Entanglement and non-unitary dynamics of the Sachdev-Ye-Kitaev models

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HKU-UCAS young physicist forum

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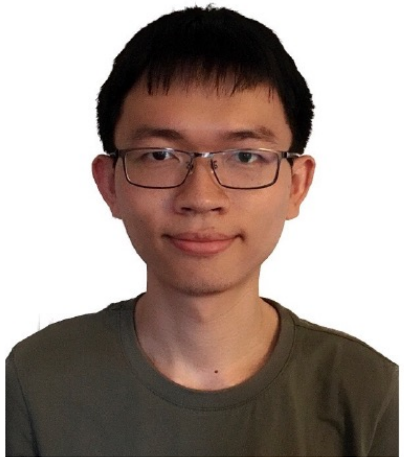
GORDON AND BETTY MOORE FOUNDATION

Collaborators

- Quantum entanglement of the SYK models, PRB 97 (24), 245126 (2018)
- Subsystem Renyi entropy of thermal ensembles for SYK-like models, SciPost Phys 8, 094 (2020)
- Non-unitary dynamics of SYK chain, SciPost Phys 10 (2), 048 (2021)
- Measurement-induced phase transition in the monitored SYK model, PRL 127 140601 (2021)



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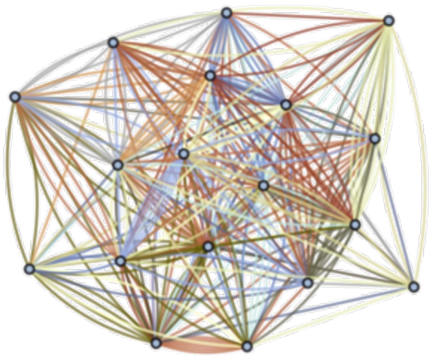
Theme: towards building interesting phases and transitions in strongly correlated systems…

Outline

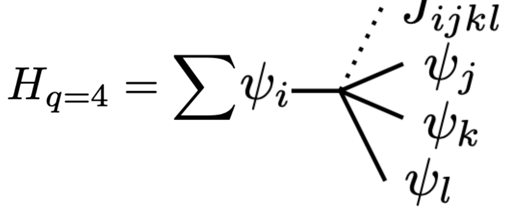
- Entanglement of the 0D SYK model
- Entanglement dynamics of 1D SYK chain (*interesting* phases; *boring* transition)
- Entanglement dynamics of another 1D SYK chain (*interesting* phases *and* transitions; with connection to *measurement induced phase transitions*)

Sachdev-Ye-Kitaev: the model



$$H_{q=2} = \sum_{i,j=1}^{N} iJ_{ij}\psi_i\psi_j$$

$$H_{q=4} = \sum_{1\leq i<j<k<l\leq N} J_{ijkl}\psi_i\psi_j\psi_k\psi_l$$



- Strongly coupled (all-to-all), yet exact solvable at large N
- $q = 2$:
  - Free fermion model
  - (Single particle) random matrix theory
- $q \geq 4$:
  - Non-Fermi liquid without quasiparticles
  - Non-zero extensive entropy at zero temperature
  - Maximally chaotic

q-body interaction
ψ : majorana fermions
J_{ijkl}: gaussian random variable

$$\overline{J^2_{i_1 i_2 \ldots i_q}} = \frac{(q-1)!J^2}{N^{q-1}}$$

Talks by Sachdev; Maldacena, Stanford, 2016; Rosenhaus, 2019

We will be focus on entanglement properties of SYK. Why?

Entanglement characterizes quantum correlation – natural to look at in SYK.

All-to-all coupling. ⇒ Maximal entanglement?

"Almost!"

Maximally chaotic. ⇒ Maximal entanglement?

Entanglement = information. How can we use SYK for information purposes?

A chain of SYK.

Properly designed coupling and measurement, can store/release information.

Dynamics: phase diagram and transition.

Entanglement entropy (EE)

Renyi EE: $$S_A^{(n)} = \frac{1}{1-n} \ln \mathrm{Tr}_A \rho_A^n$$

von Neumann EE: $$S_A = -\mathrm{Tr}_A(\rho_A \ln \rho_A)$$

$$\rho_A = \mathrm{Tr}_B \rho$$

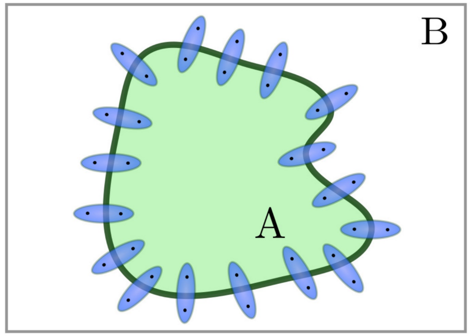
$$\rho = \frac{1}{Z} e^{-\beta H}$$

$$Z = \mathrm{Tr} e^{-\beta H}$$

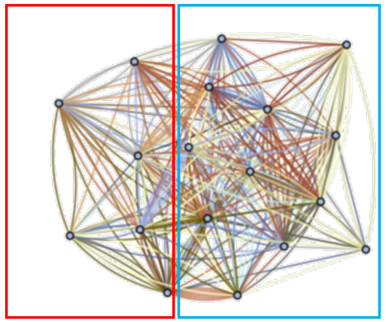
<u>Basic characterization:</u>

EE vs λ

$$\lambda = |A|/N$$



Area law



A B

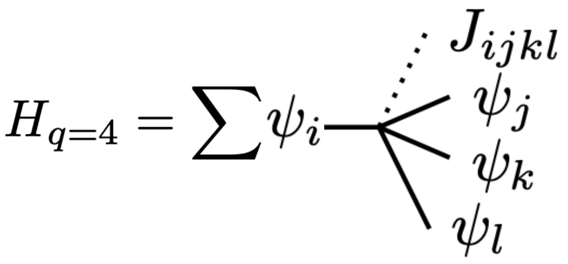
Volume law

Large-N calculation for SYK (q = 4)

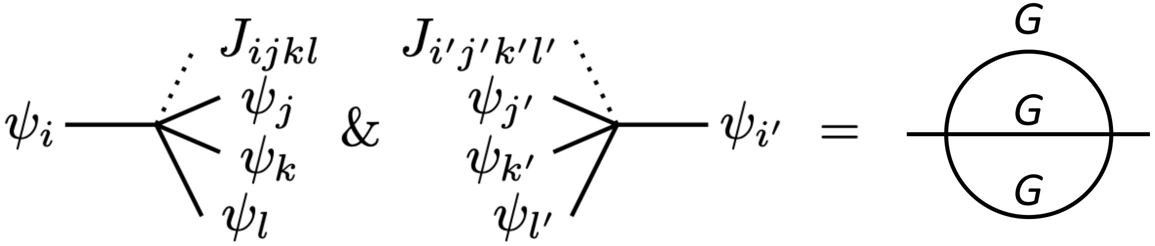
Partition function

$$Z(\{J_{i_1\ldots i_q}\}) = \mathrm{Tr}e^{-\beta H_q}$$

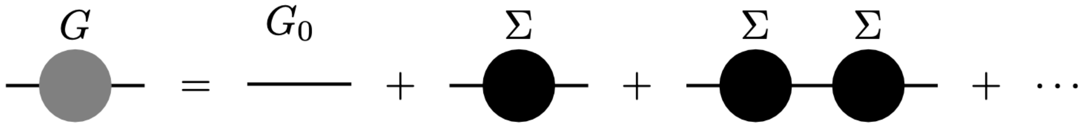
$$= \int [D\psi_i] e^{-\int_0^\beta d\tau \left(\frac{1}{2}\sum_i \psi_i \partial_\tau \psi_i + H_q^{\{J_{i_1\ldots i_q}\}}[\{\psi_i\}]\right)}$$

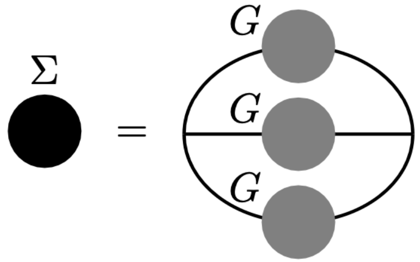


Average over disorder J_{ijkl} to get $\overline{Z}$:



$$\boxed{\begin{aligned} &\overline{Z} = \int DGD\Sigma e^{-NS_q[G,\Sigma]} \\ &G = \frac{1}{\partial_\tau \delta(\tau-\tau') - \Sigma} \\ &\Sigma(\tau,\tau') = J^2 G^{q-1}(\tau,\tau') \end{aligned}}$$





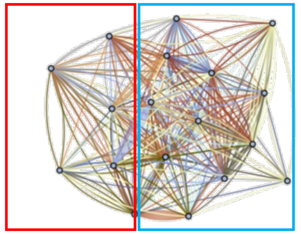
Large *N* saddle point action

$$\boxed{S_q[\Sigma, G] = \frac{1}{2}\ln \mathrm{Det}(\partial_\tau \delta(\tau - \tau') - \Sigma) + \frac{1}{2}\int d\tau d\tau' \left(-\Sigma(\tau,\tau')G(\tau,\tau') + \frac{J^2}{q}G^q(\tau,\tau')\right)}$$

$$G(\tau, \tau') = \frac{1}{N}\sum_i \langle \mathcal{T}\psi_i(\tau)\psi_i(\tau')\rangle$$

"a quantum mechanical problem"

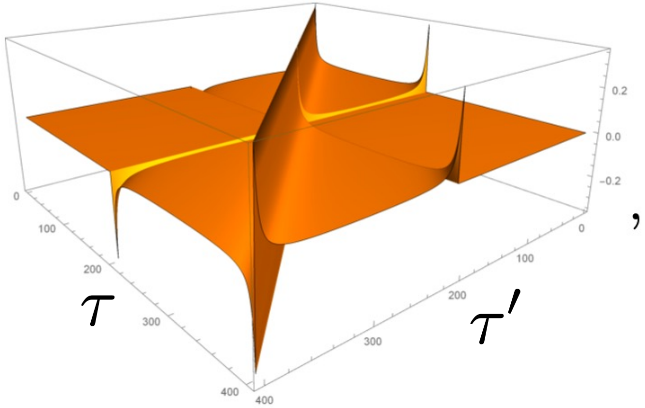
Calculating EE in SYK

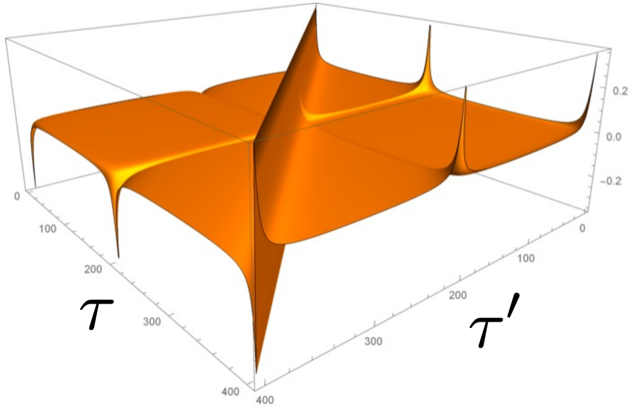


$$\overline{S_A^{(n)}} = \frac{1}{1-n}(\overline{\ln Z_n} - n\overline{\ln Z}) \approx \frac{1}{1-n}(\ln \overline{Z_n} - n \ln \overline{Z})$$

$$\overline{Z_n} = \int DG_A DG_B D\Sigma_A D\Sigma_B e^{-NS_n[\Sigma_A,\Sigma_B,G_A,G_B]}$$

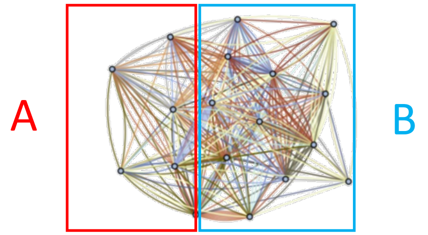
$$\boxed{\begin{aligned} G_A = (\partial_\tau^A - \Sigma_A)^{-1}, \quad G_B = (\partial_\tau^B - \Sigma_B)^{-1}, \\ \Sigma_A = \Sigma_B = J^2(\lambda G_A + (1-\lambda)G_B)^{q-1} \end{aligned}}$$

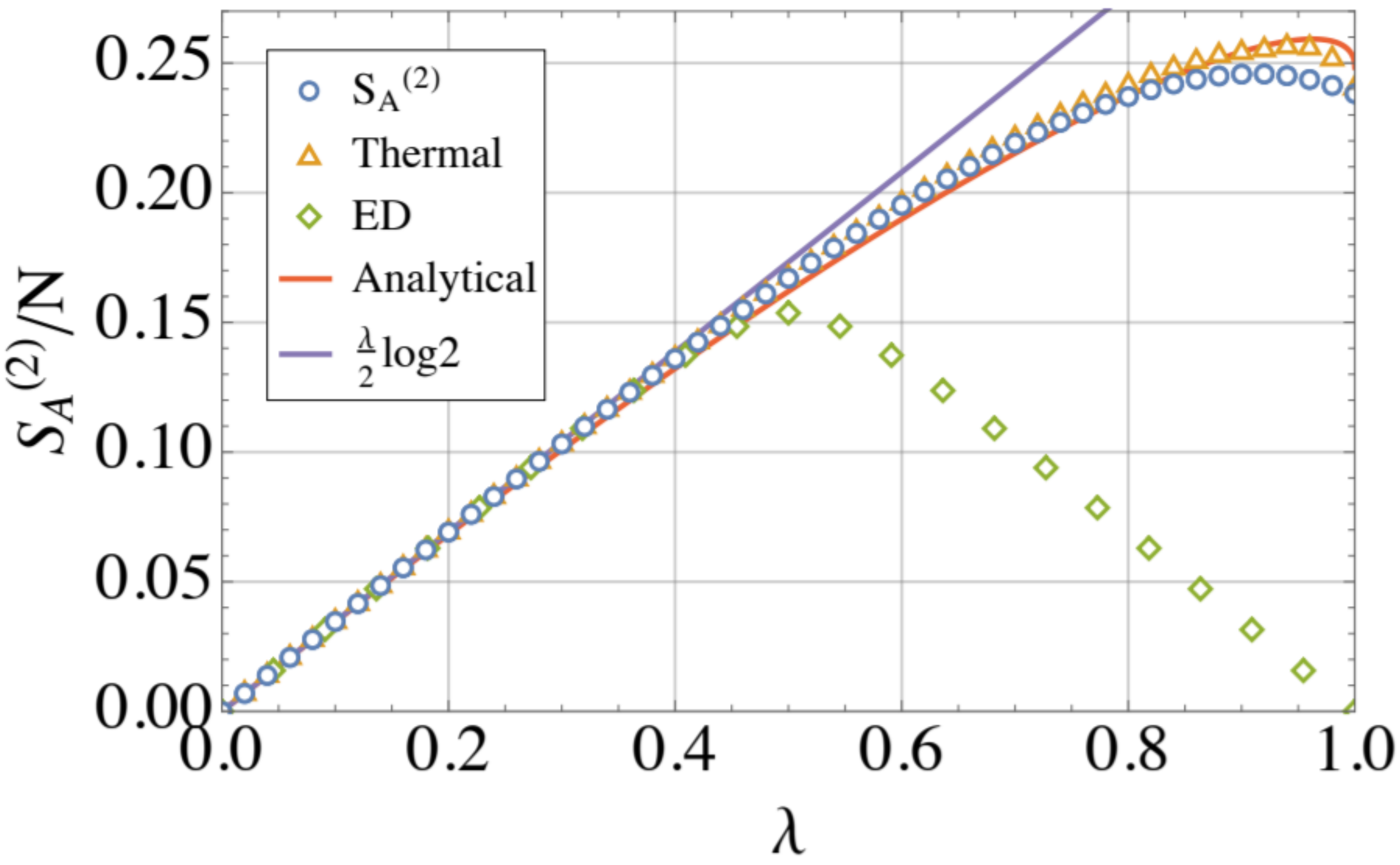
$G_A =$ 

$, \ G_B =$ 

Zhang, CL, Chen, Scipost (2020)

EE of SYK4: various results





ED (*N*=44, *T*=0, von Neumann)
CL, Xiao, Balents, PRB (2018)

Analytical (von Neumann)
Huang, Gu, PRD (2019)

$$S \sim \left[\frac{1}{2}\ln 2 - \frac{1}{16}\arcsin^2\left(\lambda^{\frac{3}{2}}\frac{E}{E_0}\right)\right]M$$

Thermal 2nd Renyi, Saddle point

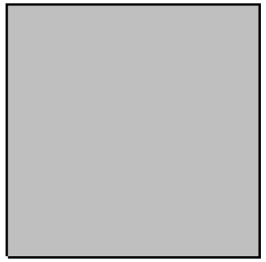
$S_A^{(2)}$ 2nd Renyi, Saddle point
with βJ = 50

Zhang, CL, Chen, Scipost (2020)

EE of free fermion

$$\mathcal{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\dagger$$

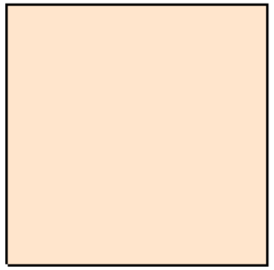
$$\mathbf{U}^T =$$



$$H_{\mathrm{SYK2}} = \sum_{1\leq i<j\leq N} iJ_{ij}\chi_i\chi_j = c^\dagger \mathcal{H} c$$

$$\mathbf{C}_{ij} = \langle c_i^\dagger c_j \rangle$$

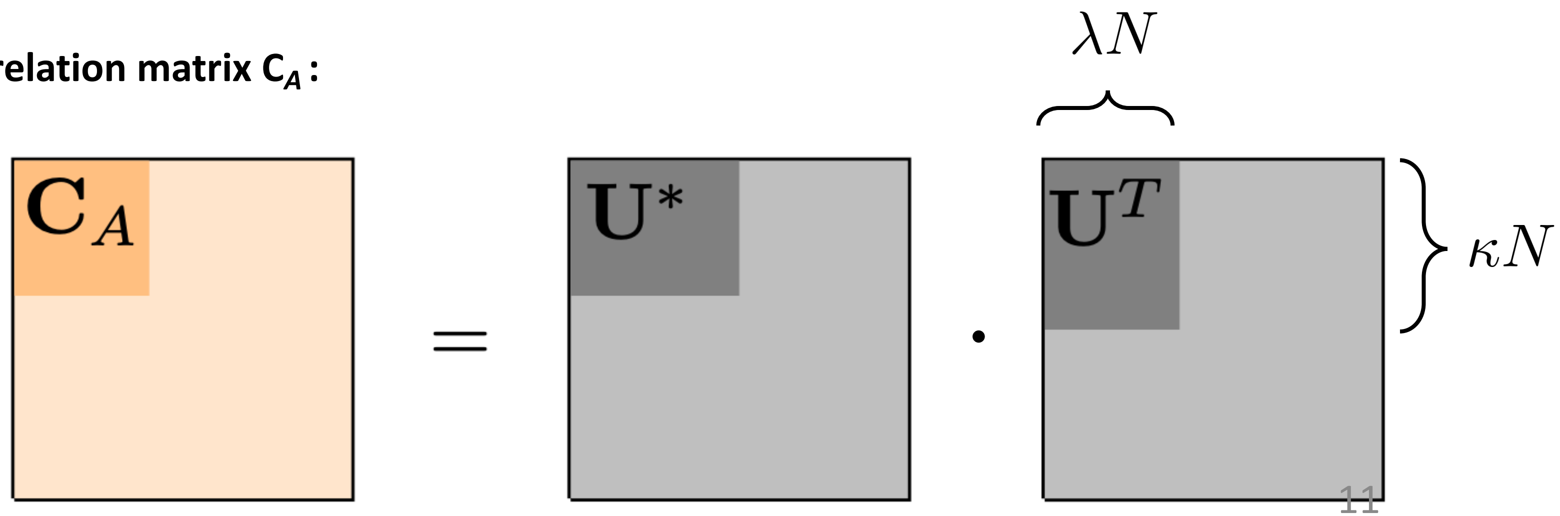
$$\mathbf{C} =$$



Free fermion EE can be calculated from two-point correlation function:

$$S_A = -\sum_{i=1}^{m} \varepsilon_i \ln \varepsilon_i + (1-\varepsilon_i)\ln(1-\varepsilon_i)$$ (sum of Binomial entropy)

Where ε_i are eigenvalue of the truncated correlation matrix C_A:



Vidal, Latorre, Rico, Kitaev, PRL 2003

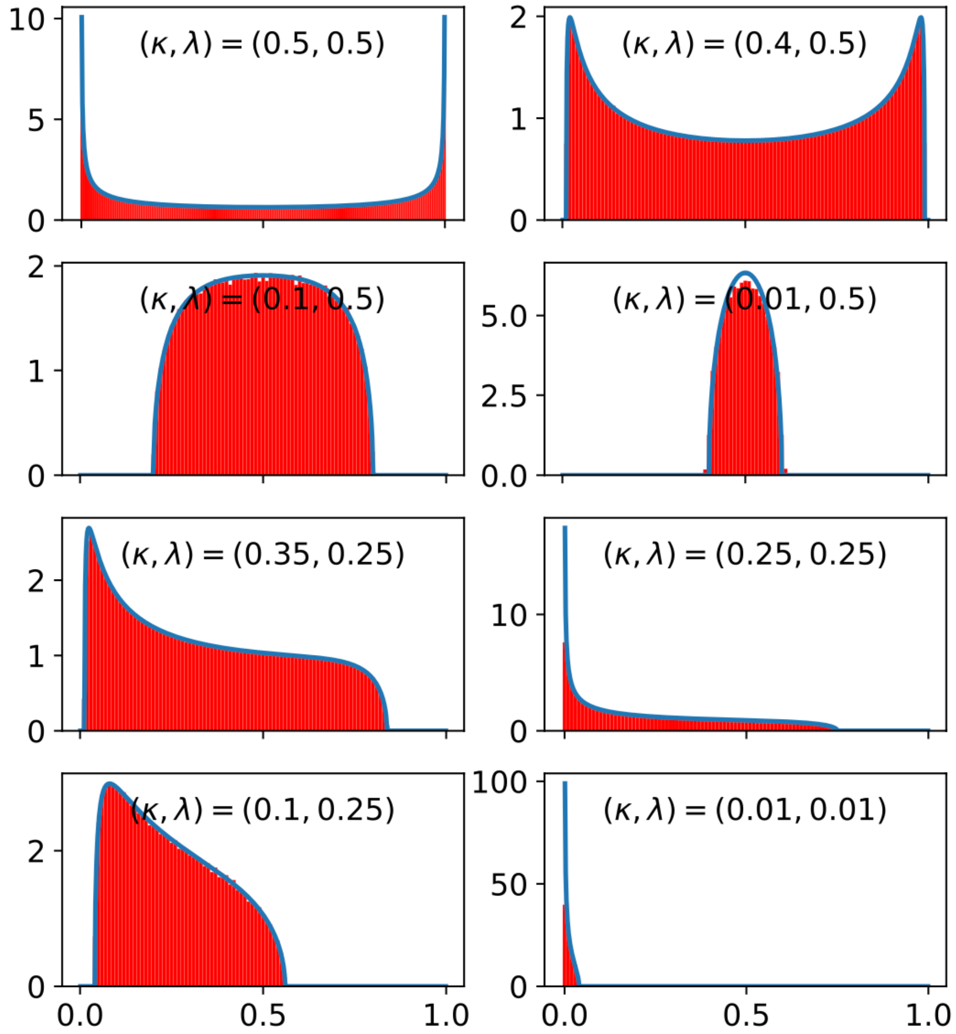
EE of SYK2 and RMT

- $\mathcal{H}$ belongs to gaussian unitary ensemble.
- $\mathbf{U}$ belongs to the "Haar ensemble".
- **$\mathbf{C}_A$ belongs to the $\boldsymbol{\beta}$-Jacobi ensemble!**

$$\boxed{f_{\kappa,\lambda}(x) = \frac{1}{2\pi\lambda}\frac{\sqrt{(\lambda_+ - x)(x-\lambda_-)}}{x(1-x)}1_{[\lambda_-,\lambda_+]}}$$

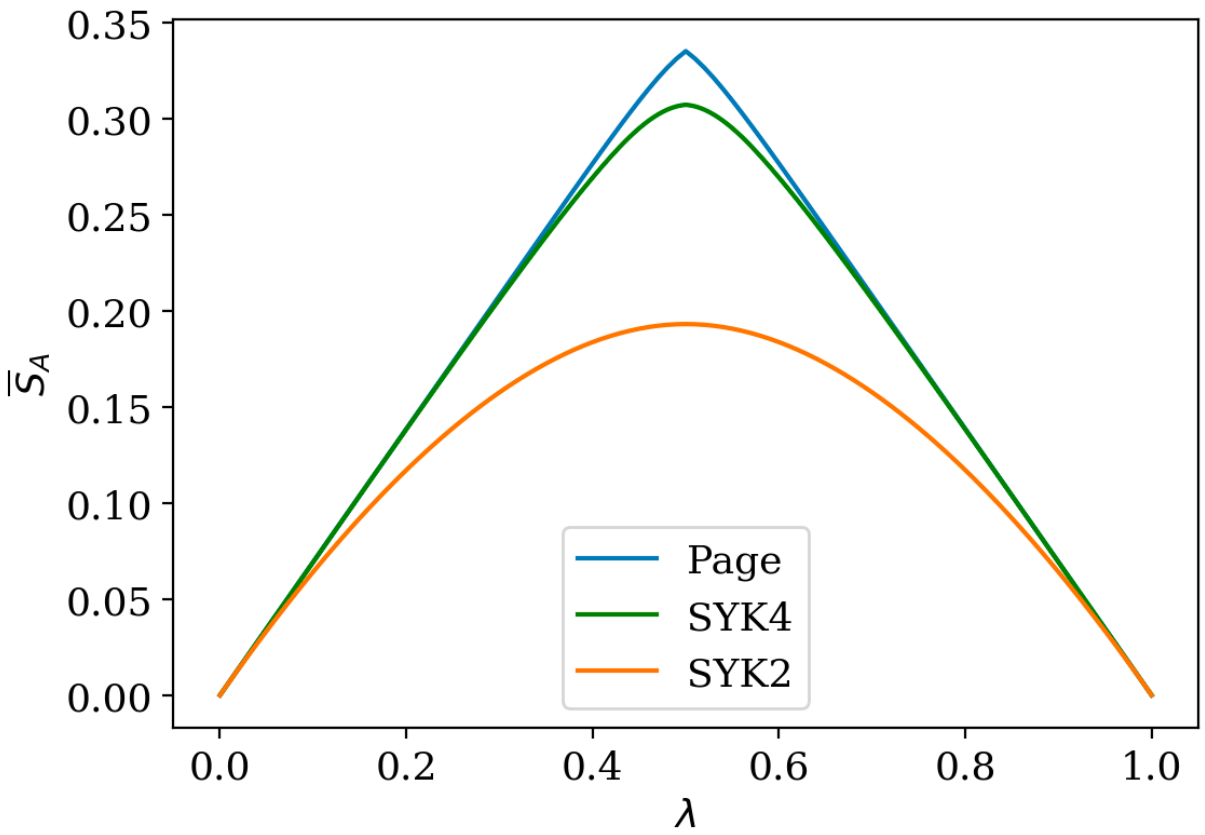
$$S_A/N = \int \underbrace{(-x\ln x - (1-x)\ln(1-x))}_{h(x):\ \text{binomial entropy}} f(x)dx$$

Eigenvalue density $f(x)$ of $\mathbf{C}_A$:



Wachter, 1980

EE of SYK: final results



Half-system EE:

Maximal entropy of a many-body eigenstate!

$$S_{\text{Page}} = \frac{N}{2}\ln 2 - \frac{1}{2} \approx 0.347N$$

$$S_{\text{SYK}_4} \approx \frac{N}{2}\left(\ln 2 - \frac{1}{8}\arcsin^2\frac{1}{2\sqrt{2}}\right) \approx 0.338N$$

$$S_{\text{SYK}_2} = \frac{N}{2}(2\ln 2 - 1) \approx 0.193N$$

Maximal entropy of a free-fermion eigenstate!

Gu, Huang, PRD (2019)
CL, Chen, Balents, PRB (2018)

Food for thought…

- We have at hand a strongly interacting model that is large-N solvable.
- This allows us to compute entanglement entropy.
- Nice result along the way: RMT and free fermion entanglement.
- Volume law reflects high entanglement (but in a 0+1D system…)

Q: Can we study entanglement in a more **"realistic"** SYK-type system?

"Realistic" = Spatial structure + phases and phase transitions

SYK dynamics

- Evolution of a quantum state under SYK like Hamiltonian *H*

$$|\psi(t)\rangle = e^{-iHt}|\psi(t=0)\rangle$$

where H may consist of a Hermitian and *anti-Hermitian* part.

- Study the entanglement structure of the state $|\psi(t)\rangle$.

- Phases (volume law or area law) and phase transitions (1st or 2nd order)

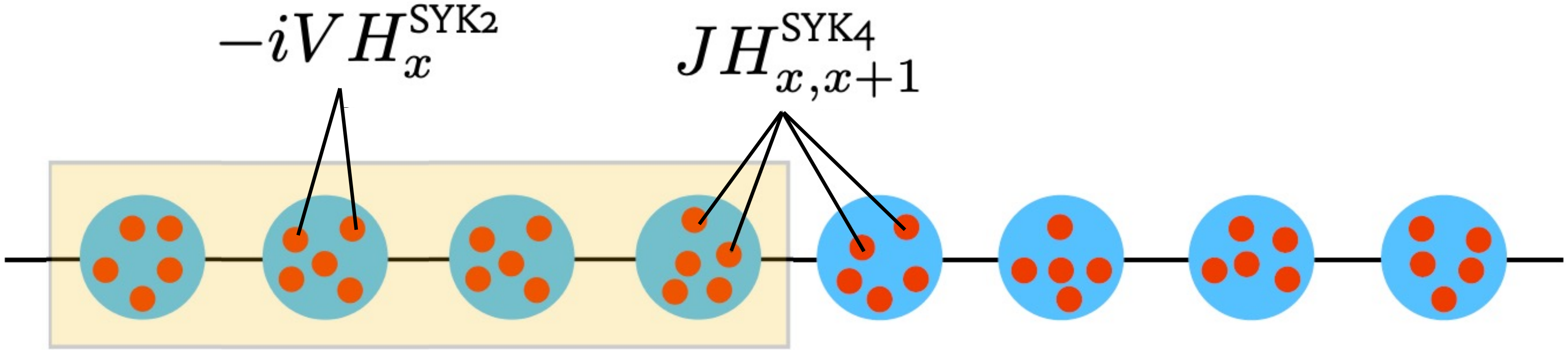
SYK (4,2)-chain: Hamiltonian dynamics

Time evolution $$|\psi(t)\rangle = \frac{1}{\sqrt{Z(t)}} e^{-iHt} |\{0\}\rangle$$

$$Z(t) = \langle\{0\}|e^{iH^\dagger t} e^{-iHt}|\{0\}\rangle$$

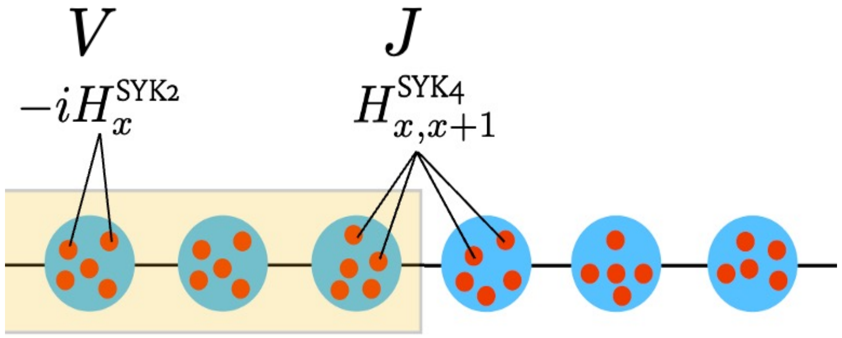
under $$H = J\sum_x H^{\mathrm{SYK4}}_{x,x+1} - iV\sum_x H^{\mathrm{SYK2}}_x$$

Anti-hermitian

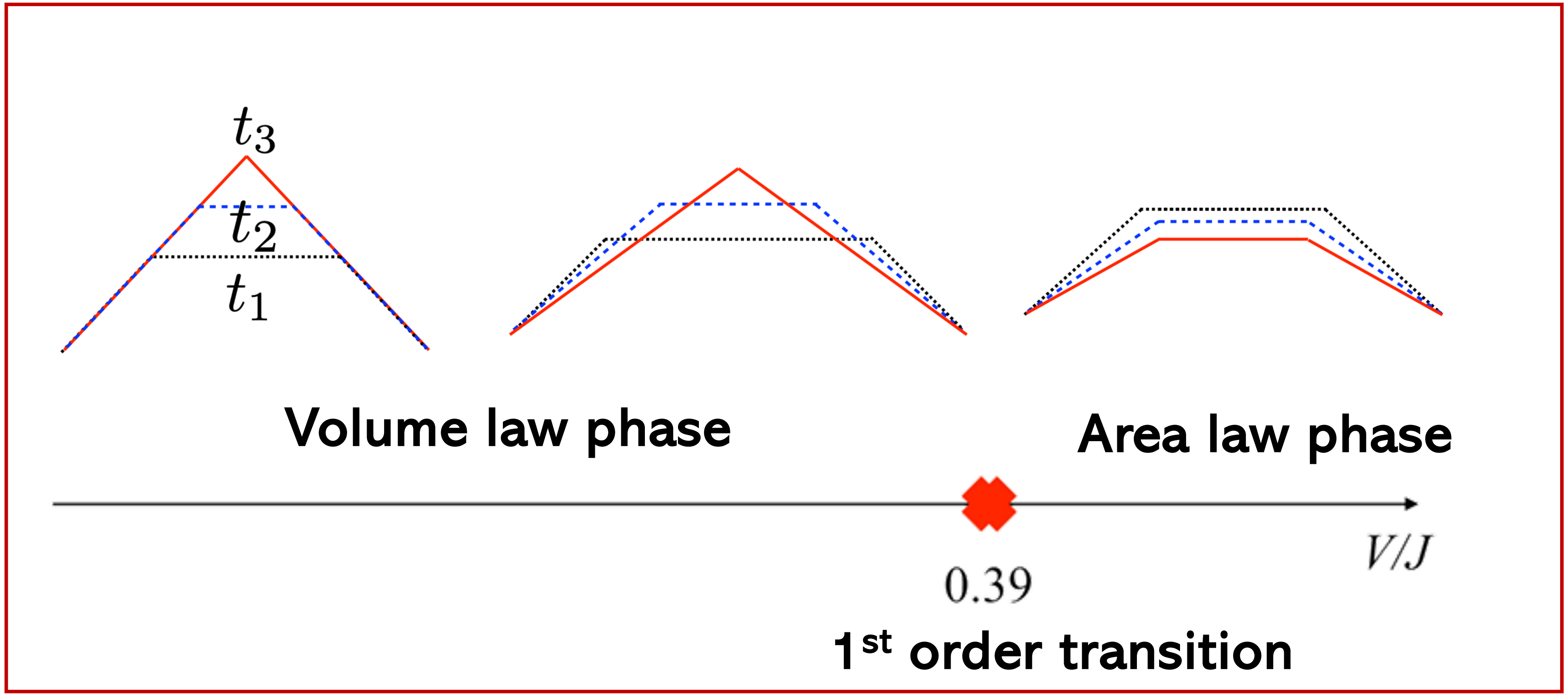


CL, Zhang, Chen, Scipost (2021)

Steady state phase diagram

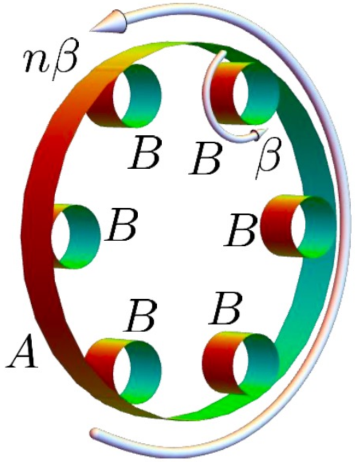


Phase diagram determined by the 2nd Renyi entropy at late times:



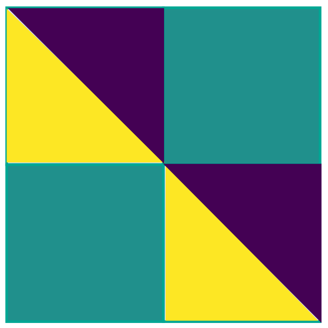
CL, Zhang, Chen, Scipost (2021)

Phases: volume law vs area law

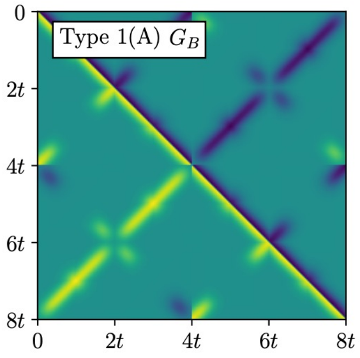


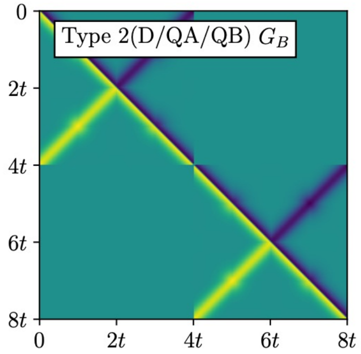
Iteration of Green's function

$$G(\tau,\tau') = \frac{1}{N}\sum_i \langle \mathcal{T}\psi_i(\tau)\psi_i(\tau')\rangle$$



Replica non-diagonal
(or replica symmetry breaking)
or **replica wormwhole**





Replica diagonal
(replica symmetric)

Food for thought…

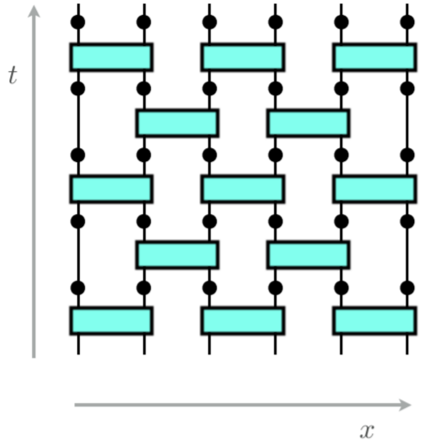
- Using SYK, we have constructed a 1D strongly correlated model that is still large N solvable.

- Entanglement now has good meaning (spatial quantum correlation), and can be used to distinguish phases (volume law and area law).

- But 1st order transition (an artifact of large N) is not natural.

Q: An SYK-type system with 2nd order phase transition?

Digression: measurement induced phase transition

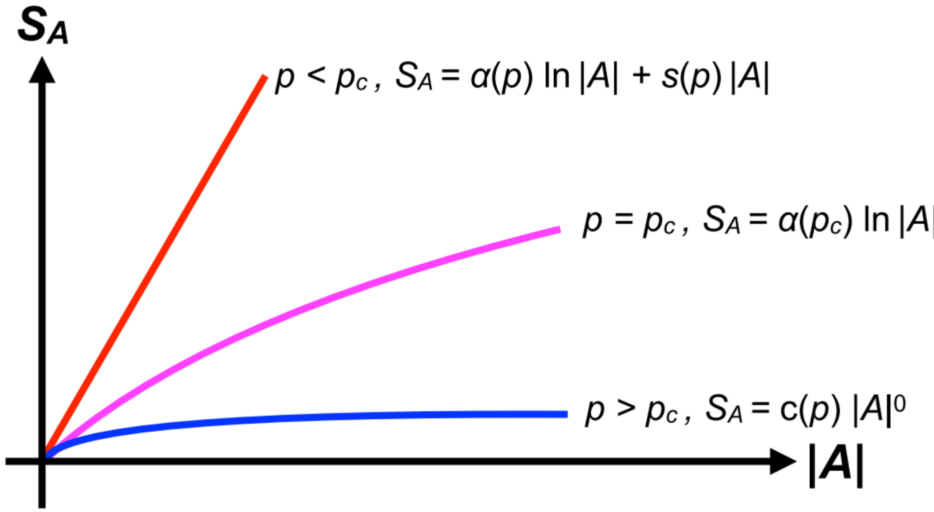


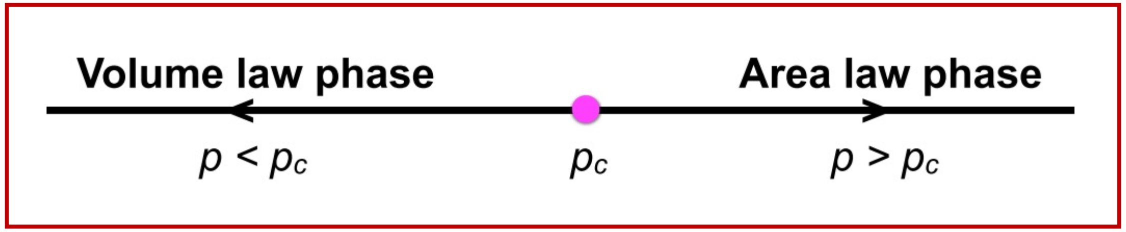
- Random unitary gate

 $: |\psi\rangle \to U|\psi\rangle$

- Random measurement

 $: |\psi\rangle \to \dfrac{P_i|\psi\rangle}{||P_i|\psi\rangle||}$





Li, Chen, Fisher, PRB (2018), Skinner, Ruhman, Nahum, PRX (2019), Chan, Nandkishore et al., PRB (2019), Choi, Bao et al., PRL (2020), Gullans, Huse, PRX (2020), Jian, You et al., PRB (2020), Bao, Choi, Altman, Ann. of Phys (2021) and many others…

SYK chain dynamics

$$\boxed{\rho(t+\delta t;\boldsymbol{\mu}) = K_{\boldsymbol{\mu}}\, U\, \rho(t)\, U^\dagger\, K_{\boldsymbol{\mu}}^\dagger}$$

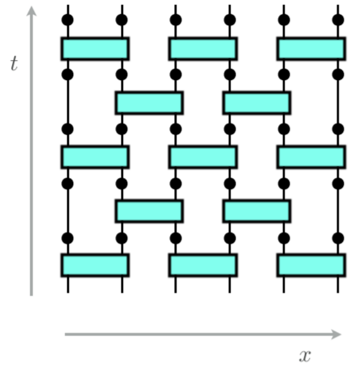
- Unitary part:

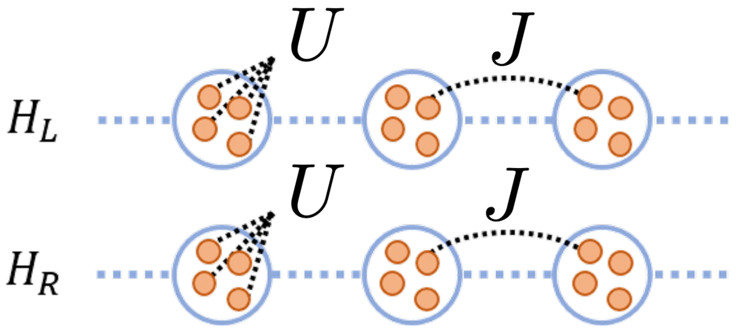
$$U = e^{-i\delta t(H_L+H_R)}$$

- Measurement (continuous monitoring)

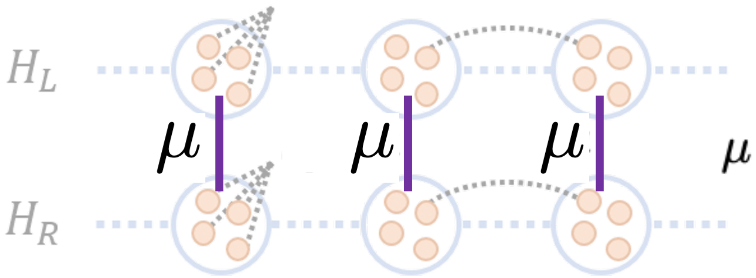
$$K_\mu^{x,i} \in \{(1-p)I, pM_1^{x,i}, pM_2^{x,i}\}$$

$$\{M_1^{x,i}, M_2^{x,i}\} = \left\{\pi_{x,i}^- + \sqrt{1-s^2}\pi_{x,i}^+, s\pi_{x,i}^+\right\}$$





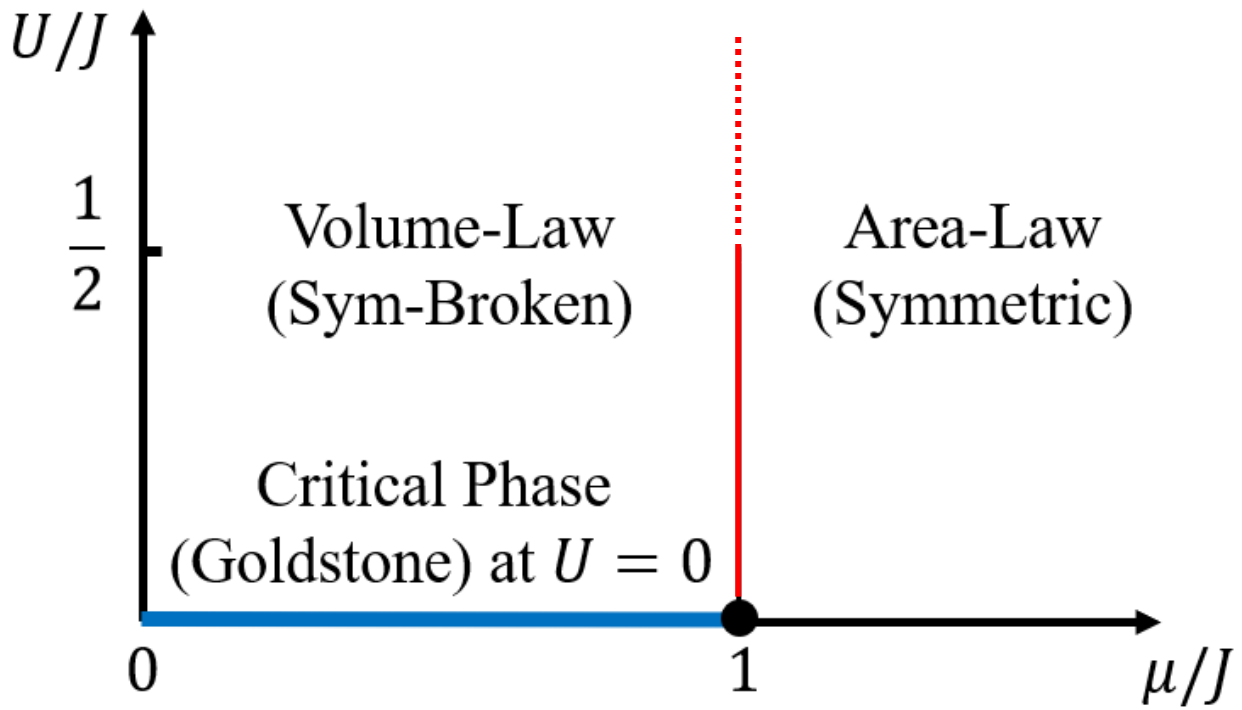
Onsite SYK4 + hopping SYK2

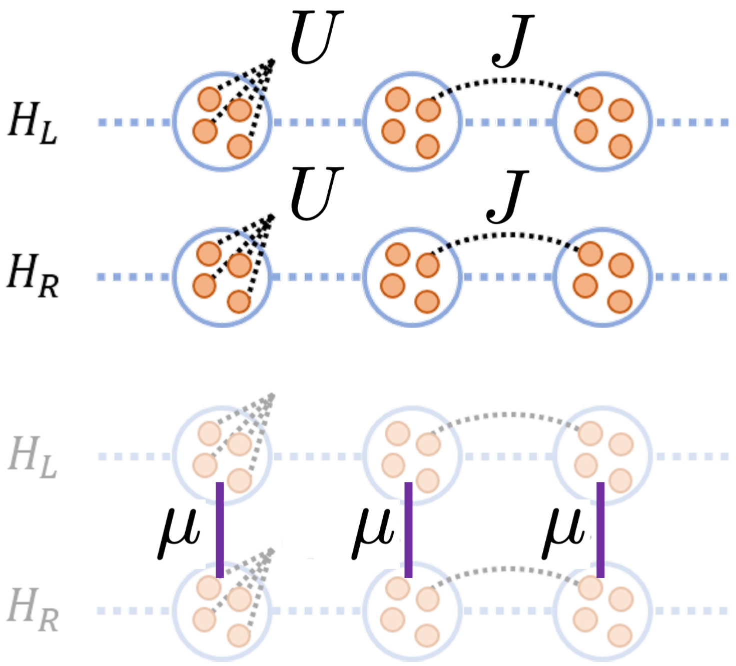


$\mu = sp\sqrt{\delta t}$

$\pi_{x,i}^\pm \equiv$ Parity of the Majorana twins in L/R chains

Large-N phase diagram





- Noninteracting case (U = 0): breaking of symmetry: O(2) x O(2)
- Interacting case (U > 0): breaking of symmetry: Z_4
- Red line: 2^{nd} order (solid) vs 1^{st} order phase transition (dotted)

Jian, CL, Chen, Swingle, Zhang, PRL (2021)

Enlarged symmetry in replica space (2nd Renyi)

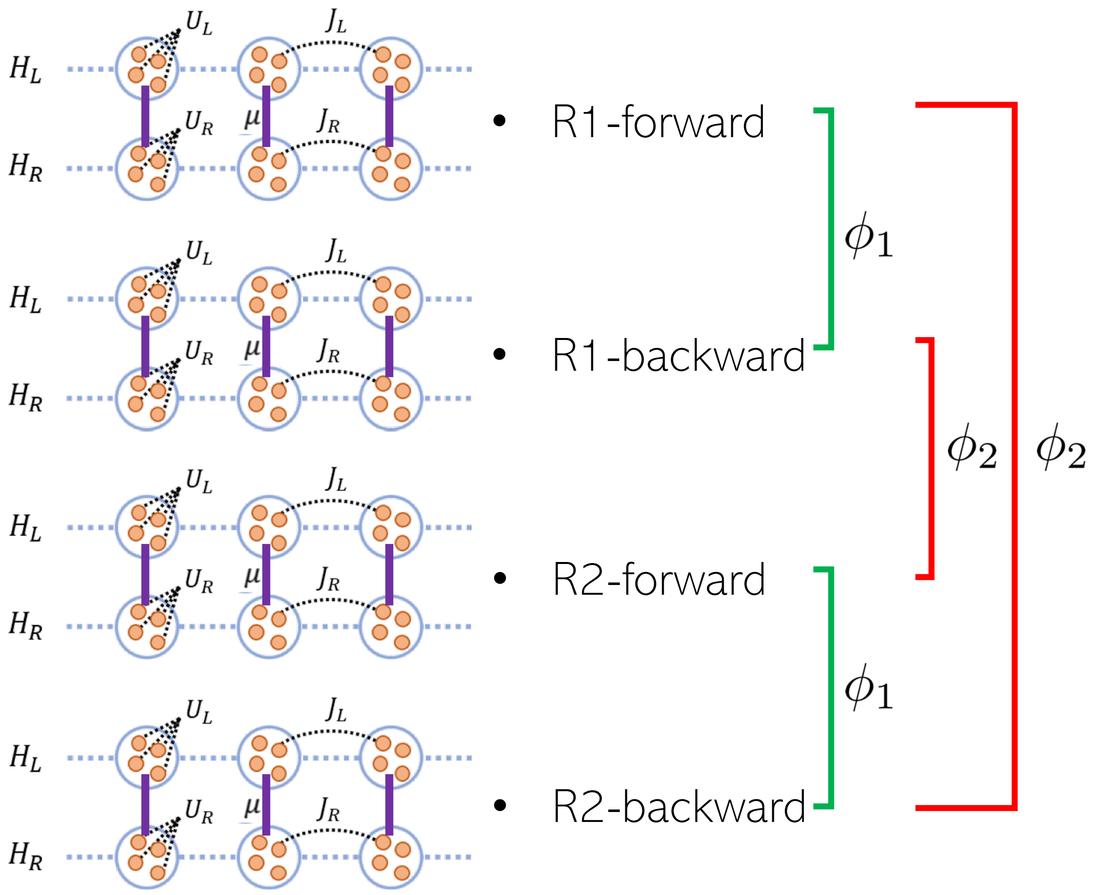
$$\rho(t+\delta t)\otimes\rho(t+\delta t) = (K_\mu U\rho(t)U^\dagger K_\mu^\dagger)\otimes(K_\mu U\rho(t)U^\dagger K_\mu^\dagger)$$

Full symmetry = symmetry of one copy x permutation symmetry for the four copies

In the non-interacting case ($U=0$): full symmetry is

O(2) x O(2),

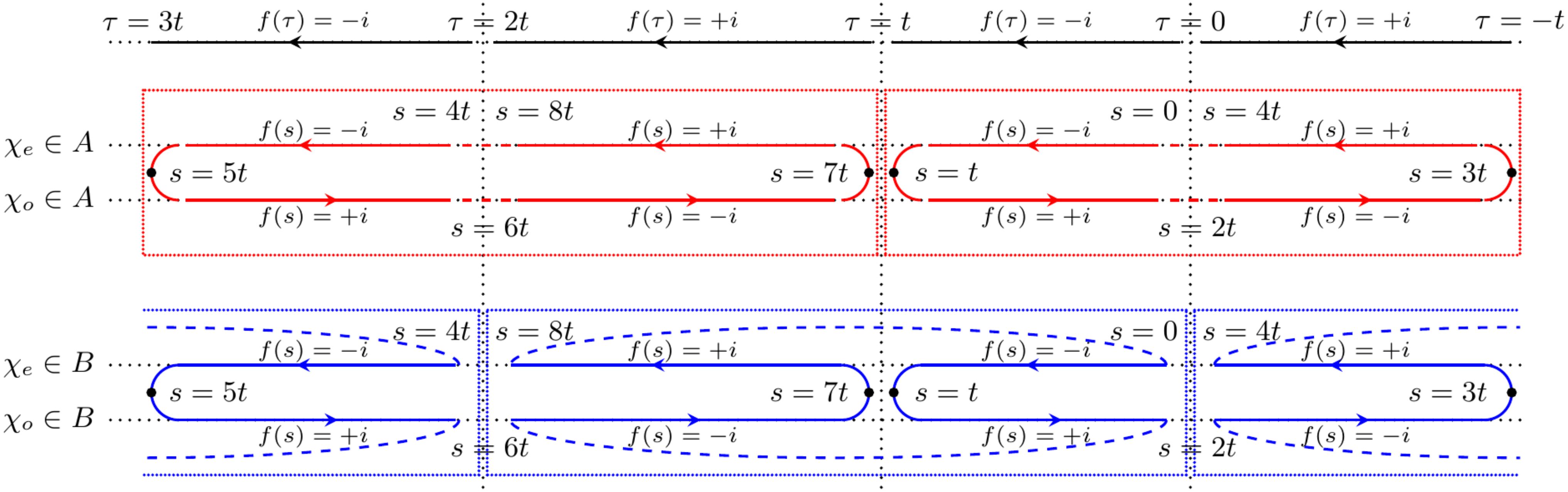
i.e. rotation of the forward contours and rotation of the replica 1 contours



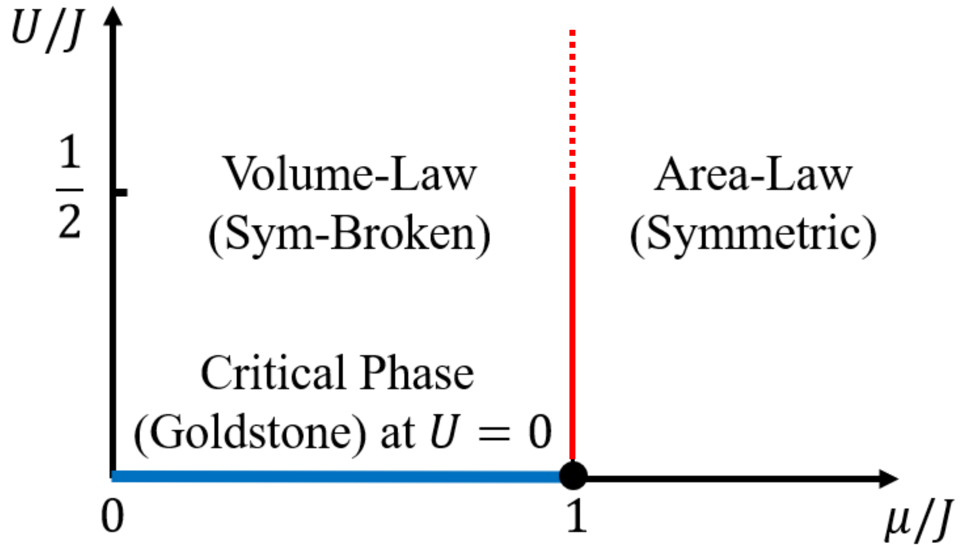
Zhang, Jian, CL, Chen, Quantum (2021)
Jian, CL, Chen, Swingle, Zhang, PRL (2021)

Time contour

$$\cdots \langle\{0\}|e^{itJH^p - tVH^q}|\phi_B^{n-1}\rangle \sum_{\phi_B^n} \langle\phi_B^n|e^{-itJH^p-tVH^q}|\{0\}\rangle\langle\{0\}|e^{itJH^p-tVH^q}|\phi_B^n\rangle\langle\phi_B^1|e^{-itJH^p-tVH^q}|\{0\}\rangle$$



Field theory for the 2nd Renyi entropy



This explains the parentheses.

But what is the physical meaning of volume-law/area-law?

$$\boxed{\frac{I_{\rm eff}}{N} = \int dtdx\left(\frac{1}{2}(\partial\vec{\phi})^2 + r\vec{\phi}^2 + \lambda(\vec{\phi}^2)^2 + \lambda'(\phi_1^4+\phi_2^4)\right)}$$

$$\phi_1 = \delta G^{\text{R1-f;R1-b}} + \delta G^{\text{R2-f;R2-b}}$$

$$\phi_2 = \delta G^{\text{R1-f;R2-b}} + \delta G^{\text{R2-b;R1-f}}$$

ϕ^4 theory!

$$r=\frac{1}{2}\mu(\mu-J),\quad \lambda=\frac{\sqrt{2}}{8}\mu^{\frac{5}{2}}J^{-\frac{1}{2}},\quad \lambda'=-\frac{2U}{\mu}\lambda.$$

Jian, CL, Chen, Swingle, Zhang, PRL (2021)

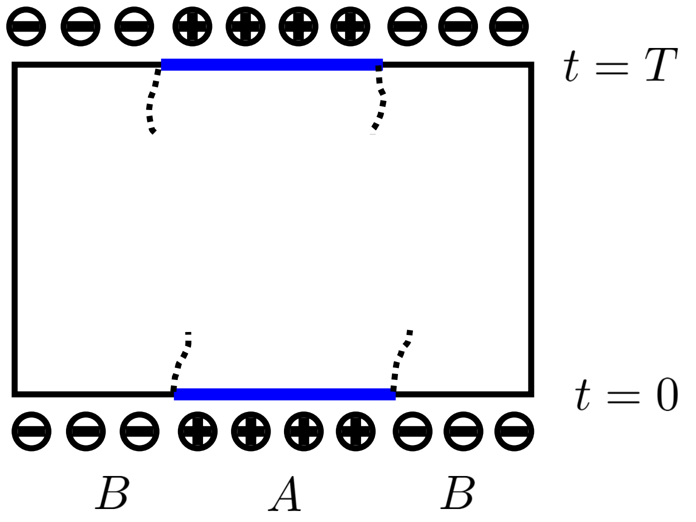
Stat. mech. Picture of 2nd Renyi entropy

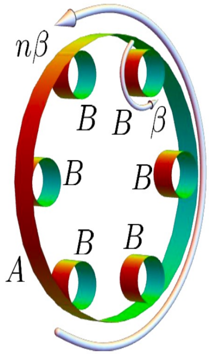
$$S_A^{(n)} = \frac{1}{1-n} \log \frac{\mathrm{Tr}_A((\mathrm{Tr}_B \rho)^n)}{(\mathrm{Tr}\rho)^n}$$

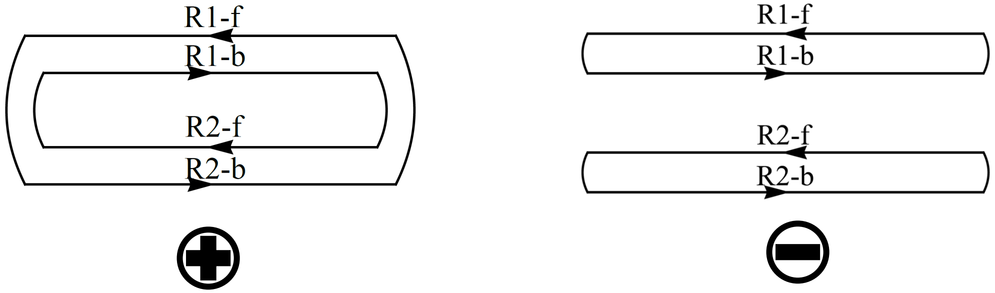
$$= I_{\mathrm{eff}}(A) - I_{\mathrm{eff}}(A = \emptyset)$$

- Renyi EE = Free energy of the ϕ^4 field theory with a particular boundary conditions (set by the definition of Renyi EE).
- In a symmetry broken phase, free energy = domain wall energy (discrete symmetry) or vortex energy (continuous symmetry).
- Domain wall energy gives volume law; vortex energy gives logarithmic law. Symmetric phase is gapped, giving area law.

Zhang, JHEP (2020)







$\phi_1 = 0, \phi_2 > 0$

$\phi_1 > 0, \phi_2 = 0$

Twisted BC
(subsystem A in Numerator)

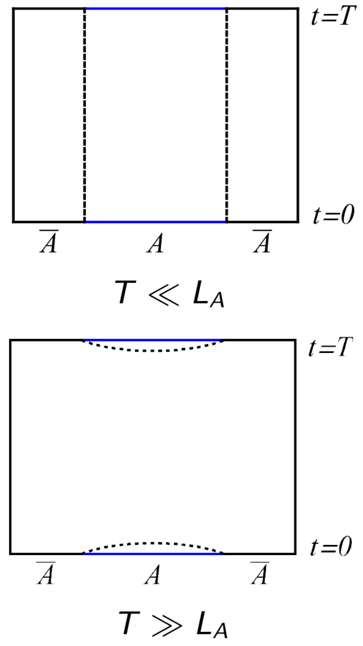
Untwisted BC
(Subsystem B in numerator; denominator)

More details about the volume law phase

- Early and late time scaling (capillary wave theory) :

$$S_A^{(2)} = \begin{cases} 2(N\sigma T + \log T/2), & T \ll L_A \\ \\ 2(N\sigma L_A + 3/2 \log L_A), & T \gg L_A \end{cases},$$

Li, Fisher, PRB (2021)

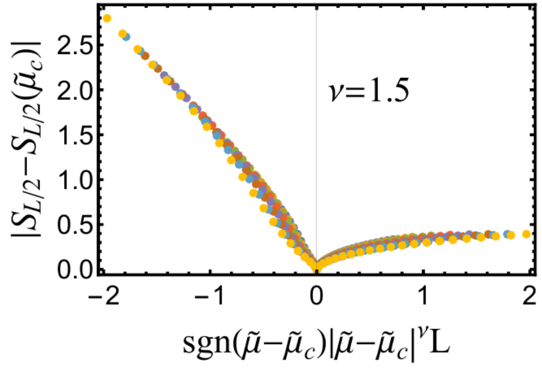


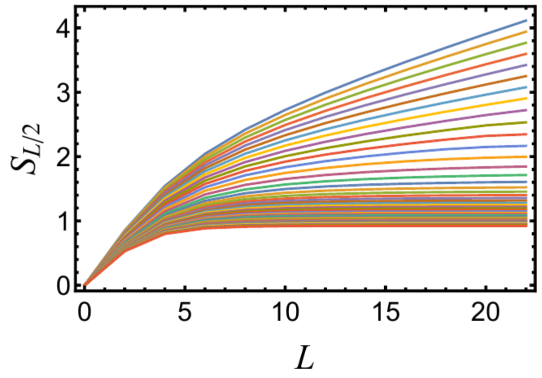
subleading log term due to transverse fluctuations.

- Line tension expression:

$$\sigma \sim (JU)^{\frac{1}{2}} \left(\frac{J-\mu}{\mu - 2U} \right)^{3/2}$$

- Critical exponent confirmed numerically



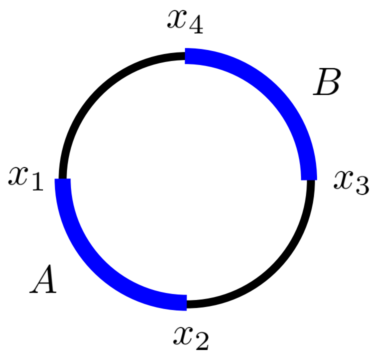


Conformal behavior at the critical point

- Mutual information as a function of cross ratio:

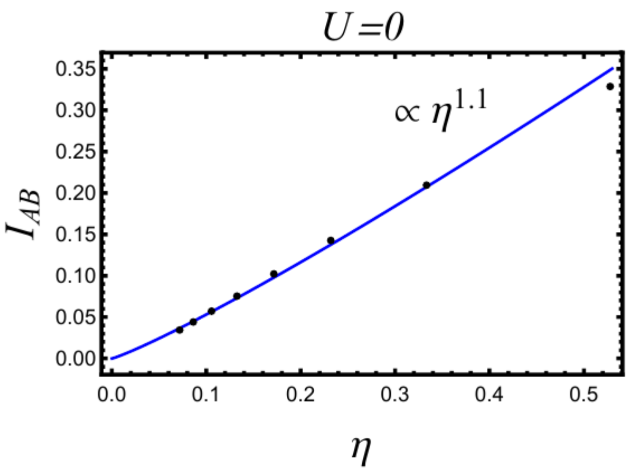
$$I_{AB}(\eta) = S^{(2)}_{AB} - S^{(2)}_{A} - S^{(2)}_{B}$$

$$\propto \eta^{\Delta}$$



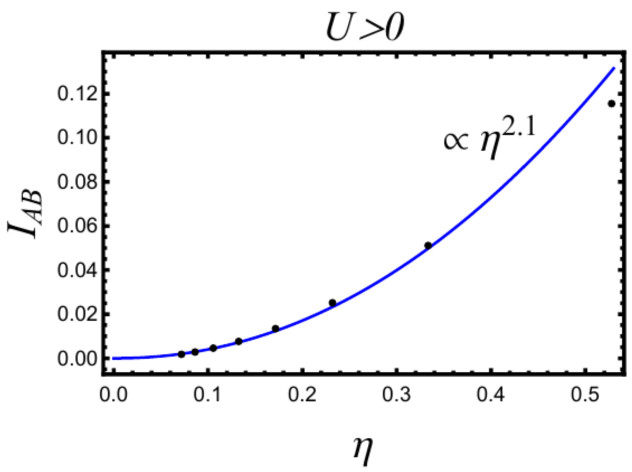
$$\eta = \frac{\sin\frac{\pi}{L}|x_1 - x_2| \sin\frac{\pi}{L}|x_3 - x_4|}{\sin\frac{\pi}{L}|x_1 - x_4| \sin\frac{\pi}{L}|x_2 - x_3|}$$

Theory prediction: Δ = 1



Noninteracting case

Theory prediction: Δ = 2



Interacting case

Compared with

$$\Delta_{\text{Clifford}} = 2.16 \pm 0.03$$

$$\Delta_{\text{Haar}} = 1.89 \pm 0.26$$

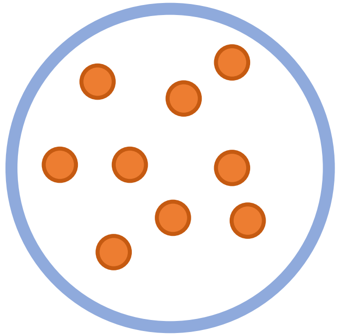
Li, Chen, Fisher, PRB (2019)
Shi, Dai, Lu, arXiv 2012.00040

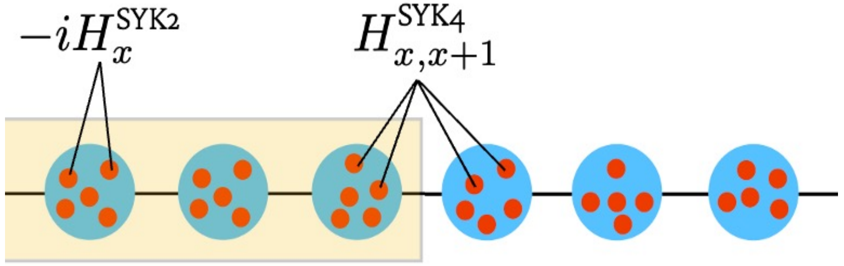
Summary

- Large N solution of SYK as a 0D strongly correlated model
- Maximal free fermion eigenstate entanglement and RMT
- Ways to generalize to 1d SYK chain with interesting phases, while maintaining large N solvability
- Stat. mech. interpretation: EE = free energy with twisted boundary conditions; volume law phase = symmetry breaking phase
- EE in the volume law phase understood as domain wall energy
- EE in the critical phase understood as vortex energy
- A large N model for measurement induced phase transition

Outlook

- Finite N effects? (Relevant perturbation to phase transition?)
- Quenched/annealed disorder?
- Free fermion case: connecting the saddle point formalism with RMT?
- How to get to von Neumann entropy (n-->1)?





Thanks!