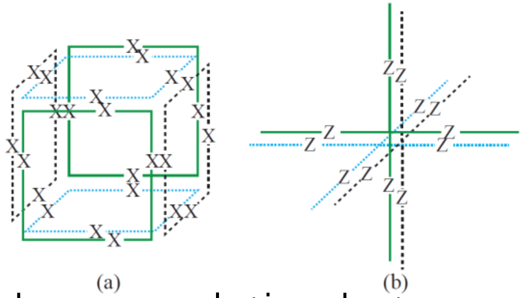
Foliated fracton orders urge us to to use a new definition of phases for such system: two states belong to the same 3D Type-I fracton phase when they can be connected by local unitary transformations **combined with addition/removal of 2D pure topological ordered states.**

[23] Wilbur Shirley, Kevin Slagle, Zhenghan Wang, and Xie Chen, Fracton Models on General Three-Dimensional Manifolds, Phys. Rev. X 8, 031051 (2018).
[28] Wilbur Shirley, Kevin Slagle, Xie Chen, Fractional excitations in foliated fracton phases, Annals of Physics Volume 410, November 2019, 167922.

Hierarchy of long range entanglement patterns

- Pure and fracton topological orders

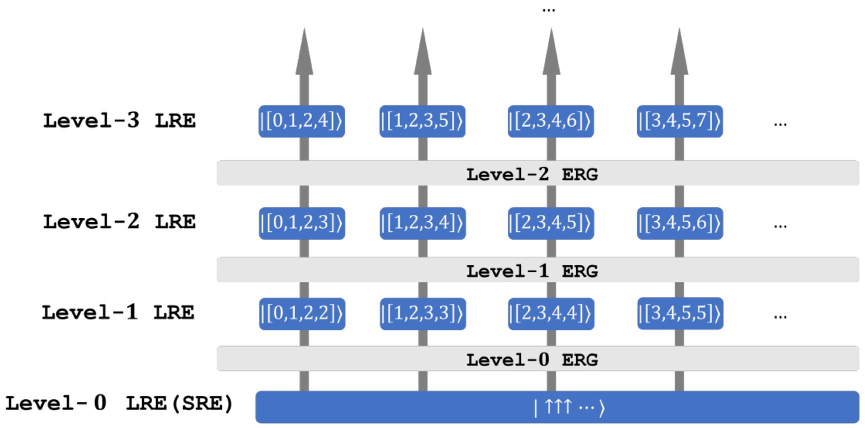
Such foliated fracton orders that depends on the foliation structure shows that their properties can be understood based on the more familiar pure topological orders.



Besides, it shows a relation between fracton and pure topological orders: a fracton order may be constructed from pure topological orders in a non-trivial way (i.e. inequivalent to decoupled stacks), just like pure topological orders constructed from decoupled spins.

- Hierarchy of LRE states

In [5], We investigated the possibility of further using X-cube ground states to build more complex orders, and use the result to build more complex orders, and so on…



We obtain a series of infinite towers of models, where a level-n ground state can be constructed from level-$(n-1)$ ground states.

[5] MYL, Peng Ye, Hierarchy of Entanglement Renormalization and Long-Range Entangled States, arXiv:2211.14136 [quant-ph].
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Summary

- Topological orders are characterized by long range entanglement patterns.

- Fracton orders show complicated long range entanglement patterns, where geometry is also involved.

- A generalized entanglement renormalization scheme allows us to understand fracton orders with pure topological orders.

Thanks for your attention!

Outline

- 1. Topological orders and toric code model
  - 1.1 From symmetry breaking to topological order
  - 1.2 2D toric code model
  - 1.3 3D toric code model
- 2. Fracton (topological) orders and X-cube model
  - 2.1 From (pure) topological order to fracton order
  - 2.2 X-cube model
- 3. Foliated fracton order theory of Type-I fracton orders
- 4. Summary

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Reviews:

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