Unconventional excitonic orders and excitonic spectra

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Outline

1. **Introduction to excitons**
   - BEC-BCS crossover
2. **Excitonic spin superfluidity with spin-charge conversion**

   Yeyang Zhang and Ryuichi Shindou, Phys. Rev. Lett. **128**, 066601 (2022)
   - Model and phases
   - Goldstone modes and Josephson effects
   - Spin-orbit coupling
3. **Antiparticles of excitons in semimetals**

   Lingxian Kong, Ryuichi Shindou, and **Yeyang Zhang**, Phys. Rev. B **106**, 235145 (2022)
   - Polology and Bethe-Salpeter equation
   - Effective field theory

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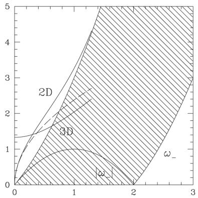
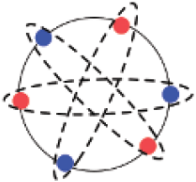
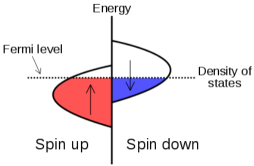
Collective modes of electrons



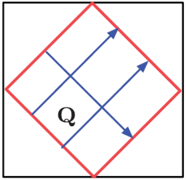
Plasmon $\rho_{\boldsymbol{q}} = \sum_{\boldsymbol{k}} \boldsymbol{c}_{\boldsymbol{k}}^{\dagger} \boldsymbol{c}_{\boldsymbol{k}+\boldsymbol{q}}$



Cooper pair $\Delta_{\boldsymbol{q}} = \sum_{\boldsymbol{k}} f(\boldsymbol{k}; \boldsymbol{q}) c_{\uparrow, \boldsymbol{k}+\boldsymbol{q}} c_{\downarrow, -\boldsymbol{k}}$



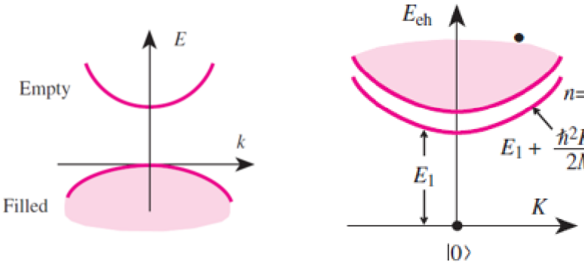
Itinerant magnetism $S_{i,\boldsymbol{q}} = \sum_{\boldsymbol{k}} \boldsymbol{c}_{\boldsymbol{k}}^{\dagger} \boldsymbol{\sigma}_i \boldsymbol{c}_{\boldsymbol{k}+\boldsymbol{q}}$



CDW/SDW $S_{0/i,\boldsymbol{q}} = \sum_{\boldsymbol{k}} \boldsymbol{c}_{\boldsymbol{k}}^{\dagger} \boldsymbol{\sigma}_{0/i} \boldsymbol{c}_{\boldsymbol{k}+\boldsymbol{q}}$

Quantum Theory of the Electron Liquid (2005); *Ultracold Atomic Physics* (2021)

Exciton: pair of an electron and a hole



- Bound states in semiconductors:

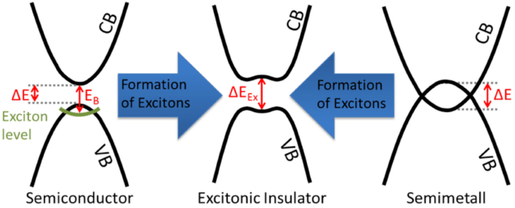
$$[-\frac{\nabla^2_{\boldsymbol{R_e}}}{2m_e} - \frac{\nabla^2_{\boldsymbol{R_h}}}{2m_h} + E_g - \frac{e^2}{\epsilon|\boldsymbol{R_e} - \boldsymbol{R_h}|}]\Phi(\boldsymbol{R_e}, \boldsymbol{R_h}) = E\Phi(\boldsymbol{R_e}, \boldsymbol{R_h}), \quad (1)$$

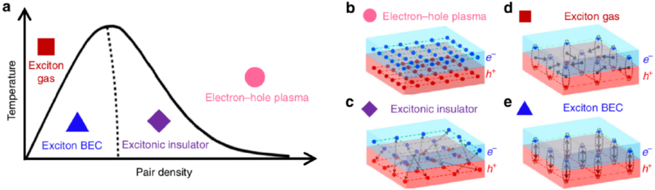
$$-\frac{\nabla^2_{\boldsymbol{R}}}{2M}\psi(\boldsymbol{R}) = E_R\psi(\boldsymbol{R}), \quad [-\frac{\nabla^2_{\boldsymbol{r}}}{2\mu} - \frac{e^2}{\epsilon r} + |E_g|]\phi(\boldsymbol{r}) = E_r\phi(\boldsymbol{r}), \quad (2)$$

$$E_{nlm;\boldsymbol{K}} = \frac{\boldsymbol{K}^2}{2M} + |E_g| - \frac{R^*}{n}. \quad (3)$$

Fundamentals of Semiconductors (2010)

Excitonic phase diagram

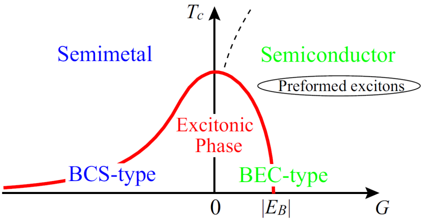




- Exciton annihilation operators: $\phi_{n,\boldsymbol{q}} = \sum_{\boldsymbol{k}} f_n(\boldsymbol{k};\boldsymbol{q}) b_{\boldsymbol{k}}^{\dagger} a_{\boldsymbol{k}+\boldsymbol{q}}$.
  Electron/hole annihilation operators: $a_{\boldsymbol{k}}/b_{-\boldsymbol{k}}^{\dagger}$.

J. Phys. Condens. Matter **30**, 305602 (2018); Nat. Commun. **8**, 1971 (2017)

BEC-BCS crossover



- Hamiltonian:

$$H = \sum_{\vec{k}} \epsilon_a(\vec{k}) a^\dagger_{\vec{k}} a_{\vec{k}} + \sum_{\vec{k}} \epsilon_b(\vec{k}) b^\dagger_{\vec{k}} b_{\vec{k}} + \frac{1}{\Omega} \sum_{\vec{k},\vec{k}',\vec{q}} V(\vec{q}) b^\dagger_{\vec{k}+\vec{q}} b_{\vec{k}} a^\dagger_{\vec{k}'-\vec{q}} a_{\vec{k}'}, \tag{4}$$

$$\epsilon_a(\vec{k}) = \frac{\vec{k}^2}{2m} - E_g, \quad \epsilon_b(\vec{k}) = -\frac{\vec{k}^2}{2m} + E_g, \quad V(\vec{q}) = \frac{4\pi e^2/K}{\vec{q}^2 + \kappa^2}, \quad E_g \equiv -\frac{G}{2}. \tag{5}$$

- Order parameter:

$$\Delta(\vec{k}) \equiv -\frac{1}{\Omega} \sum_{\vec{k}'} V(\vec{k} - \vec{k}') \left\langle b^\dagger_{\vec{k}'} a_{\vec{k}'} \right\rangle. \tag{6}$$

T. Kaneko, Ph.D. Dissertation (2016)

BEC-BCS crossover

- Mean-field Hamiltonian:

$$\bar{H} = \sum_{\vec{k}} \epsilon_a(\vec{k}) a^\dagger_{\vec{k}} a_{\vec{k}} + \sum_{\vec{k}} \epsilon_b(\vec{k}) b^\dagger_{\vec{k}} b_{\vec{k}} + \sum_{\vec{k}} (\Delta(\vec{k}) a^\dagger_{\vec{k}} b_{\vec{k}} + \mathrm{h.c.}) + \epsilon_0, \quad (7)$$

where $\epsilon_0 = \sum_{\vec{k}} \Delta(\vec{k}) \left\langle a^\dagger_{\vec{k}} b_{\vec{k}} \right\rangle$.

- Gap equation:

$$\Delta(\vec{k}) = \frac{1}{\Omega} \sum_{\vec{k'}} V(\vec{k} - \vec{k'}) \frac{\Delta(\vec{k'})}{2E_{\vec{k'}}} \tanh \frac{\beta E_{\vec{k'}}}{2}, \quad (8)$$

where

$$E_{\vec{k}} \equiv \sqrt{\epsilon^2_{\vec{k}} + |\Delta(\vec{k})|^2}, \quad \epsilon_{\vec{k}} \equiv \frac{\hbar^2 \vec{k}^2}{2m} - E_g. \quad (9)$$

- Difficulties come from the $\vec{k}$-dependence of $\Delta(\vec{k})$.

BEC limit

- Consider $E_g < 0$ (semiconductor case, $\kappa = 0$), $|\Delta(\vec{k})| \ll |E_g|$. At $T = 0$,

$$\Delta(\vec{k}) = \frac{1}{\Omega}\sum_{\vec{k'}} V(\vec{k} - \vec{k'})\frac{\Delta(\vec{k'})}{2E_{\vec{k'}}}. \quad (10)$$

Define $\psi(\vec{k}) \equiv \frac{\Delta(\vec{k})}{2E_{\vec{k}}}$, we get

$$[(\frac{\vec{k}^2}{m} + 2|E_g|)^2 + 4|\Delta(\vec{k})|^2]^{1/2}\psi(\vec{k}) = \frac{1}{\Omega}\sum_{\vec{k'}} V(\vec{k} - \vec{k'})\psi(\vec{k'}). \quad (11)$$

- Near $\vec{k} = \vec{0}$, $V(\vec{k} - \vec{k'})$ dominates, $|\Delta(\vec{k})| \approx \Delta_0 \ll |E_g|$.

$$[\frac{\vec{k}^2}{m} + 2|E_g| + \frac{\Delta_0^2}{|E_g|}]\psi(\vec{k}) = \frac{1}{\Omega}\sum_{\vec{k'}} V(\vec{k} - \vec{k'})\psi(\vec{k'}). \quad (12)$$

BEC limit

- Consider the following equation in $\vec{k}$-space,

$$[\frac{\vec{k}^2}{m}+|E|]\phi(\vec{k})=\frac{1}{\Omega}\sum_{\vec{k}'}V(\vec{k}-\vec{k}')\phi(\vec{k}'). \tag{13}$$

In $\vec{r}$-space, Eq. (13) becomes

$$[-\frac{\nabla^2}{2\mu}-\frac{e^2}{Kr}]\phi(\vec{r})=-|E|\phi(\vec{r}), \quad \frac{1}{\mu}\equiv\frac{2}{m}. \tag{14}$$

- Suppose the ground-state energy of Eq. (14) is $-|E_B|$. $\Delta(\boldsymbol{k})=0$ when $|E_B|<2|E_g|$. For $0<|E_B|-2|E_g|\ll|E_g|$,

$$\Delta_0\approx\sqrt{(|E_B|-2|E_g|)|E_B|}\approx|E_B|\sqrt{\frac{1}{2}(1-\frac{2|E_g|}{|E_B|})}. \tag{15}$$

BCS limit

- Consider $E_g > 0$ (semimetal case), $|\Delta(\vec{k})| \ll |E_g|$.

$$\epsilon(\vec{k}) = \frac{\vec{k}^2}{2m} - \frac{k_F^2}{2m}, \quad k_F \equiv \sqrt{2mE_g}, \tag{16}$$

$$\Delta(\vec{k}) = \int \frac{\mathrm{d}^3\vec{k}'}{(2\pi)^3} \frac{4\pi e^2/K}{|\vec{k}-\vec{k}'|^2+\kappa^2} \frac{\Delta(\vec{k}')}{2E_{\vec{k}'}} \tanh\frac{\beta E_{\vec{k}'}}{2}. \tag{17}$$

- When $k_F \gg \kappa$, we apply the BCS approximation:

$$\Delta(\vec{k}) = \begin{cases} \Delta_0 & |k-k_F| < k_c \\ 0 & |k-k_F| > k_c \end{cases}. \tag{18}$$

At $T=0$, taking Eq. (18) into Eq. (17), we get

$$\begin{aligned}\Delta(k) &= \frac{4\pi e^2/K}{(2\pi)^3}\Delta_0 \int_{k_F-k_c}^{k_F+k_c} dk' \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{k'^2 \sin\theta}{k^2+k'^2-2kk'\cos\theta+\kappa^2}\frac{1}{2E_{k'}} \\ &= \frac{e^2/K}{2\pi k}\Delta_0 \int_{k_F-k_c}^{k_F+k_c} k' dk' \frac{1}{2E_{k'}} \ln\left[\frac{(k+k')^2+\kappa^2}{(k-k')^2+\kappa^2}\right]. \end{aligned} \tag{19}$$

BCS limit

- For $k_c \ll k_F$, the k-dependence of $\Delta(k)$ at $T = 0$ is

$$\Delta(k) \approx \frac{e^2/K}{2\pi k}\Delta_0 \ln[\frac{(k+k_F)^2+\kappa^2}{(k-k_F)^2+\kappa^2}]\int_0^{\frac{k_F k_c}{m}} d\xi \frac{m}{\sqrt{\xi^2+\Delta_0^2}}$$

$$\approx \frac{me^2/K}{2\pi k}\Delta_0 \ln[\frac{(k+k_F)^2+\kappa^2}{(k-k_F)^2+\kappa^2}]\ln(\frac{2k_F k_c}{m\Delta_0}). \quad (20)$$

  If we choose $\Delta(k_F \pm k_c) = \frac{\Delta(k_F)}{2}$, we have $k_c \approx \sqrt{2\kappa k_F} \ll k_F$, so the BCS approximation is for a narrow peak.

- Taking $k = k_F$ into Eq. (20), we get the order parameter Δ_0:

$$1 \approx \frac{me^2/K}{2\pi k_F}\ln(\frac{4k_F^2}{\kappa^2})\ln(\frac{2k_F k_c}{m\Delta_0}). \quad (21)$$

Sov. Phys. JETP **21**, 790 (1965)

BCS limit

- At critical temperature β_c, take $k = k_F$,

$$1 = \frac{e^2/K}{2\pi k_F} \int_{k_F-k_c}^{k_F+k_c} \mathrm{d}k' \frac{k'}{2\epsilon'_k} \tanh\frac{\beta_c \epsilon_{k'}}{2} \ln[\frac{(k_F+k')^2+\kappa^2}{(k_F-k')^2+\kappa^2}]. \quad (22)$$

We apply a similar approximation as for $T = 0$,

$$1 \approx \frac{e^2/K}{2\pi k_F} \ln(\frac{4k_F^2}{\kappa^2}) \int_{k_F-k_c}^{k_F+k_c} \frac{k'\mathrm{d}k'}{2\epsilon_{k'}} \tanh\frac{\beta_c \epsilon_{k'}}{2}. \quad (23)$$

By changing the integral argument, we have

$$1 = \frac{me^2/K}{2\pi k_F} \ln(\frac{4k_F^2}{\kappa^2})([\ln\xi \tanh\xi]_0^{\frac{k_F k_c}{2m}\beta_c} - \int_0^{\frac{k_F k_c}{2m}\beta_c} \mathrm{d}\xi \frac{\ln\xi}{\cosh^2\xi}). \quad (24)$$

k_F is large, so we can take $\frac{k_F k_c}{2m}\beta_c \to \infty$ to get

$$1 \approx \frac{me^2/K}{2\pi k_F} \ln(\frac{4k_F^2}{\kappa^2}) \ln(\frac{2\mathrm{e}^\gamma}{\pi} \frac{k_F k_c}{m} \beta_c). \quad (25)$$

BCS limit

- Critical temperature T_c:

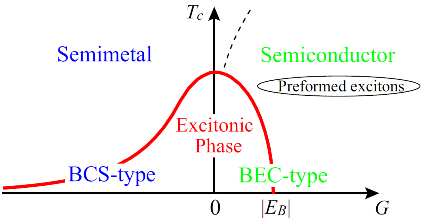
$$k_B T_c = \frac{2\mathrm{e}^{\gamma}}{\pi}\frac{k_F k_c}{m}\exp\Big[-\frac{\frac{2\pi k_F}{me^2/K}}{\ln(\frac{4k_F^2}{\kappa^2})}\Big]$$
$$= \frac{\mathrm{e}^{\gamma}}{\pi^{3/2}}\frac{me^4}{K^2}\Big(\frac{2E_g}{|E_B|}\Big)^{\frac{3}{4}}\exp\Big[-\frac{\pi\sqrt{\frac{2E_g}{|E_B|}}}{\ln(\pi\sqrt{\frac{2E_g}{|E_B|}})}\Big] = \frac{\Delta_{0,T=0}}{\pi\mathrm{e}^{-\gamma}}, \quad (26)$$

where we use a relation from RPA,

$$\frac{4k_F^2}{\kappa^2} = \frac{2\pi k_F}{me^2/K} = \pi\sqrt{\frac{2E_g}{|E_B|}}. \quad (27)$$

BCS limit

- Because of the $\vec{k}$-dependence of $V(\vec{k})$, T_c decreases when k_F increase (i.e. when E_g increases).
  In superconductors, $V(\vec{r}) = U\delta(\vec{r})$ with an energy cutoff ω_D, T_c increases when k_F increases.
  In both cases, we have $\frac{2\Delta_{0,T=0}}{k_B T_c} = 2\pi e^{-\gamma} \approx 3.53$.
- T_c remains finite for large k_F is because of Fermi surface nesting ($m_a = m_b$, $\mu = 0$).



Modern Condensed Matter Physics (2019); T. Kaneko, Ph.D. Dissertation (2016)

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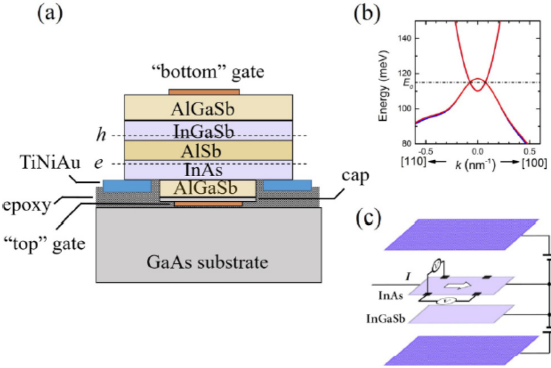
Electron-hole double-layer systems



- Advantages: approximate $U(1) \times U(1)$ charge symmetry; separation of electron and hole currents; engineering flexibility of the two bands.
- Disadvantage: quasi-long-range orders?

Phys. Rev. B **99**, 085307 (2019)

Model

- Electron-hole double layers with in-plane magnetic exchange fields (H_a and H_b):

$$\begin{aligned} K \equiv& H - \mu N \\ =& \int \mathrm{d}^2\vec{r}\boldsymbol{a}^\dagger(\vec{r})[(-\frac{\hbar^2\nabla^2}{2m_a} - E_g - \mu)\boldsymbol{\sigma}_0 + H_a\boldsymbol{\sigma}_x]\boldsymbol{a}(\vec{r}) \\ &+ \int \mathrm{d}^2\vec{r}\boldsymbol{b}^\dagger(\vec{r})[(\frac{\hbar^2\nabla^2}{2m_b} + E_g - \mu)\boldsymbol{\sigma}_0 + H_b\boldsymbol{\sigma}_x]\boldsymbol{b}(\vec{r}) \\ &+ g\sum_{\sigma,\sigma'=\uparrow,\downarrow}\int \mathrm{d}^2\vec{r}\, a^\dagger_\sigma(\vec{r})b^\dagger_{\sigma'}(\vec{r})b_{\sigma'}(\vec{r})a_\sigma(\vec{r}). \end{aligned} \tag{28}$$

  $\boldsymbol{a}$: electron in the electron layer; $\boldsymbol{b}$: electron in the hole layer.

- Exciton pairing: $\phi_\mu \equiv \frac{g}{2}\left\langle \boldsymbol{b}^\dagger\boldsymbol{\sigma}_\mu\boldsymbol{a}\right\rangle$ ($\mu = 0, x, y, z$).
  Pseudospin singlet: $\mu = 0$; Pseudospin triplets: $\mu = x, y, z$.

Phys. Rev. B **100**, 035130 (2019)

Effective theory of the excitonic field

- Habbard-Stactonovich transformation:

$$\begin{aligned}\mathcal{Z} = & \int \mathcal{D}[\boldsymbol{a}^\dagger, \boldsymbol{a}, \boldsymbol{b}^\dagger, \boldsymbol{b}]\mathcal{D}[\phi^\dagger_\mu, \phi_\mu]\exp[(\boldsymbol{a}^\dagger, \boldsymbol{b}^\dagger)\boldsymbol{G}_0^{-1}\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}] \\ & \times \exp\{-\sum_{k,\mu}[\frac{2}{g}\phi^\dagger_\mu(k)\phi_\mu(k) - \phi^\dagger_\mu(k)O_\mu(k) - O^\dagger_\mu(k)\phi_\mu(k)]\} \\ = & \int \mathcal{D}[\boldsymbol{a}^\dagger, \boldsymbol{a}, \boldsymbol{b}^\dagger, \boldsymbol{b}]\mathcal{D}[\phi^\dagger_\mu, \phi_\mu]\exp[(\boldsymbol{a}^\dagger, \boldsymbol{b}^\dagger)\boldsymbol{G}^{-1}[\phi^\dagger_\mu, \phi_\mu]\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}] \\ & \times \exp\{-\sum_{k,\mu}[\frac{2}{g}\phi^\dagger_\mu(k)\phi_\mu(k)\} \\ = & \mathcal{N}\int \mathcal{D}[\phi^\dagger_\mu, \phi_\mu]\exp\{-\sum_{k,\mu}[\frac{2}{g}\phi^\dagger_\mu(k)\phi_\mu(k)\}\exp\{\mathrm{Tr}\ln \boldsymbol{G}^{-1}[\phi^\dagger_\mu, \phi_\mu]\},\end{aligned} \tag{29}$$

where $O_\mu(k) \equiv \frac{1}{\sqrt{\hbar\beta\Omega}}\sum_q \boldsymbol{b}^\dagger_q \boldsymbol{\sigma}_\mu \boldsymbol{a}_{q+k}$.

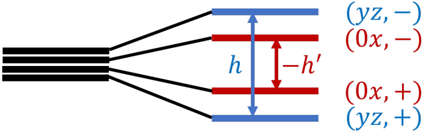
Effective theory of the excitonic field

- Effective Lagrangian for $\vec{\Phi} \equiv (-\mathrm{i}\phi_0, \phi_x, \phi_y, \phi_z) \equiv \vec{\Phi}' + \mathrm{i}\vec{\Phi}''$:

$$\mathcal{L} = -\left(\alpha - \frac{2}{g}\right)|\vec{\Phi}|^2 - \gamma\Big[\big(\vec{\Phi}'^2\big)^2 + \big(\vec{\Phi}''^2\big)^2 + 6\vec{\Phi}'^2\vec{\Phi}''^2 - 4\big(\vec{\Phi}'\cdot\vec{\Phi}''\big)^2\Big] + \lambda|\nabla\vec{\Phi}|^2 - 2h\big(\Phi'_y\Phi''_z - \Phi'_z\Phi''_y\big) + 2h'\big(\Phi'_0\Phi''_x - \Phi'_x\Phi''_0\big). \tag{30}$$

  $\gamma < 0$, $\lambda > 0$. h and h' are proportional to exchange fields.

- Four-fold pseudospin degeneracy lifted:



yz: transverse phase; 0x: longitudinal phase.

Mean-Field solutions of the excitonic field

- $|h| > |h'|$ for transverse (yz) phase:

$$\vec{\phi}_{\perp}(\theta,\varphi,\varphi_0) = \rho\cos\theta(\cos\varphi_0\,\vec{e}_y + \sin\varphi_0\,\vec{e}_z) + \mathrm{i}\rho\sin\theta[\cos(\varphi+\varphi_0)\,\vec{e}_y + \sin(\varphi+\varphi_0)\,\vec{e}_z]. \quad (31)$$

- $|h| < |h'|$ for longitudinal (0x) phase:

$$\vec{\phi}_{\|}(\theta,\varphi,\varphi_0) = \rho[-\sin\theta\cos(\varphi+\varphi_0)\,\vec{e}_0 + \cos\theta\sin\varphi_0\,\vec{e}_x] + \mathrm{i}\rho[\cos\theta\cos\varphi_0\,\vec{e}_0 + \sin\theta\sin(\varphi+\varphi_0)\,\vec{e}_x]. \quad (32)$$

- Exchange-field dependence (for small exchange fields):

$$\tilde{h} \equiv \sin\varphi\sin 2\theta = \mathrm{h} \equiv \begin{cases} h/h_c & \text{for } \vec{\phi}_{\perp} \\ -h'/h_c & \text{for } \vec{\phi}_{\|}. \end{cases} \quad (33)$$

φ: angle between $\vec{\Phi}'$ and $\vec{\Phi}''$. θ: proportion of $|\vec{\Phi}'|$ and $|\vec{\Phi}''|$.

- $\rho = \sqrt{\frac{1}{2|\gamma|}(\alpha - \frac{2}{g})}$: $|\vec{\Phi}|$ fixed. φ_0: arbitrary rotational phase.

Symmetries and Goldstone modes

- Two spontaneously broken global symmetries.
  $U(1)$ spin rotational symmetry:

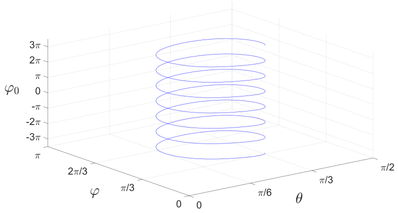
$$\vec{\phi}(\theta, \varphi, \varphi_0) \to \vec{\phi}(\theta, \varphi, \varphi_0 + \delta\varphi_0),$$
$$\boldsymbol{a} \to \mathrm{e}^{\mathrm{i}\varphi_a \boldsymbol{\sigma}_x} \boldsymbol{a}, \quad \boldsymbol{b} \to \mathrm{e}^{\mathrm{i}\varphi_b \boldsymbol{\sigma}_x} \boldsymbol{b} = \mathrm{e}^{\mp\mathrm{i}(\varphi_a + \delta\varphi_0)\boldsymbol{\sigma}_x} \boldsymbol{b} \tag{34}$$

  Upper/Lower sign (in $\pm$ or $\mp$) for yz/0x phase.
  $U(1)$ gauge (charge) symmetry:

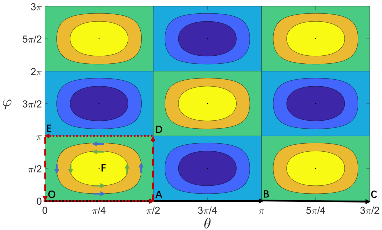
$$\vec{\phi}(\theta, \varphi, \varphi_0) \to \mathrm{e}^{\mathrm{i}\psi} \vec{\phi}(\theta, \varphi, \varphi_0) = \vec{\phi}(\theta'(\psi), \varphi'(\psi), \varphi_0'(\psi)),$$
$$\boldsymbol{a} \to \mathrm{e}^{\mathrm{i}\psi_a} \boldsymbol{a}, \quad \boldsymbol{b} \to \mathrm{e}^{\mathrm{i}\psi_b} \boldsymbol{b} = \mathrm{e}^{\mathrm{i}(\psi_a - \psi)} \boldsymbol{b} \tag{35}$$

Symmetries and Goldstone modes

The curve $(\theta(\psi), \varphi(\psi), \varphi_0(\psi))$:



(a)



(b)

(a): When ψ increases, $(\theta(\psi), \varphi(\psi), \varphi_0(\psi))$ goes down the curve.
(b): The projection of the curve on the (θ, φ) plane. $\sin 2\theta \sin\varphi = \mathsf{h}$ is satisfied when changing ψ. So $\vec{\phi}(\varphi_0, \theta, \varphi)|_{\sin 2\theta \sin\varphi = \mathsf{h}} = \vec{\phi}(\varphi_0, \psi)$.

- Two Goldstone modes: φ_0 and ψ $\Rightarrow$ spin and charge superfluidity

Quantum-dot junction model

- Action of a Josephson junction with two domains:

$$\begin{aligned}\mathcal{S}[\boldsymbol{a}_i, \boldsymbol{a}_i^\dagger, \boldsymbol{b}_i, \boldsymbol{b}_i^\dagger, \psi_i, \varphi_{0i}; V_{Cd}, V_{Sd}] &= \mathcal{S}_T[\boldsymbol{a}_i, \boldsymbol{a}_i^\dagger, \boldsymbol{b}_i, \boldsymbol{b}_i^\dagger] \\ &+ \mathcal{S}_{\mathrm{mf}}[\boldsymbol{a}_i, \boldsymbol{a}_i^\dagger, \boldsymbol{b}_i, \boldsymbol{b}_i^\dagger, \psi_i, \varphi_{0i}; V_{Cd}, V_{Sd}]. \end{aligned} \tag{36}$$

Domain index: $i = 1, 2$. Layer index: $d = a, b$.
Energy-level index: α. Spin index: bold font.
V_{Cd}: charge voltage. V_{Sd}: spin voltage.

$$\begin{aligned}\mathcal{S}_{\mathrm{mf}} = \int \mathrm{d}\tau \sum_{i=1,2} \sum_{\alpha} \{ &\boldsymbol{a}_{i\alpha}^\dagger [\hbar\partial_\tau + \boldsymbol{H}_{a\alpha} - \mu - \mathrm{i}\frac{\eta_i}{2} e(V_{Ca} + V_{Sa}\boldsymbol{\sigma}_x)]\boldsymbol{a}_{i\alpha} \\ + &\boldsymbol{b}_{i\alpha}^\dagger [\hbar\partial_\tau + \boldsymbol{H}_{b\alpha} - \mu - \mathrm{i}\frac{\eta_i}{2} e(V_{Cb} + V_{Sb}\boldsymbol{\sigma}_x)]\boldsymbol{b}_{i\alpha} \\ - &[\vec{\phi}_\lambda(\psi_i, \varphi_{0i}) \cdot \boldsymbol{a}_{i\alpha}^\dagger \vec{\boldsymbol{\sigma}} \boldsymbol{b}_{1\alpha} + \mathrm{h.c.}] \}, \end{aligned} \tag{37}$$

$$\mathcal{S}_T = \int \mathrm{d}\tau \sum_{\alpha\beta} [\boldsymbol{a}_{1\alpha}^\dagger T_{\alpha\beta}^{(a)} \boldsymbol{a}_{2\beta} + \boldsymbol{b}_{1\alpha}^\dagger T_{\alpha\beta}^{(b)} \boldsymbol{b}_{2\beta} + \mathrm{h.c.}]. \tag{38}$$

$\eta_1 = -\eta_2 = 1$, $\boldsymbol{H}_{d\alpha} \equiv E_{d\alpha}\boldsymbol{\sigma}_0 + H_d\boldsymbol{\sigma}_x$. Tunneling matrices: $T_{\alpha\beta}^{(d)}$.

Condensed Matter Field Theory (2010)

Effective theory of the Josephson junction

$$
\begin{aligned}
&\mathcal{Z}[V_{Cd}, V_{Sd}] \equiv \int \mathcal{D}\psi_i \mathcal{D}\varphi_{0i} \mathcal{D}\mathbf{\Psi}^\dagger \mathcal{D}\mathbf{\Psi} \mathrm{e}^{-\mathcal{S}[\mathbf{\Psi},\mathbf{\Psi}^\dagger,\psi_i,\varphi_{0i};V_{Cd},V_{Sd}]} \\
&= \int \mathcal{D}N_C \mathcal{D}N_S \mathcal{D}\psi_i \mathcal{D}\varphi_{0i} \mathcal{D}\mathbf{\Psi}^\dagger \mathcal{D}\mathbf{\Psi} \\
&\quad \times \delta(N_C - \sum_{i\alpha} \eta_i \boldsymbol{b}_{i\alpha}^\dagger \boldsymbol{b}_{i\alpha}/2) \delta(N_S - \sum_{i\alpha} \eta_i \boldsymbol{b}_{i\alpha}^\dagger \boldsymbol{\sigma}_x \boldsymbol{b}_{i\alpha}/2) \mathrm{e}^{-\mathcal{S}[\mathbf{\Psi},\mathbf{\Psi}^\dagger,\psi_i,\varphi_{0i};V_{Cd},V_{Sd}]} \\
&= \int \mathcal{D}\mu_C \mathcal{D}\mu_S \mathcal{D}N_C \mathcal{D}N_S \mathcal{D}\psi_i \mathcal{D}\varphi_{0i} \mathcal{D}\mathbf{\Psi}^\dagger \mathcal{D}\mathbf{\Psi} \\
&\quad \times \mathrm{e}^{\mathrm{i}\int \mathrm{d}\tau [\mu_C (N_C - \sum_{i\alpha} \eta_i \boldsymbol{b}_{i\alpha}^\dagger \boldsymbol{b}_{i\alpha}/2) + \mu_S (N_S - \sum_{i\alpha} \eta_i \boldsymbol{b}_{i\alpha}^\dagger \boldsymbol{\sigma}_x \boldsymbol{b}_{i\alpha}/2)] - \mathcal{S}[\mathbf{\Psi},\mathbf{\Psi}^\dagger,\psi_i,\varphi_{0i};V_{Cd},V_{Sd}]} \\
&= \int \mathcal{D}\mu_C \mathcal{D}\mu_S \mathcal{D}N_C \mathcal{D}N_S \mathcal{D}\psi_i \mathcal{D}\varphi_{0i} \mathcal{D}\mathbf{\Psi}^\dagger \mathcal{D}\mathbf{\Psi} \mathrm{e}^{\mathrm{i}\int \mathrm{d}\tau (\mu_C N_C + \mu_S N_S) - \mathcal{S}[\mathbf{\Psi},\mathbf{\Psi}^\dagger,\psi_i,\varphi_{0i};V_C - i\mu_C, V_S - i\mu_S]} \\
&= \int \mathcal{D}\mu_C \mathcal{D}\mu_S \mathcal{D}N_C \mathcal{D}N_S \mathcal{D}\psi_i \mathcal{D}\varphi_{0i} \mathrm{e}^{\mathrm{i}\int \mathrm{d}\tau (\mu_C N_C + \mu_S N_S) + \mathrm{Tr}\ln \boldsymbol{\mathcal{G}}_\mu^{-1}[\psi_i,\varphi_{0i};V_C - i\mu_C, V_S - i\mu_S]} \\
&= \int \mathcal{D}N_C \mathcal{D}N_S \mathcal{D}\tilde{\psi} \mathcal{D}\tilde{\varphi}_0 \mathrm{e}^{-\mathcal{S}_{\mathrm{eff}}[\tilde{\psi},\tilde{\varphi}_0,N_C,N_S;V_C,V_S]}, \qquad (39)
\end{aligned}
$$

where a saddle point is taken at the last step.

Effective theory of the Josephson junction

- Effective action:

$$\begin{aligned}
&\mathcal{S}_{\text{eff}}[\tilde{\psi},\tilde{\varphi}_0,N_C,N_S;V_C,V_S]\\
&=\int \mathrm{d}\tau\Big[\mathrm{i}N_C(-\hbar\dot{\tilde{\psi}}(\tau)-eV_C)+\mathrm{i}N_S(\mp\hbar\dot{\tilde{\varphi}}_0(\tau)-eV_S)\\
&-\hbar I_0\Big(\cos\big(\tilde{\psi}(\tau)-\frac{e}{\hbar c}\Psi\big)\cos\big(\tilde{\varphi}_0(\tau)\big)\\
&+\bar{h}_\pm\sin\big(\tilde{\psi}(\tau)-\frac{e}{\hbar c}\Psi\big)\sin\big(\tilde{\varphi}_0(\tau)\big)\Big)\Big].
\end{aligned} \tag{40}$$

Magnetic Flux: Ψ.
Phase differences: $\tilde{\psi}\equiv\psi_1-\psi_2$, $\quad\tilde{\varphi}_0\equiv\varphi_{01}-\varphi_{02}$.
Voltages: $V_C\equiv V_{Cb}-V_{Ca}$, $\quad V_S\equiv V_{Sb}\pm V_{Sa}$.
Currents: $I_C\equiv I_{Cb}=-I_{Ca}=e\partial_t N_C$, $\quad I_S\equiv I_{Sb}=\pm I_{Sa}=e\partial_t N_S$.

Josephson effects

- First Josephson equations:

$$I_C = -eI_0[\sin(\tilde{\psi} - \frac{e}{\hbar c}\Psi)\cos\tilde{\varphi}_0 - \bar{h}_\pm \cos(\tilde{\psi} - \frac{e}{\hbar c}\Psi)\sin\tilde{\varphi}_0], \quad (41)$$

$$\pm I_S = -eI_0[\sin\tilde{\varphi}_0 \cos(\tilde{\psi} - \frac{e}{\hbar c}\Psi) - \bar{h}_\pm \cos\tilde{\varphi}_0 \sin(\tilde{\psi} - \frac{e}{\hbar c}\Psi)]. \quad (42)$$

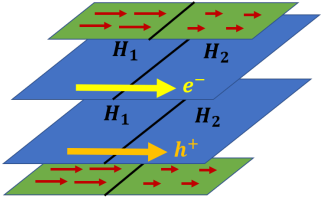
- Second Josephson equations:

$$\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t} = -\frac{e}{\hbar}V_C, \quad \frac{\mathrm{d}\tilde{\varphi}_0}{\mathrm{d}t} = \mp\frac{e}{\hbar}V_S. \quad (43)$$

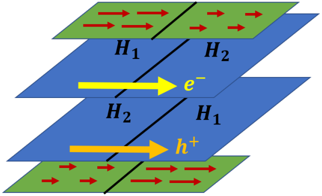
- Spin voltage $V_S = \frac{1}{2}(V_\uparrow - V_\downarrow) \Rightarrow$ charge current $I_C = I_\uparrow + I_\downarrow$

Josephson effects

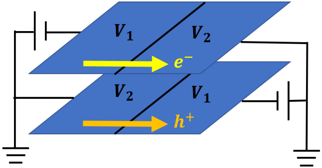
- Four ways to induce charge current:



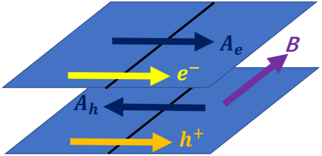
(a) By spin voltage for *yz* phase



(b) By spin voltage for 0*x* phase



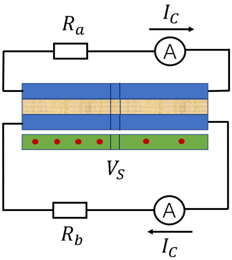
(c) By charge voltage



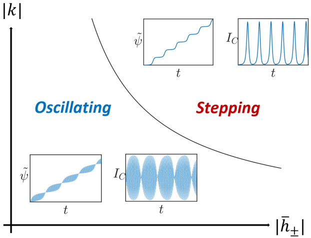
(d) By time-dependent magnetic flux

Device setup

- Two circuits: $V_C = I_{Ca} R_a - I_{Cb} R_b$, $\quad k \equiv \frac{e I_0}{V_S}(R_a + R_b)$.



Circuit setup



Time dependence of $\tilde{\psi}$ and I_C

Blue: electron layer and hole layer; Yellow: insulating layer.
Green: magnetic substrates.
The density of red dots: strength of magnetic polarizations.

- The phase difference $\tilde{\psi}$ has an oscillating or stepping behavior.
- The spin voltage V_S can be measured from the period of the charge current I_C.

Spin-Orbit coupling

- Rashba SOC in the electron layer:

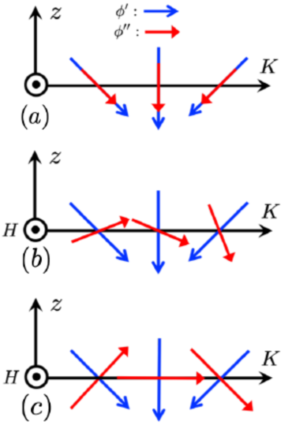
$$\hat{H}_R = \xi_e \int \mathrm{d}^2\vec{r}\boldsymbol{a}^\dagger(\vec{r})(-\mathrm{i}\partial_y\boldsymbol{\sigma}_x + \mathrm{i}\partial_x\boldsymbol{\sigma}_y)\boldsymbol{a}(\vec{r}). \tag{44}$$

- Effective Lagrangian:

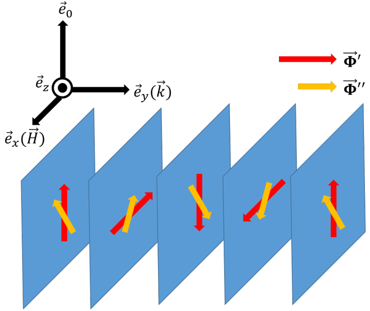
$$\begin{aligned}\delta\mathcal{L} = &- D(\Phi'_z\partial_x\Phi'_x - \Phi'_x\partial_x\Phi'_z + \Phi'_z\partial_y\Phi'_y - \Phi'_y\partial_y\Phi'_z) \\ &- D(\Phi'_0\partial_x\Phi'_y - \Phi'_y\partial_x\Phi'_0 + \Phi'_x\partial_y\Phi'_0 - \Phi'_0\partial_y\Phi'_x) \\ &- D(\Phi''_z\partial_x\Phi''_x - \Phi''_x\partial_x\Phi''_z + \Phi''_z\partial_y\Phi''_y - \Phi''_y\partial_y\Phi''_z) \\ &- D(\Phi''_0\partial_x\Phi''_y - \Phi''_y\partial_x\Phi''_0 + \Phi''_x\partial_y\Phi''_0 - \Phi''_0\partial_y\Phi''_x).\end{aligned} \tag{45}$$

- The yz/0x phase (transverse/longitudinal phase) is substituted by a helicoidal/helical phase carrying nonzero momentum.
- The $U(1)$ phase φ_0 in the mean-field solutions is substituted by $\varphi_0 - Ky \equiv \varphi_0 - \frac{D}{2\lambda}y$.
- Spin rotational symmetry in the hole layer is spontaneously broken.
  $\Rightarrow$ Spin and charge superfluidity

Phases with translational symmetry breaking



Helicoidal phase



Helical phase

Phys. Rev. B **100**, 035130 (2019)

Conserved currents

- A countinuous symmetry ($\phi_\nu \to \phi_\nu + \epsilon\Delta\phi_\nu$) leads to a Noether's current:
$$J_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\nu)}\Delta\phi_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\nu^*)}\Delta\phi_\nu^*. \tag{46}$$
- Charge current (for $\phi_\nu \to \phi_\nu - \mathrm{i}\epsilon\phi_\nu$):
$$J_i^C = -\frac{\lambda h_c}{|\gamma|}[(\partial_i\psi - \frac{e}{\hbar c}\tilde{A}_i) - \mathrm{h}\partial_i\varphi_0]. \tag{47}$$
- Spin current (for $\phi_y \to \phi_y \pm \epsilon\phi_z, \quad \phi_z \to \phi_z \mp \epsilon\phi_y,$ $-\mathrm{i}\phi_0 \to -\mathrm{i}\phi_0 \pm \epsilon\phi_x, \quad \phi_x \to \phi_x \mp \epsilon(-\mathrm{i}\phi_0)$):
$$J_i^S = \mp\frac{\lambda h_c}{|\gamma|}[\partial_i\varphi_0 - \mathrm{h}(\partial_i\psi - \frac{e}{\hbar c}\tilde{A}_i)]. \tag{48}$$

Gauge fields: $\tilde{A}_i \equiv A_{b,i} - A_{a,i}$. ($\mu = 0, x, y$ and $i = x, y$.)

An Introduction to Quantum Field Theory (1995)

Outline

1. **Introduction to excitons**
   - BEC-BCS crossover
2. **Excitonic spin superfluidity with spin-charge conversion**

   Yeyang Zhang and Ryuichi Shindou, Phys. Rev. Lett. **128**, 066601 (2022)
   - Model and phases
   - Goldstone modes and Josephson effects
   - Spin-orbit coupling
3. **Antiparticles of excitons in semimetals**

   Lingxian Kong, Ryuichi Shindou, and **Yeyang Zhang**, Phys. Rev. B **106**, 235145 (2022)
   - Polology and Bethe-Salpeter equation
   - Effective field theory

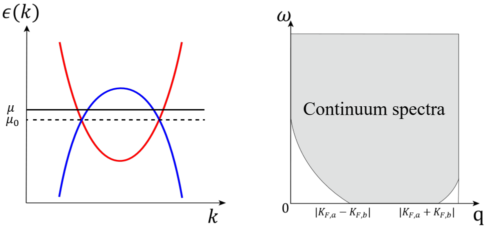
Model

- Normal-state semimetals ($E_g > 0$) with $U(1) \times U(1)$ symmetry at $T = 0$:

$$K_0 = \sum_{\boldsymbol{k}} \left[\left(\frac{k^2}{2m_a} - \frac{E_g}{2} - \mu \right) a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + \left(-\frac{k^2}{2m_b} + \frac{E_g}{2} - \mu \right) b_{\boldsymbol{k}}^\dagger b_{\boldsymbol{k}} \right] + \frac{1}{2\Omega} \sum_{\boldsymbol{q}} v(\boldsymbol{q}) \rho(\boldsymbol{q}) \rho(-\boldsymbol{q}), \tag{49}$$

$$\rho(\boldsymbol{q}) = \sum_{\boldsymbol{k}} \left(a_{\boldsymbol{k}+\boldsymbol{q}}^\dagger a_{\boldsymbol{k}} + b_{\boldsymbol{k}+\boldsymbol{q}}^\dagger b_{\boldsymbol{k}} \right), \quad v(\boldsymbol{q}) = \begin{cases} 4\pi/q^2 & \text{for 3D} \\ 2\pi/q & \text{for 2D.} \end{cases} \tag{50}$$

- $\mu \neq \mu_0$: excitons near $\boldsymbol{q} = 0$ are not damped by the continuum spectra.



Polology

- Excitonic Green's function:

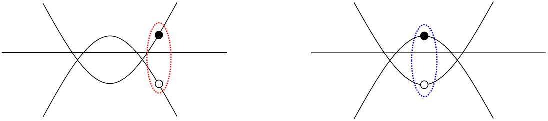
$$G^{ex}(\boldsymbol{x}-\boldsymbol{x}', t-t')_{\boldsymbol{y}\boldsymbol{y}'} = -(-i)^2 \langle \mathcal{T}\{a_{\boldsymbol{x}}(t) b^{\dagger}_{\boldsymbol{x}+\boldsymbol{y}}(t) b_{\boldsymbol{x}'+\boldsymbol{y}'}(t') a^{\dagger}_{\boldsymbol{x}'}(t')\}\rangle. \tag{51}$$

- Lehmann representation:

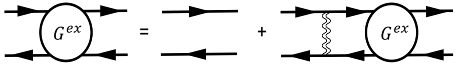
$$\begin{aligned} G^{ex}(\boldsymbol{q},\omega)_{\boldsymbol{k}\boldsymbol{k}'} = & \sum_n \frac{i\Omega\langle 0|b^{\dagger}_{\boldsymbol{k}} a_{\boldsymbol{q}+\boldsymbol{k}}|n\rangle\langle n|a^{\dagger}_{\boldsymbol{q}+\boldsymbol{k}'} b_{\boldsymbol{k}'}|0\rangle}{\omega-(E_n-E_0)+i0^+} \\ & -\sum_{n'} \frac{i\Omega\langle 0|a^{\dagger}_{\boldsymbol{q}+\boldsymbol{k}'} b_{\boldsymbol{k}'}|n'\rangle\langle n'|b^{\dagger}_{\boldsymbol{k}} a_{\boldsymbol{q}+\boldsymbol{k}}|0\rangle}{\omega+(E_{n'}-E_0)-i0^+}. \end{aligned} \tag{52}$$

$|0\rangle$: ground state with particle numbers (N_a, N_b).
$|n\rangle$: excitons with (N_a+1, N_b-1), energy $E_n - E_0$. Positive poles.
$|n'\rangle$: antiexcitons with (N_a-1, N_b+1), energy $E_{n'} - E_0$. Negative poles.

- For semiconductors, $N_a = 0$. No negative poles.



Bethe-Salpeter equation



- Bethe-Salpeter equation with ladder approximation:

$$\widetilde{G}^{ex}(\boldsymbol{q},\omega)_{\boldsymbol{k}\boldsymbol{k}'} = \widetilde{G}_0^{ex}(\boldsymbol{q},\omega)_{\boldsymbol{k}\boldsymbol{k}'} - \frac{1}{\Omega}\sum_{\boldsymbol{k}_1\boldsymbol{k}_2} \widetilde{G}_0^{ex}(\boldsymbol{q},\omega)_{\boldsymbol{k}\boldsymbol{k}_1} w(\boldsymbol{k}_1 - \boldsymbol{k}_2) \widetilde{G}^{ex}(\boldsymbol{q},\omega)_{\boldsymbol{k}_2\boldsymbol{k}'}, \tag{53}$$

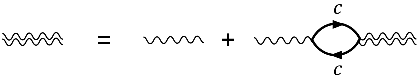
$i\Omega\widetilde{G}^{ex}(\boldsymbol{q},\omega)_{\boldsymbol{k}\boldsymbol{k}'} \equiv G^{ex}(\boldsymbol{q},\omega)_{\boldsymbol{k}\boldsymbol{k}'}$, $i\Omega\widetilde{G}_0^{ex}(\boldsymbol{q},\omega)_{\boldsymbol{k}\boldsymbol{k}'} \equiv G_0^{ex}(\boldsymbol{q},\omega)_{\boldsymbol{k}\boldsymbol{k}'}$.

- Eigenvalues and eigenvectors:

$$\widetilde{G}^{ex}(\boldsymbol{q},\omega)^{-1}|\phi_j(\boldsymbol{q},\omega)\rangle = \xi_j(\boldsymbol{q},\omega)|\phi_j(\boldsymbol{q},\omega)\rangle. \tag{54}$$

Zeros of $\xi_j(\boldsymbol{q},\omega)$ are the poles of Green's function.

Bethe-Salpeter equation



- Effective interaction with RPA:

$$w(\boldsymbol{q}) = \frac{v(\boldsymbol{q})}{1 - v(\boldsymbol{q})\Pi_0(0,0)} = \frac{v(\boldsymbol{q})}{1 - v(\boldsymbol{q})\sum_{c=a,b}\Pi_0^c(0,0)}, \tag{55}$$

  where $\Pi_0^c(0,0)$ are static limits of electron polarization functions.
- Excitonic free Green's function:

$$\begin{aligned} G_0^{ex}(\boldsymbol{q},\omega)_{\boldsymbol{kk'}} &= \Omega\delta_{\boldsymbol{kk'}} \int \frac{d\omega_1}{2\pi} G_0^a(\boldsymbol{k}+\boldsymbol{q},\omega_1+\omega) G_0^b(\boldsymbol{k},\omega_1) \\ &= i\Omega\delta_{\boldsymbol{kk'}} \left\{ \frac{\theta(|\boldsymbol{k}+\boldsymbol{q}| - K_{F,a})\theta(|\boldsymbol{k}| - K_{F,b})}{\omega - [\epsilon_a(\boldsymbol{k}+\boldsymbol{q}) - \epsilon_b(\boldsymbol{k})] + i0^+} \right. \\ &\quad \left. - \frac{\theta(K_{F,a} - |\boldsymbol{k}+\boldsymbol{q}|)\theta(K_{F,b} - |\boldsymbol{k}|)}{\omega - [\epsilon_a(\boldsymbol{k}+\boldsymbol{q}) - \epsilon_b(\boldsymbol{k})] - i0^+} \right\}. \end{aligned} \tag{56}$$

Patial wave expansion

- The Green's function is given by eigenvalues and eigenvectors,

$$\widetilde{G}^{ex}(\boldsymbol{q},\omega)=\sum_{j}|\phi_j(\boldsymbol{q},\omega)\rangle\xi_j(\boldsymbol{q},\omega)^{-1}\langle\phi_j(\boldsymbol{q},\omega)|. \tag{57}$$

- For $\boldsymbol{q}=0$, there is a rotational symmetry ($\mathcal{R}$),

$$G^{ex}(0,\omega)_{\boldsymbol{kk'}}=G^{ex}(0,\omega)_{\mathcal{R}(\boldsymbol{k})\mathcal{R}(\boldsymbol{k'})}\,. \tag{58}$$

The Green's function is expanded by spherical harmonics in 3D,

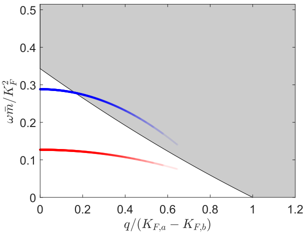
$$-iG^{ex}(0,\omega)_{\boldsymbol{kk'}}=\sum_{nlm}\frac{Y_{lm}(\theta,\varphi)f_{nl}(\omega;k)f_{nl}(\omega;k')Y^{*}_{lm}(\theta',\varphi')}{\xi_{nl}(\omega)}, \tag{59}$$

and by trigonometric functions in 2D,

$$-iG^{ex}(0,\omega)_{\boldsymbol{kk'}}=\sum_{nm}\frac{f_{nm}(\omega;k)f_{nm}(\omega;k')e^{im(\varphi-\varphi')}}{\xi_{nm}(\omega)}. \tag{60}$$

$U(1) \times U(1)$ symmetric cases

- Semiconductors: The Bethe-Salpeter equation becomes the two-body Schroedinger equation. All poles are positive.
- Semimetals: $G_0^{ex}(\boldsymbol{q}, \omega)^{-1}_{\boldsymbol{kk}'}$ (kinetic energy) contains non-analytic theta functions by many-body effects. Negative poles are physical.
- Realizations of the $U(1) \times U(1)$ symmetry: electron-hole double layers without inter-layer hopping; two bands carrying different quantum numbers (e.g. spins); incommensurate indirect semimetals.



Lowest bands of s-wave excitons (red) and antiexcitons (blue)

$U(1) \times U(1)$ asymmetric case

- Only one $U(1)$: inter-band hopping is allowed or condensation happens.

$$\hat{\mathcal{G}}^{ex}(\boldsymbol{x}-\boldsymbol{x}', t-t')_{\boldsymbol{y}\boldsymbol{y}'} \equiv -(-i)^2 \times \begin{pmatrix} \langle 0|\mathcal{T}\{\gamma(\boldsymbol{x},\boldsymbol{y};t)\gamma^\dagger(\boldsymbol{x}',\boldsymbol{y}';t')\}|0\rangle & \langle 0|\mathcal{T}\{\gamma(\boldsymbol{x},\boldsymbol{y},t)\gamma(\boldsymbol{x}',\boldsymbol{y}';t')\}|0\rangle \\ \langle 0|\mathcal{T}\{\gamma^\dagger(\boldsymbol{x},\boldsymbol{y},t)\gamma^\dagger(\boldsymbol{x}',\boldsymbol{y}',t')\}|0\rangle & \langle 0|\mathcal{T}\{\gamma^\dagger(\boldsymbol{x},\boldsymbol{y},t)\gamma(\boldsymbol{x}',\boldsymbol{y}',t')\}|0\rangle \end{pmatrix}, \quad (61)$$

with $\gamma(\boldsymbol{x},\boldsymbol{y};t) \equiv \beta^\dagger_{\boldsymbol{x}+\boldsymbol{y}}(t)\alpha_{\boldsymbol{x}}(t)$, $\alpha_{\boldsymbol{k}}$ and $\beta_{\boldsymbol{k}}$ are superpositions of $a_{\boldsymbol{k}}$ and $b_{\boldsymbol{k}}$ that diagonalize the free part of Hamiltonian.

- Bosonic BdG-type Bethe-Salpeter equation:

$$\hat{\mathcal{G}}^{ex}(\boldsymbol{q},t-t')_{\boldsymbol{k}\boldsymbol{k}'} = \hat{\mathcal{G}}^{ex}_0(\boldsymbol{q},t-t')_{\boldsymbol{k}\boldsymbol{k}'} + \frac{i}{\Omega^2}\sum_{\bar{\boldsymbol{k}},\bar{\boldsymbol{k}}'}\int d\bar{t}\,\hat{\mathcal{G}}^{ex}_0(\boldsymbol{q},t-\bar{t})_{\boldsymbol{k}\bar{\boldsymbol{k}}} \times \begin{pmatrix} \boldsymbol{A}_{\bar{\boldsymbol{k}},\bar{\boldsymbol{k}}'}(\boldsymbol{q}) & \boldsymbol{B}_{\bar{\boldsymbol{k}},\bar{\boldsymbol{k}}'}(\boldsymbol{q}) \\ \boldsymbol{B}^*_{-\bar{\boldsymbol{k}},-\bar{\boldsymbol{k}}'}(-\boldsymbol{q}) & \boldsymbol{A}^*_{-\bar{\boldsymbol{k}},-\bar{\boldsymbol{k}}'}(-\boldsymbol{q}) \end{pmatrix} \hat{\mathcal{G}}^{ex}(\boldsymbol{q},\bar{t}-t')_{\bar{\boldsymbol{k}}'\boldsymbol{k}'}. \quad (62)$$

- There are pairs of positive poles $(E_n - E_0)$ and negative poles $(-E_n + E_0)$. A pair only represents one physical state (with energy $E_n - E_0$). No distinction (no definition) between excitons and antiexcitons. Negative poles are redundancy.

Frequency dependence of the eigenvalues

- Zeros of $\xi_j(0,\omega)$ are poles of the Green's function.

$$\widetilde{G}^{ex}(0,\omega)^{-1}|\phi_j(0,\omega)\rangle = \xi_j(0,\omega)|\phi_j(0,\omega)\rangle. \tag{63}$$

- For semiconductors, $\xi_j(0,\omega)$ is linear in ω. For semimetals,

$$\xi_j(0,\omega) = -\beta + \alpha\omega + \gamma\omega^2 + \ldots = \gamma(\omega-\omega_+)(\omega+\omega_-) + \ldots \tag{64}$$

The nonlinearity comes from the free Green's function,

$$[\widetilde{G}^{ex}(0,\omega)^{-1}]_{\boldsymbol{kk'}} = \delta_{\boldsymbol{kk'}}\Big\{\theta(|\boldsymbol{k}|-K_{\mathrm{out}})\Big(\omega-(\epsilon_a(\boldsymbol{k})-\epsilon_b(\boldsymbol{k}))\Big) - \theta(K_{\mathrm{in}}-|\boldsymbol{k}|)\Big(\omega-(\epsilon_a(\boldsymbol{k})-\epsilon_b(\boldsymbol{k}))\Big)\Big\} + \frac{w(\boldsymbol{k}-\boldsymbol{k'})}{\Omega}, \tag{65}$$

$$\left[\partial_\omega \widetilde{G}^{ex}(0,\omega)^{-1}\right]_{\boldsymbol{kk'}} = \delta_{\boldsymbol{kk'}}\Big\{\theta(|\boldsymbol{k}|-K_{\mathrm{out}}) - \theta(K_{\mathrm{in}}-|\boldsymbol{k}|)\Big\}, \tag{66}$$

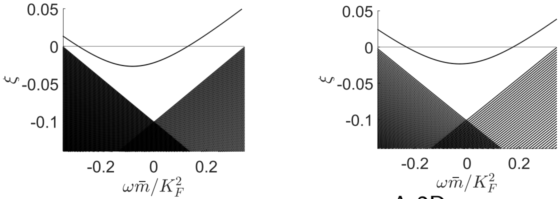
with $K_{\mathrm{out}} \equiv \max(K_{F,a},K_{F,b})$, $K_{\mathrm{in}} \equiv \min(K_{F,a},K_{F,b})$.

Frequency dependence of the eigenvalues

- Hellman-Feynman theorem:

$$\frac{d\xi_j(0,\omega)}{d\omega} = \langle\phi_j(0,\omega)|\left[\frac{d\widetilde{G}^{ex}(0,\omega)^{-1}}{d\omega}\right]|\phi_j(0,\omega)\rangle = \sum_{|\boldsymbol{k}|>K_{\mathrm{out}}} |\langle\boldsymbol{k}|\phi_j\rangle|^2 - \sum_{|\boldsymbol{k}|<K_{\mathrm{in}}} |\langle\boldsymbol{k}|\phi_j\rangle|^2. \quad (67)$$

- $\partial_\omega\xi_j|_{\omega=\omega_+} > 0$ for excitons, $\partial_\omega\xi_j|_{\omega=-\omega_-} < 0$ for antiexcitons. $\Rightarrow \gamma > 0$, $\beta > 0$, otherwise there is condensation.



A 3D case   A 2D case

Effective field theory

- Effective field theory:

$$\int dt\,\mathcal{L} = \int dt\,\varphi^{\dagger}(t)\big(-\gamma\partial_t^2 + i\alpha\partial_t - \beta\big)\varphi(t), \tag{68}$$

where the effective field is defined by

$$\varphi(t) \equiv \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} dt' \langle \boldsymbol{k}|\phi_j(0, t-t')\rangle b_{\boldsymbol{k}}^{\dagger}(t') a_{\boldsymbol{k}}(t'). \tag{69}$$

- Effective Hamiltonian:

$$\begin{aligned}\mathcal{H} &= \pi_1\partial_t\varphi_1 + \pi_2\partial_t\varphi_2 - \mathcal{L} \\ &= \frac{1}{2\lambda}\left(\pi_1^2 + \pi_2^2\right) + \frac{1}{2}\lambda\eta^2\left(\varphi_1^2 + \varphi_2^2\right) + \frac{\alpha}{2\gamma}\left(\pi_2\varphi_1 - \pi_1\varphi_2\right),\end{aligned} \tag{70}$$

with $\lambda = 2\gamma$, $\eta = \sqrt{\frac{\alpha^2}{4\gamma^2} + \frac{\beta}{\gamma}}$. The theory is two coupled harmonic oscillators.

Effective field theory

- Two bosonic modes:

$$\mathcal{H} = \nu_+ a_+^\dagger a_+ + \nu_- a_-^\dagger a_-, \tag{71}$$

with

$$a_{1,2} \equiv \sqrt{\frac{\lambda\eta}{2}} \left(\varphi_{1,2} + \frac{i}{\lambda\eta} \pi_{1,2} \right), \quad a_\pm \equiv \frac{1}{\sqrt{2}} \left(a_1 \pm i a_2 \right), \tag{72}$$

$$\nu_\pm = \sqrt{\frac{\alpha^2}{4\gamma^2} + \frac{\beta}{\gamma}} \mp \frac{\alpha}{2\gamma} = \omega_\pm > 0. \tag{73}$$

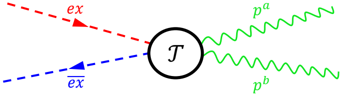
$a_\pm$ is the exciton/antiexciton annihilation operator.

- Conserved charge of the effective Lagrangian:

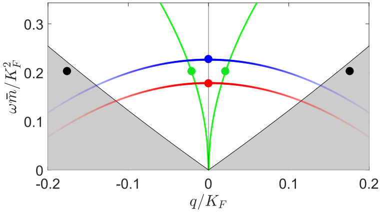
$$j^0(t) = i\gamma \left[\varphi^\dagger \left(\partial_t \varphi \right) - \left(\partial_t \varphi^\dagger \right) \varphi \right] + \alpha \varphi^\dagger \varphi = a_+^\dagger a_+ - a_-^\dagger a_-. \tag{74}$$

Excitons and antiexcitons carry conserved charges $+1$ and -1.

Charge-conserved processes



(a)



(b)

(a): A pair-annihilation process. An exciton-antiexciton pair decays to intra-band excitations of the two bands.

(b): Energy-momentum conservation of the process. A pair of a 2D s-wave exciton (red) and antiexciton (blue) can decay to either two plasmons (green) or two individual excitations (black).

Summary

Excitons offer a good platform to explore the physics of both quasiparticles and condensates:

- Excitons undergo a BEC-BCS crossover from semiconductors to semimetals.
- Excitonic superfluids with spin and charge Goldstone modes enable spin-charge conversion in electron-hole double layers.
- Excitons in semimetals are classified into particles and antiparticles carrying opposite conserved charges.

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Thanks for Your Attention!