

Thermal Tensor Network Approach for Quantum Many-Body systems

Bin-Bin Chen

Oct. 19, 2022

➤ Outline

□ 1. Finite-T tensor network methods

- 1.1 Tensor network basis
- 1.2 Series-expansion thermal tensor network
- 1.3 Exponential tensor renormalization group
- 1.4 Differentiable tensor renormalization group

□ 2. Application 1

- 2.1 Square-lattice Hubbard model
- 2.2 Triangular-lattice Hubbard model
- 2.3 Magic-angle twisted bilayer graphene model

□ 3. Application 2

- 3.1 Finite-temperature entanglement spectrum
- 3.2 disorder operator

➤ 1.1 Motivation

□ Strong Correlated systems:

- High-T superconductivity in cuprate
- Quantum spin liquid
- Magic-angle twisted bilayer graphene
- ...

“Exponential wall”



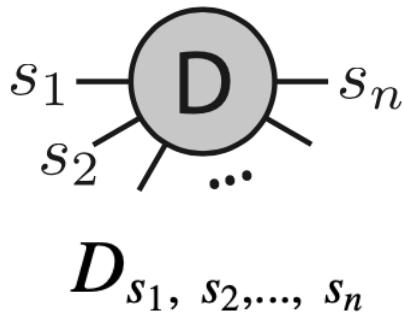
- **Quantum Monte Carlo (QMC)**
Negative sign problem
- **Tensor Network (TN)**
capture the entanglement structure

“Wheat and chessboard” problem

➤ 1.1 Tensor network basis

□ Basic terminology

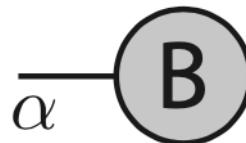
- Rank-n Tensor



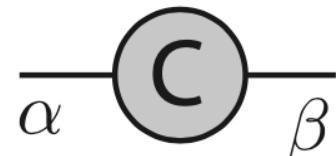
[special cases]



scalar

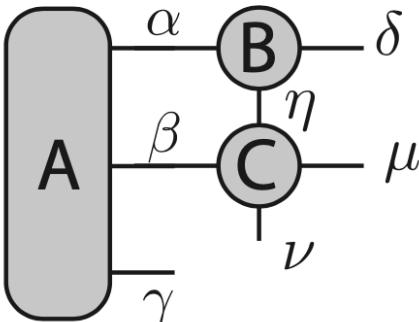


vector



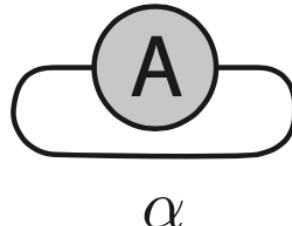
matrix

- Contraction

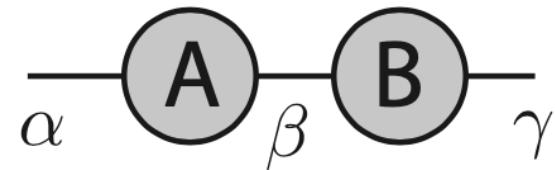


$$\sum_{\alpha, \beta, \eta} A_{\alpha, \beta, \gamma} B_{\alpha, \delta, \eta} C_{\beta, \eta, \mu, \nu}$$

[special cases]



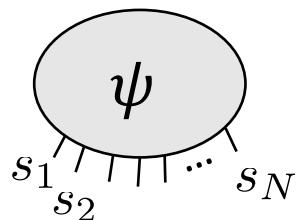
trace



matrix product

➤ 1.1 Tensor network basis

□ Manybody wavefunction

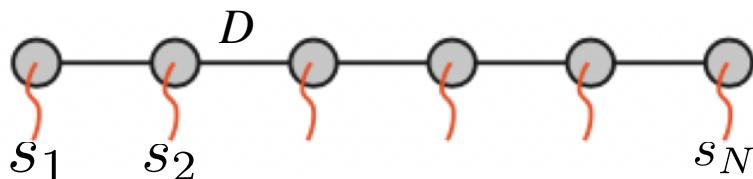


of parameters: d^N

Exponential wall!

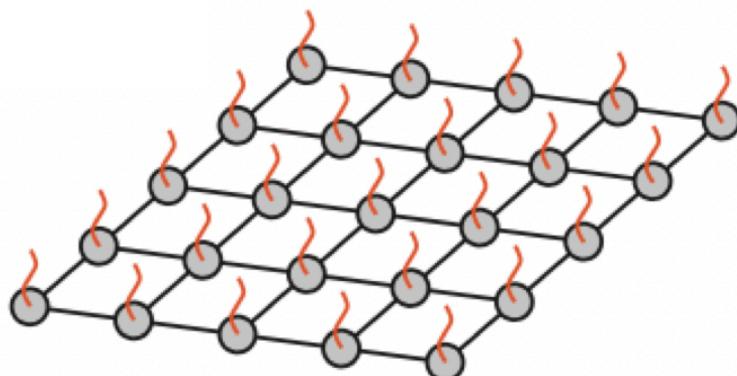
□ Typical Tensor Network

- ✓ Matrix Product State (MPS)



$\sim N \times (dD^2)$

- ✓ Projected Entangled-Pair State (PEPS)

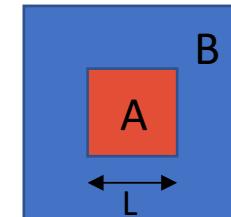


$\sim N \times (dD^4)$

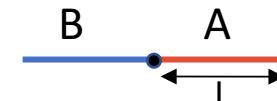
➤ 1.1 Tensor network basis

- ✓ Remarkable fact: For Hamiltonians with local interactions, the ground state entanglement entropy is governed by an “area law”(Eisert2010).

E.g., for 2D gapped systems, $S_E \sim L$



E.g., for 1D gapped systems, $S_E \sim \text{const.}$



E.g., for 1D gapless systems, $S_E \sim \ln L$



For a bond with dimension D, entanglement entropy $S_E = -\text{Tr} \rho_A \ln \rho_A \leq \ln D$

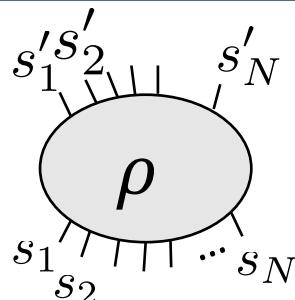
Then, for 2D gapped systems, $D \geq e^{S_E} \sim e^L$

for 1D gapped systems, $D \geq e^{S_E} \sim \text{const.}$ (independent of system size)

for 1D gapless systems, $D \geq e^{S_E} \sim L^\alpha$ (polynomial resources)

➤ 1.1 Tensor network basis

□ Density operator

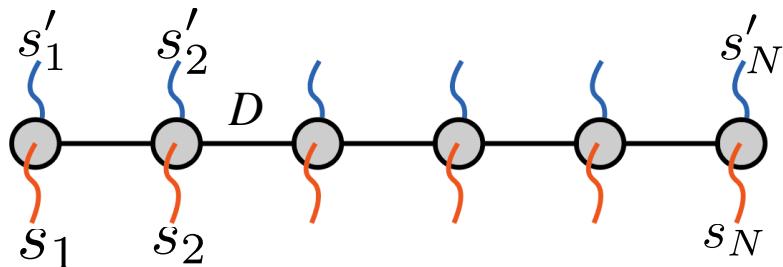


of parameters: d^{2N}

Exponential wall!

□ Typical Tensor Network

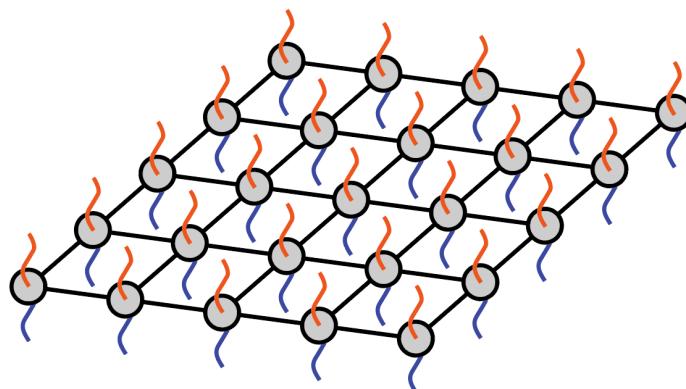
✓ Matrix Product Operator (MPO)



$$\sim N \times (d^2 D^2)$$

W. Li, et al. 2011; Dong, et al. 2017

✓ Projected Entangled-Pair Operator (PEPO)



$$\sim N \times (d^2 D^4)$$

Czarnik, et al. 2017

➤ 1.1 Path-Integral Thermal Tensor Network

Partition function for Hamiltonian with local interactions $H = \sum_i h_{i,i+1} = \sum_i S_i \cdot S_{i+1}$

$$Z = \text{Tr } e^{-\beta H} = \text{Tr } e^{-\beta \sum_i h_{i,i+1}}$$

$$= \sum_{\vec{\sigma}^1} \langle \vec{\sigma}^1 | e^{-\beta \sum_i h_{i,i+1}} | \vec{\sigma}^1 \rangle \quad | \vec{\sigma} \rangle := |\sigma_1, \sigma_2, \dots, \sigma_N \rangle$$

$$= \sum_{\vec{\sigma}^1, \vec{\sigma}^2, \dots, \vec{\sigma}^M} \langle \vec{\sigma}^1 | e^{-\frac{\beta}{M} \sum_i h_{i,i+1}} | \vec{\sigma}^M \rangle \dots \langle \vec{\sigma}^3 | e^{-\frac{\beta}{M} \sum_i h_{i,i+1}} | \vec{\sigma}^2 \rangle \langle \vec{\sigma}^2 | e^{-\frac{\beta}{M} \sum_i h_{i,i+1}} | \vec{\sigma}^1 \rangle$$

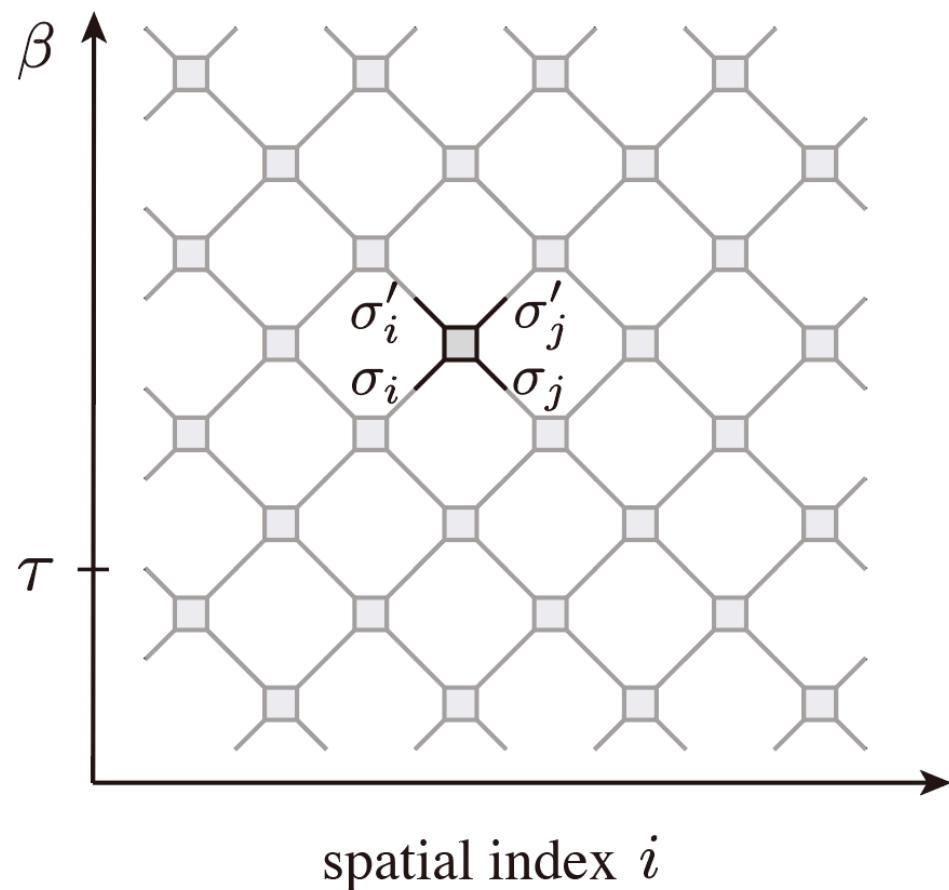
$$= \sum_{\vec{\sigma}^j} \prod_{j=1}^M \langle \vec{\sigma}^{j+1} | e^{-\tau \sum_i h_{i,i+1}} | \vec{\sigma}^j \rangle$$

$$= \sum_{\{\sigma_i^j\}} \prod_{i=1}^N \prod_{j=1}^M \langle \sigma_i^{j+1} \sigma_{i+1}^{j+1} | e^{-\tau h_{i,i+1}} | \sigma_i^j \sigma_{i+1}^j \rangle + O(\tau^2)$$

➤ 1.1 Path-Integral Thermal Tensor Network

$$Z = \text{Tr}(e^{-\beta H}) \quad \xrightarrow{\text{Trotter}} \quad Z \approx \sum_{\{\sigma_i^j\}} \prod_{i=1}^N \prod_{j=1}^M \langle \sigma_i^{j+1} \sigma_{i+1}^{j+1} | e^{-\tau h_{i,i+1}} | \sigma_i^j \sigma_{i+1}^j \rangle$$

- 1+1D Tensor Network



- Efficient contraction

Wang and Xiang, 1997

Li, et al. 2011

Dong, et al. 2017

Series-Expansion Thermal Tensor Network (SETTN)

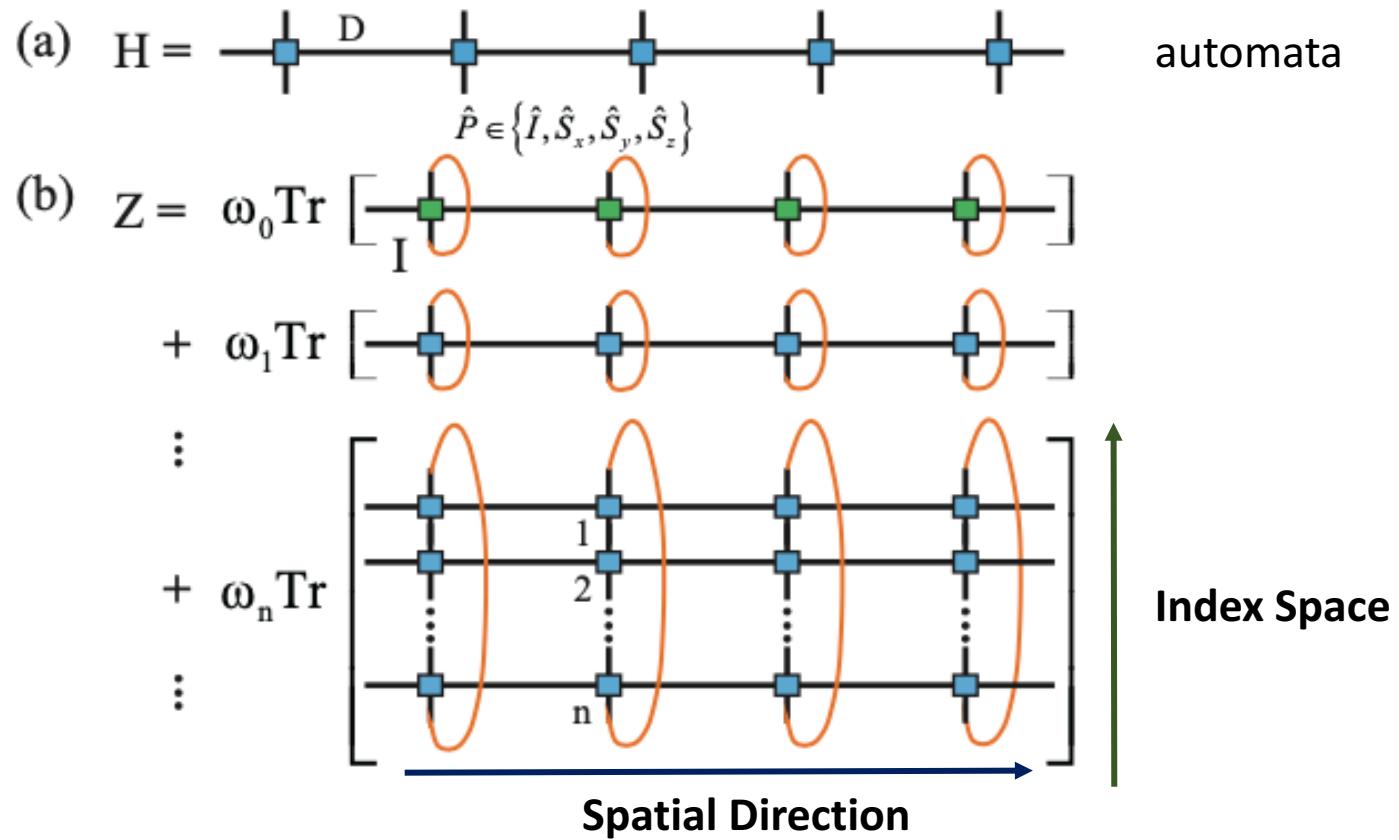
BC, Yun-Jing Liu, Ziyu Chen, Wei Li, *PRB(R)* 95, 161104

➤ 1.2 Basic Idea of SETTN

□ Taylor Expansion of Partition Function:

$$Z(\beta) = \text{Tr}(e^{-\beta H}) \simeq \sum_{n=0}^{N_c} \frac{(-\beta)^n}{n!} \text{Tr}(H^n)$$

□ Main Procedure:



➤ 1.2 Basic Idea of SETTN

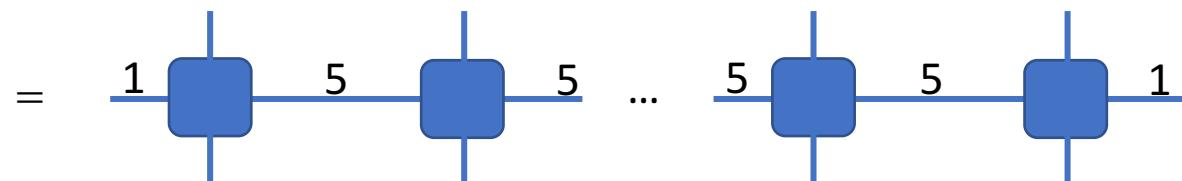
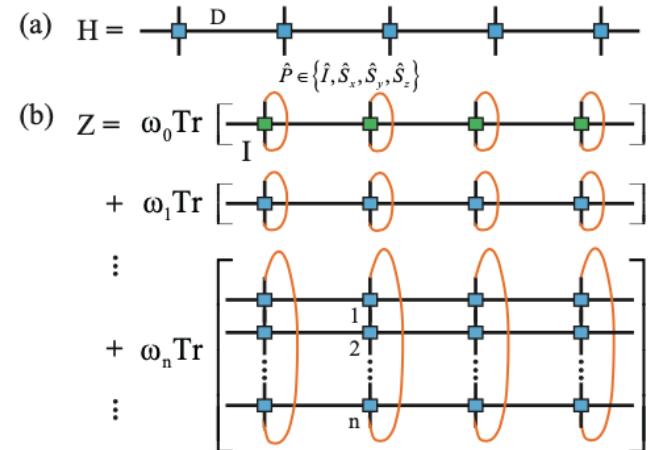
□ Taylor Expansion of Partition Function:

$$Z(\beta) = \text{Tr}(e^{-\beta H}) \simeq \sum_{n=0}^{N_c} \frac{(-\beta)^n}{n!} \text{Tr}(H^n)$$

□ MPO of Hamiltonian: (e.g Heisenberg chain)

$$H = \sum_{i=1}^{N-1} (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z)$$

$$= [I \quad S^x \quad S^y \quad S^z \quad 0] \begin{bmatrix} I & S^x & S^y & S^z & 0 \\ 0 & 0 & 0 & 0 & S^x \\ 0 & 0 & 0 & 0 & S^y \\ 0 & 0 & 0 & 0 & S^z \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \cdots \begin{bmatrix} I & S^x & S^y & S^z & 0 \\ 0 & 0 & 0 & 0 & S^x \\ 0 & 0 & 0 & 0 & S^y \\ 0 & 0 & 0 & 0 & S^z \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ S^x \\ S^y \\ S^z \\ I \end{bmatrix}$$



➤ 1.2 Basic Idea of SETTN

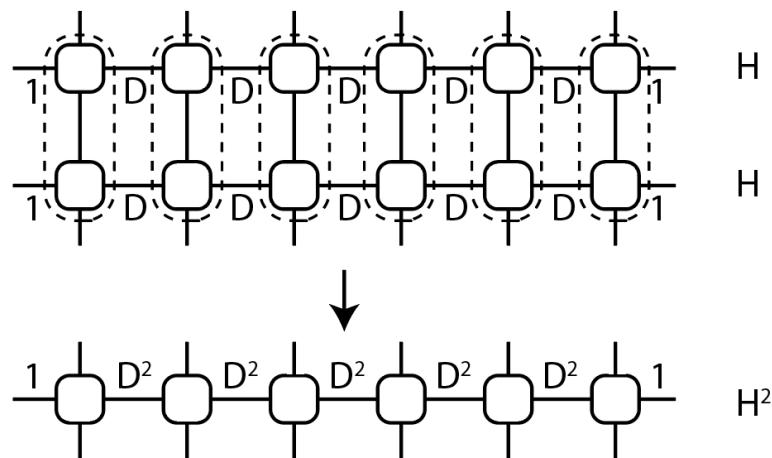
E.g. N= 4 Heisenberg chain

$$\begin{aligned}
 & [I \quad S^x \quad S^y \quad S^z \quad 0] \begin{bmatrix} I & S^x & S^y & S^z & 0 \\ 0 & 0 & 0 & 0 & S^x \\ 0 & 0 & 0 & 0 & S^y \\ 0 & 0 & 0 & 0 & S^z \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & S^x & S^y & S^z & 0 \\ 0 & 0 & 0 & 0 & S^x \\ 0 & 0 & 0 & 0 & S^y \\ 0 & 0 & 0 & 0 & S^z \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ S^x \\ S^y \\ S^z \\ I \end{bmatrix} \\
 = & [I \quad S^x \quad S^y \quad S^z \quad 0] \begin{bmatrix} I & S^x & S^y & S^z & 0 \\ 0 & 0 & 0 & 0 & S^x \\ 0 & 0 & 0 & 0 & S^y \\ 0 & 0 & 0 & 0 & S^z \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} S^x S^x + S^y S^y + S^z S^z \\ S^x I \\ S^y I \\ S^z I \\ II \end{bmatrix} \\
 = & [I \quad S^x \quad S^y \quad S^z \quad 0] \begin{bmatrix} IS^x S^x + IS^y S^y + IS^z S^z + S^x S^x I + S^y S^y I + S^z S^z I \\ S^x II \\ S^y II \\ S^z II \\ II \end{bmatrix}
 \end{aligned}$$

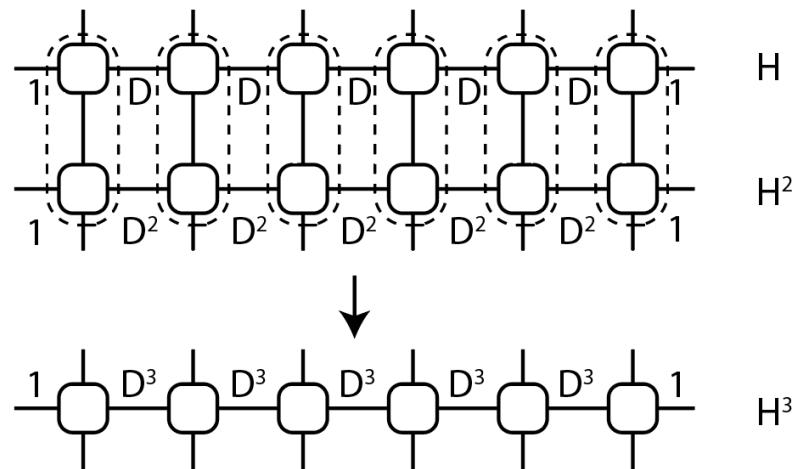
$$\begin{aligned}
 & IIS^x S^x + IIS^y S^y + IIS^z S^z \\
 = & +IS^x S^x I + IS^y S^y I + IS^z S^z I \\
 & +S^x S^x II + S^y S^y II + S^z S^z II
 \end{aligned}$$

➤ 1.2 Efficient contraction of H^n

Construction of H^2



Construction of H^3

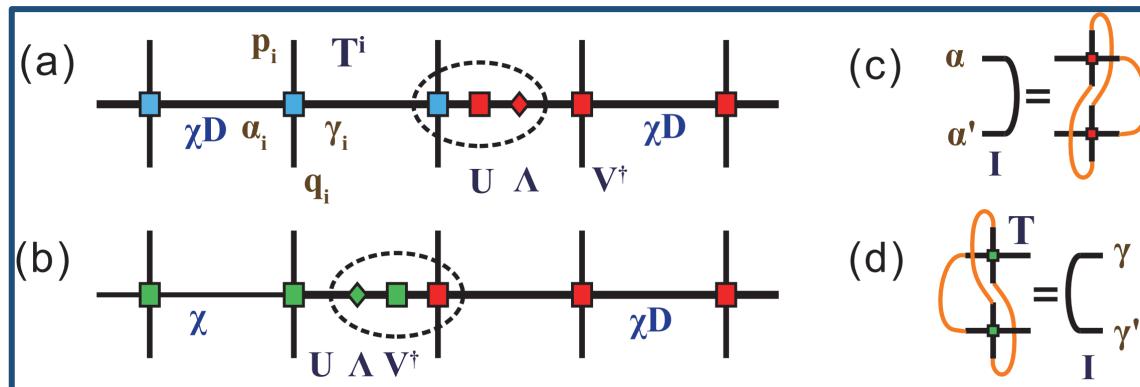


Similarly, H^n will have bond dimension D^n

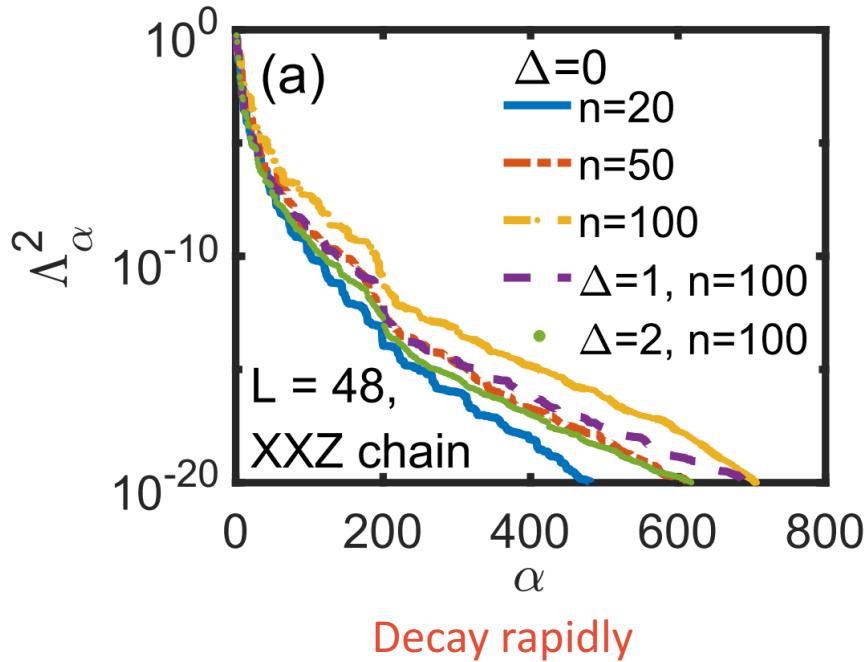
Bond dimension scales **exponentially** and will quickly become unaffordable.

➤ 1.2 Efficient contraction of H^n

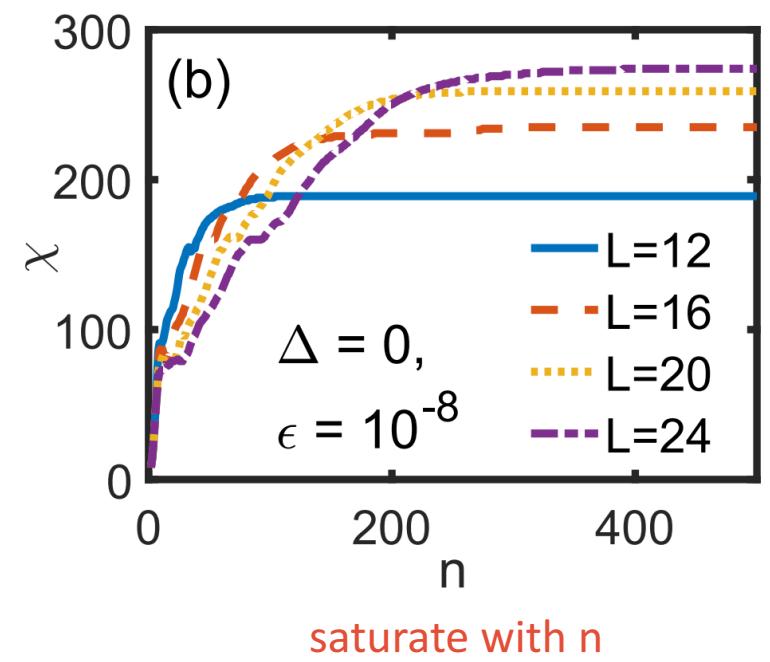
□ Canonical form:



□ “Entanglement” spectrum



□ Required bond dimension



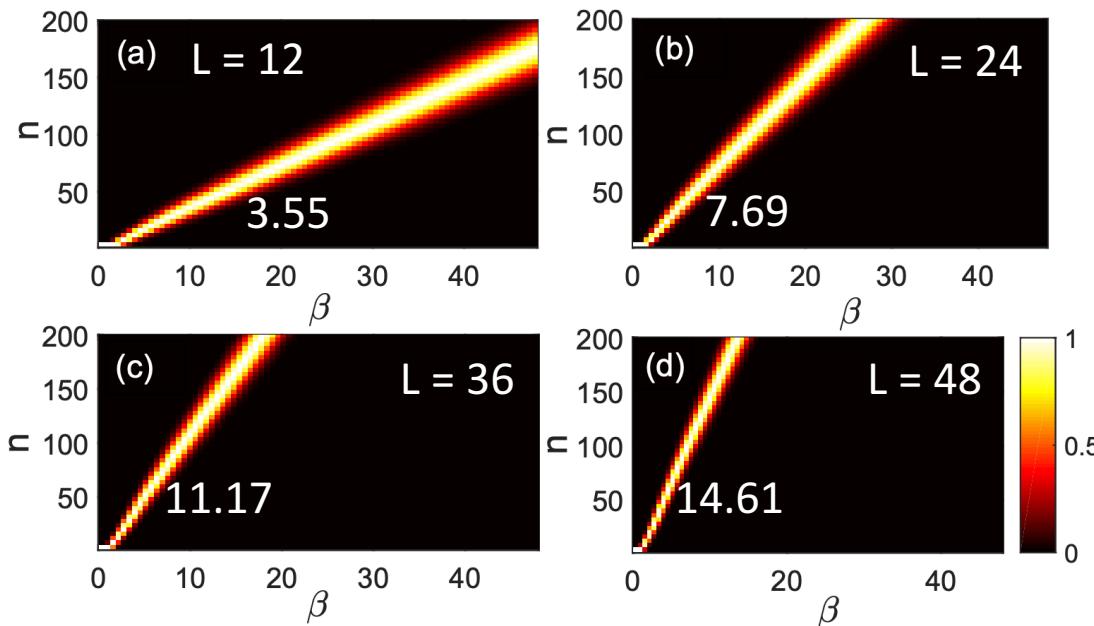
➤ 1.2 Expansion cutoff

$$Z(\beta) = \text{Tr}(e^{-\beta H}) \simeq \sum_{n=0}^{N_c} \frac{(-\beta)^n}{n!} \text{Tr}(H^n)$$

For large n, we have $\text{Tr}(H^n) \propto (E_{\ln})^n = (e_{\ln} L)^n$

Partition function is sum of $\kappa(n) = \frac{(-\beta L e_{\ln})^n}{n!} = \frac{(\beta L |e_g|)^n}{n!}$

which is most prominent around $n = \beta L |e_g|$



E.g., XY chain: $e_g = -1/\pi$

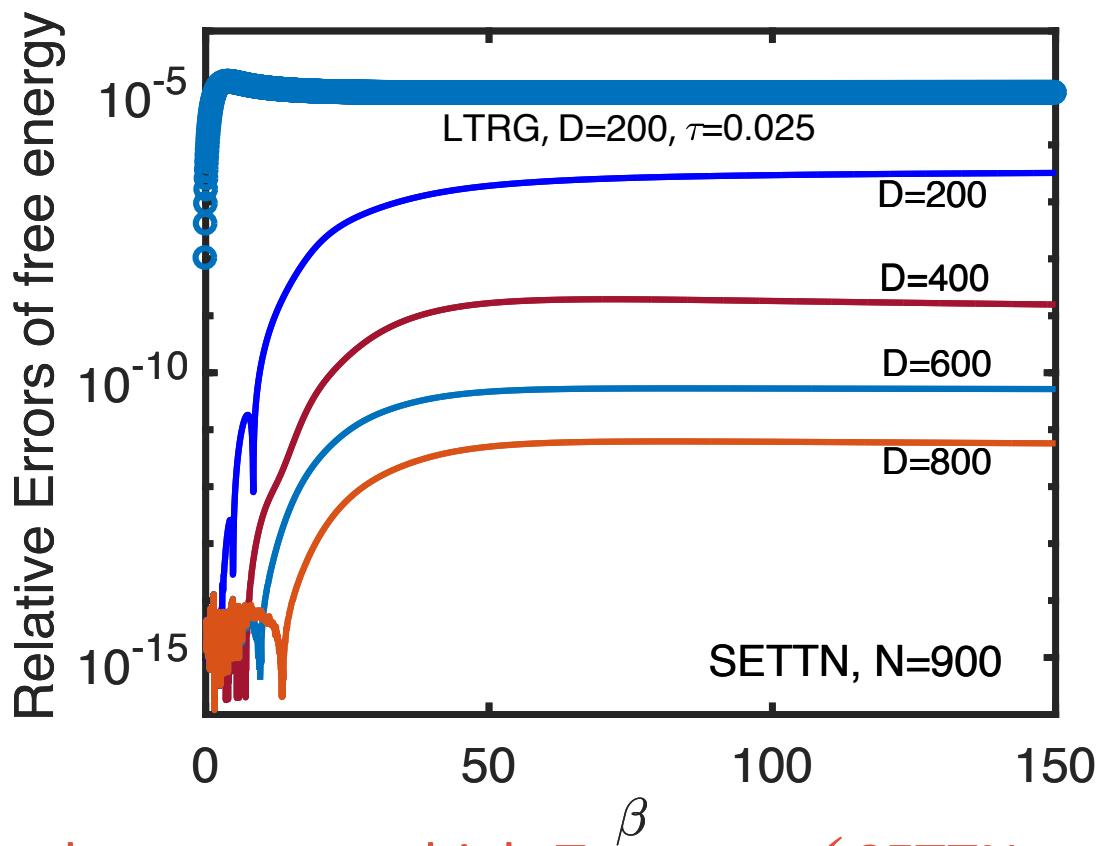
Then partition function is dominant by weight around

$$n = \beta L / \pi$$

➤ 1.2 Performance of SETTN

- L=14 XY chain

$$H = J \sum_{\langle i,j \rangle} (S_i^x S_j^x + S_i^y S_j^y)$$



✓ Extremely accurate at high-T

β

✓ SETTN outperform Trotter

➤ 1.2 Summary of SETTN

Trotter-
error free



Continuous
Time



Long-Range
Couplings



The problem is still very **challenging** for 2D system at **Low T** .

Exponential Tensor Renormalization Group (XTRG)

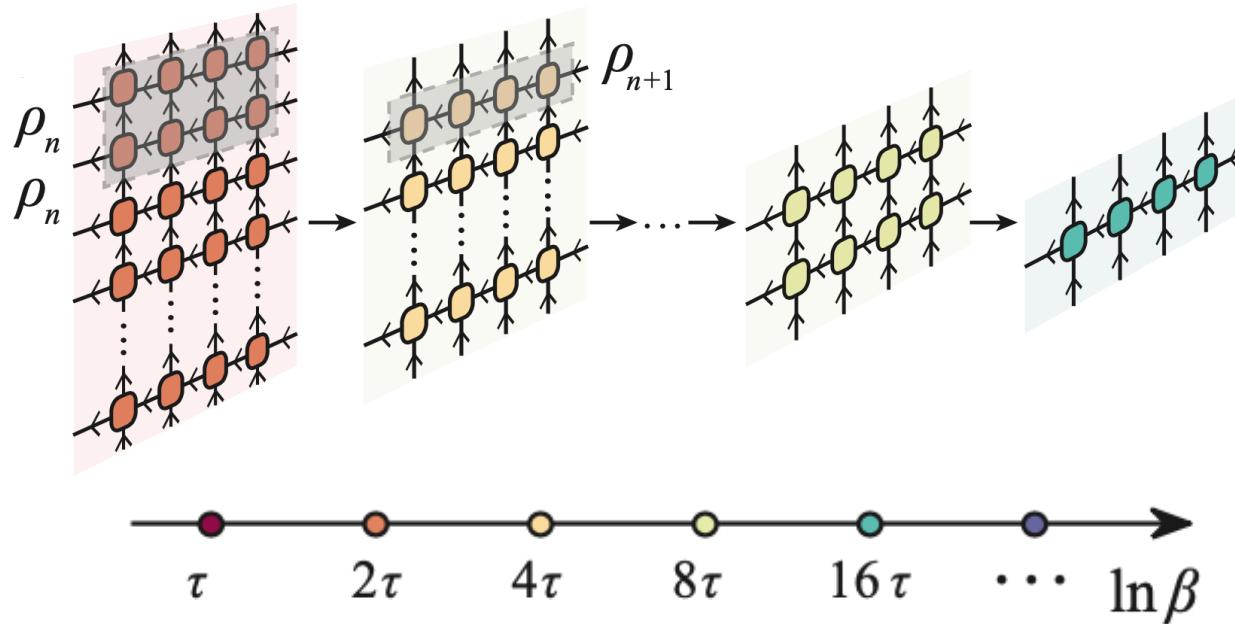
- [1] **BC**, L. Chen, Z. Chen, W. Li, A. Weichselbaum. *PRX* 8, 031082
- [2] L. Chen, D.-W. Qu, H. Li, **BC**, S.-S. Gong, J. von Delft, A. Weichselbaum, W. Li. *PRB(R)* 99, 140404
- [3] H. Li, **BC**, Z. Chen, J. von Delft, A. Weichselbaum, W. Li. *PRB(R)* 100, 045110

➤ 1.3 Basic Idea of XTRG

1) SETTN initialization at high Temperature

$$\rho_0 \equiv \rho(\tau) \quad \begin{array}{cccccc} & \uparrow & & \uparrow & & \uparrow & \\ \text{---} & \rightarrow & \text{---} & \rightarrow & \text{---} & \rightarrow & \text{---} \\ & \downarrow & & \downarrow & & \downarrow & \\ \text{---} & \rightarrow & \text{---} & \rightarrow & \text{---} & \rightarrow & \text{---} \end{array}$$

2) Exponential evolution $\rho_n \cdot \rho_n \rightarrow \rho_{n+1}$

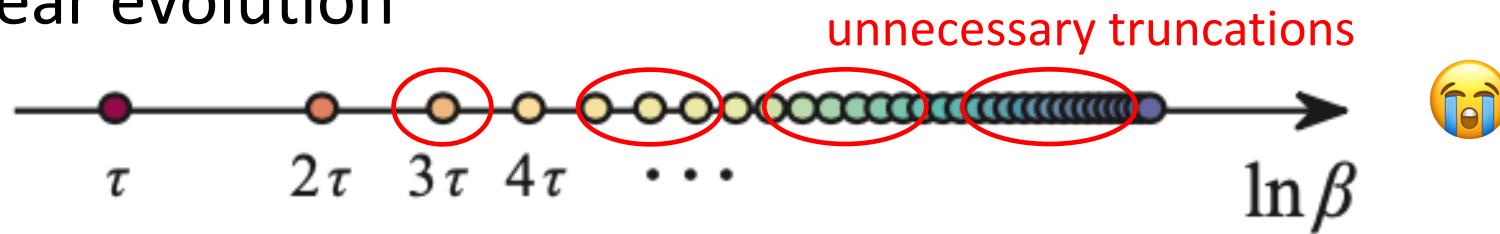


➤ 1.3 Reduce numbers of truncation steps

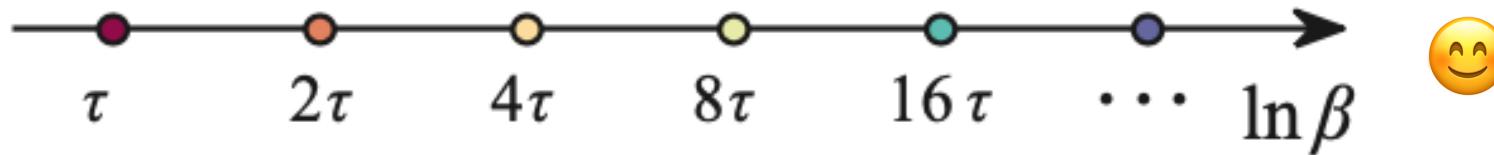
- logarithmic scaling of entanglement entropy

$$S_E \sim (c/3) \ln \beta$$

- Linear evolution



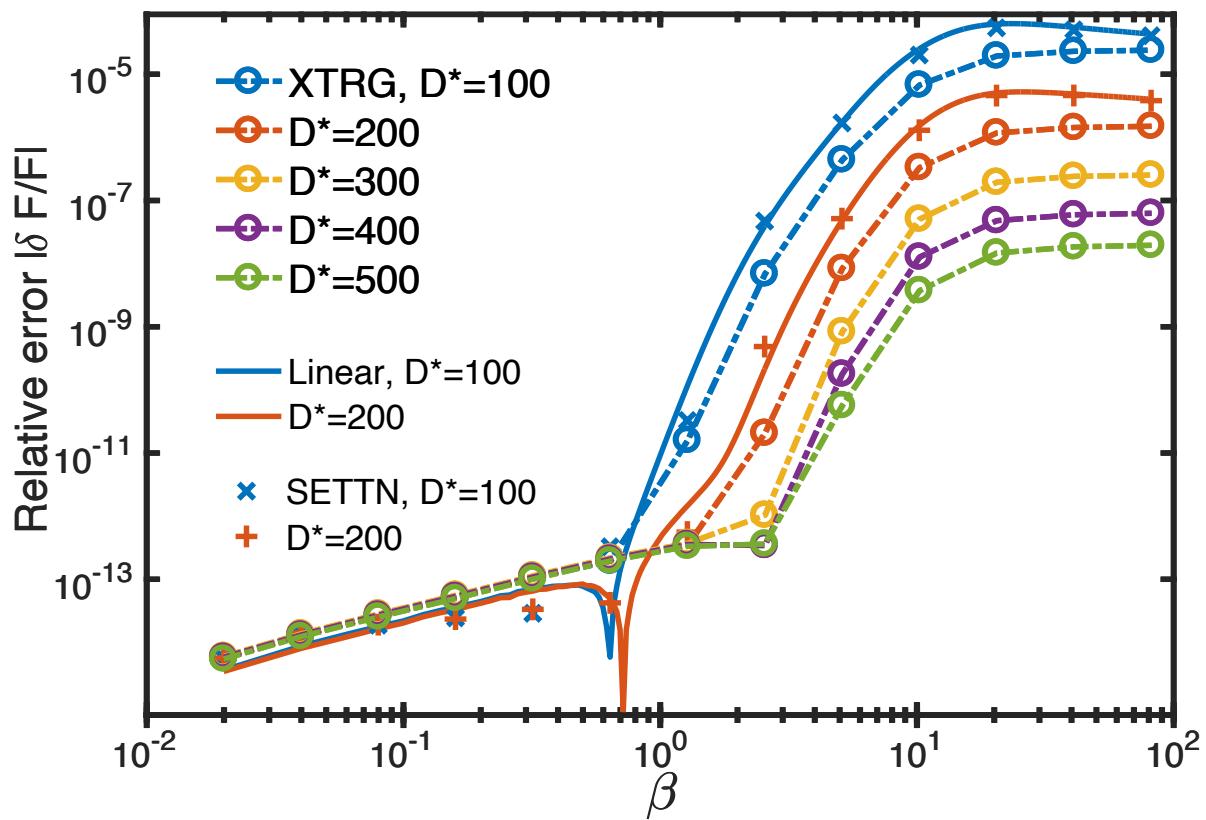
- Exponential evolution



➤ 1.3 Performance of XTRG

□ $L=18$ Heisenberg chain

$$H = J \sum_{i,j} (S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z)$$

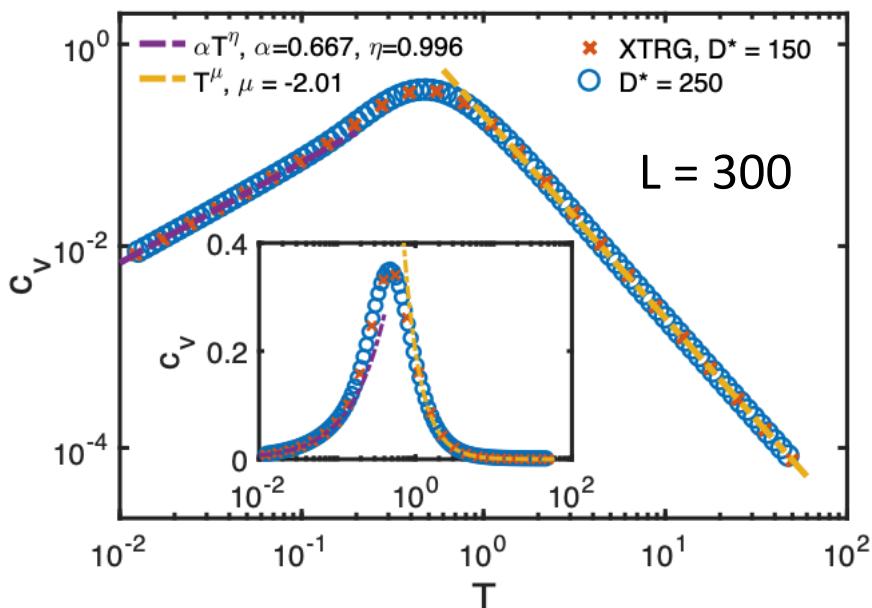


1 order of magnitude better than **Linear** scheme and also **SETTN**

➤ 1.3 More data of XTRG

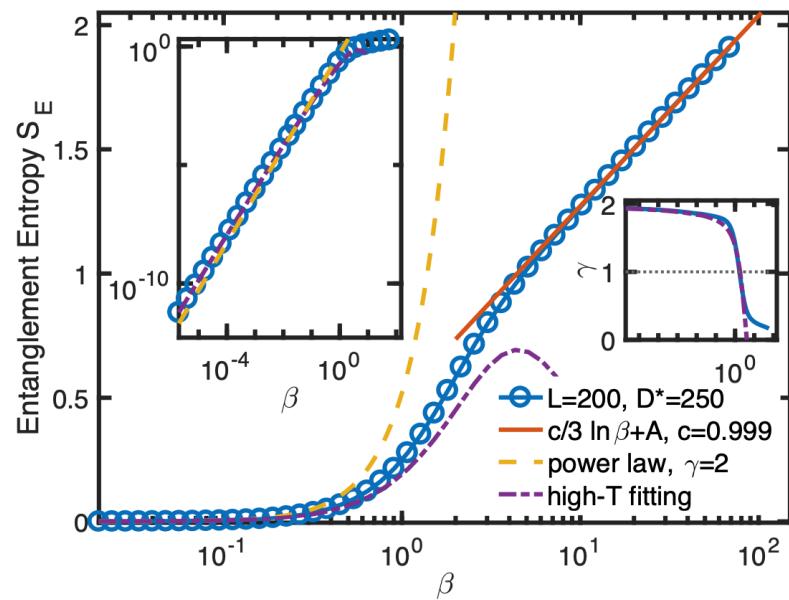
✓ Long Heisenberg chain

Extraction of central charge

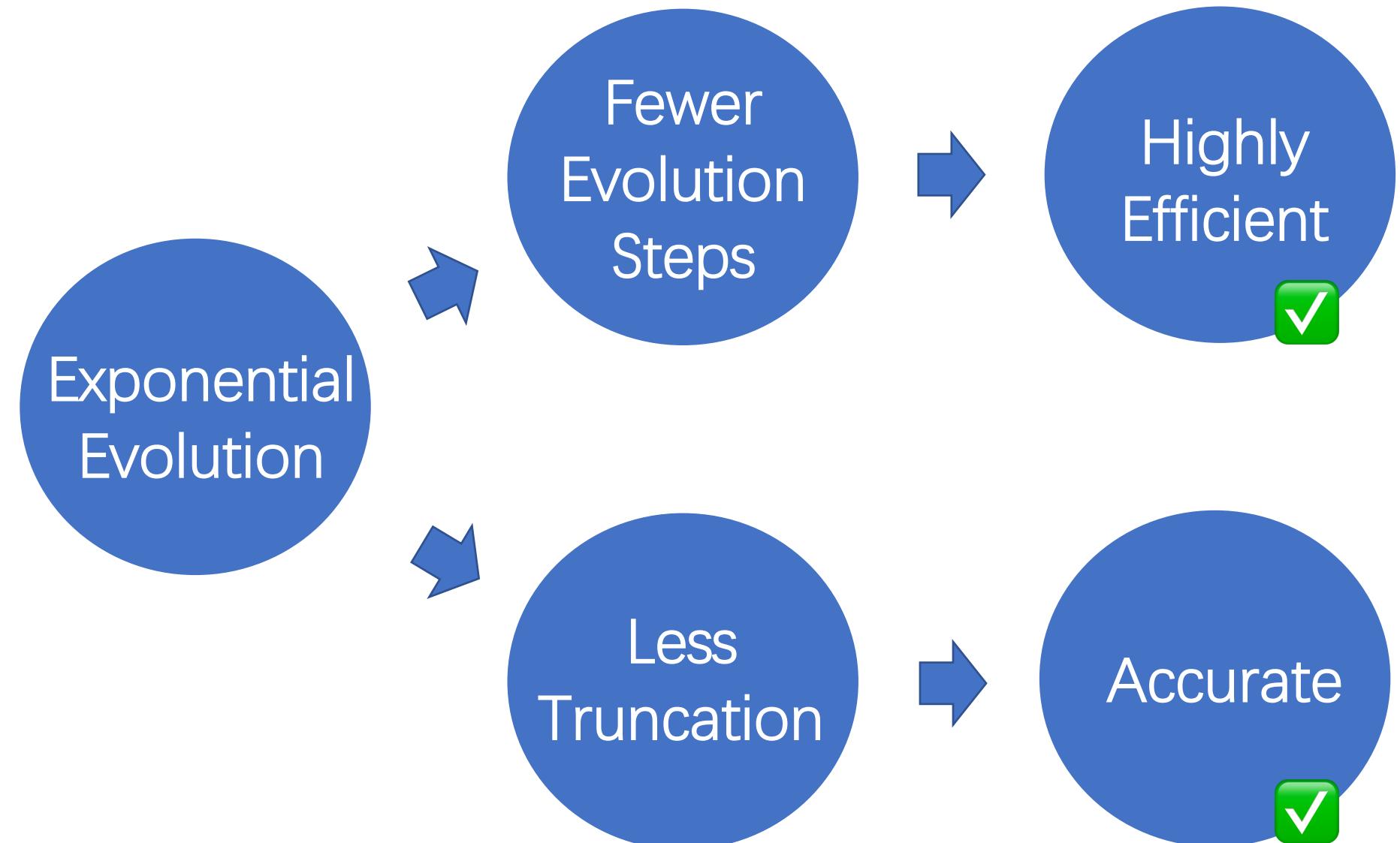


$$c_V = \alpha T^\eta \quad \alpha = \frac{\pi c}{3v}$$

logarithmic scaling of entanglement entropy
 $S_E \sim (c/3) \ln \beta$



➤ 1.3 Summary of XTRG

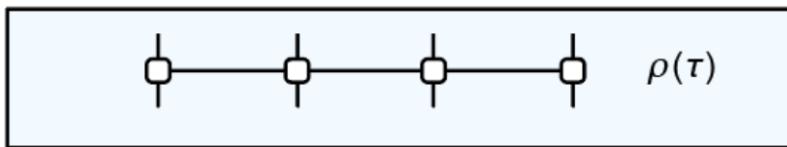


Differentiable Tensor Renormalization Group (δ TRG)

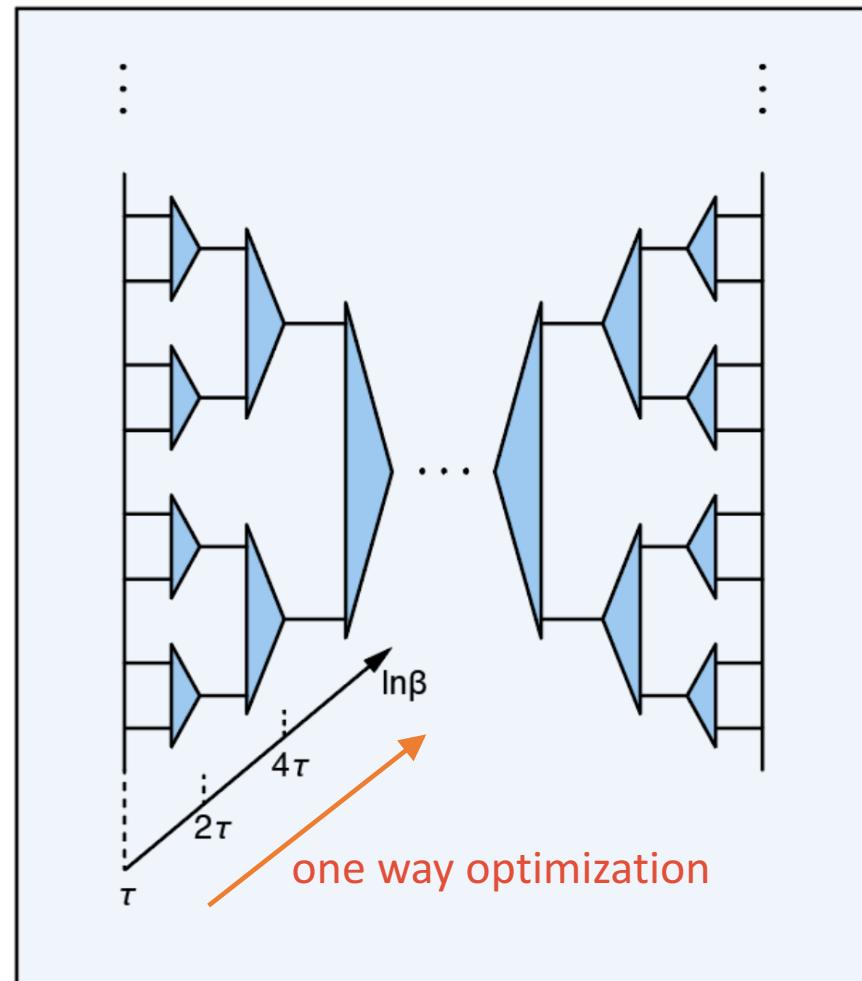
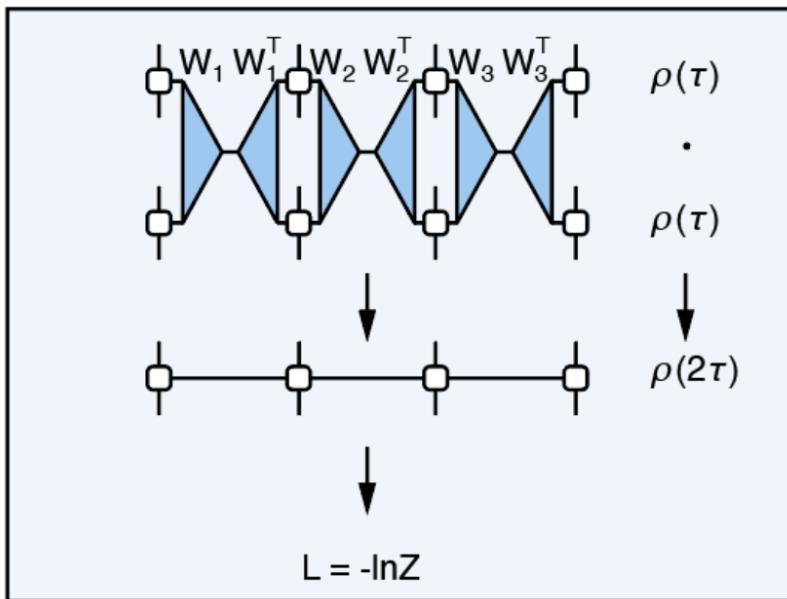
BC, Y. Gao, Y.-B. Guo, Y. Liu, H.-H. Zhao, H.-J. Liao, L. Wang, T. Xiang, W. Li, Z. Y. Xie. *PRB(R)* 101, 220409

➤ 1.4 Basic Idea of δ TRG

1) SETTN initialization at high temperature



2) Forward TRG



➤ 1.4 Basic Idea of δTRG

3) Backward optimization

By using **Automatic Differentiation**,
We can calculate environment

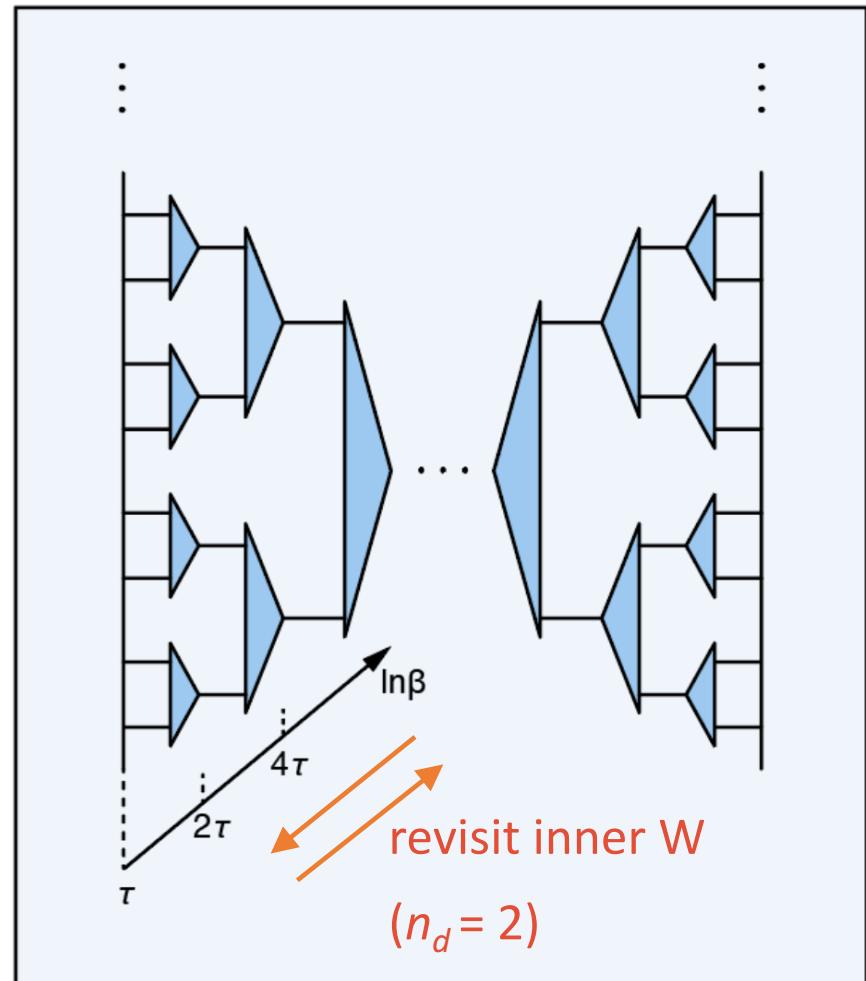
$$\frac{\partial \mathcal{L}}{\partial W_i}$$

and update W_i

4) Repeat forward and backward for all

$$W_i$$

within optimization depth n_d

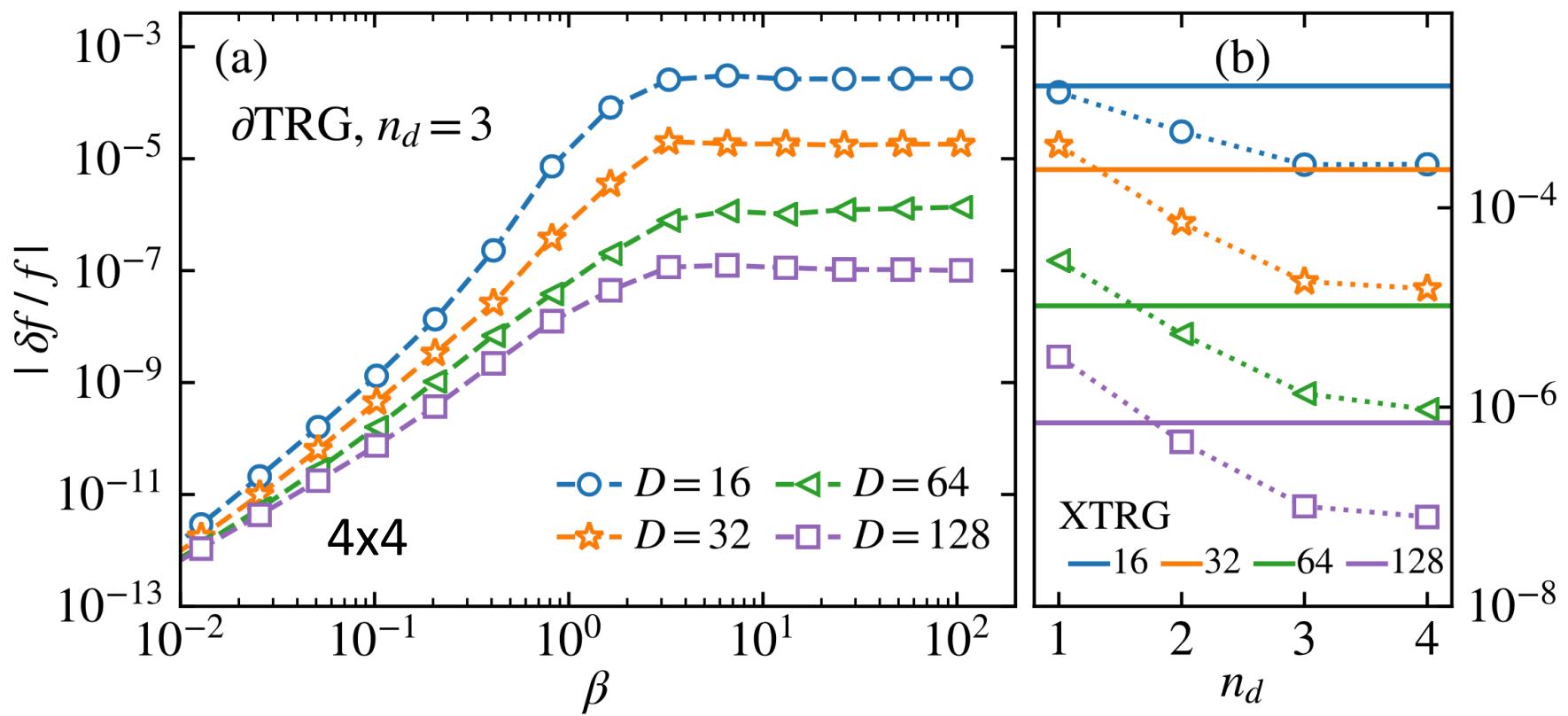


Deep optimization in thermal tensor network

➤ 1.4 2D transverse-field Ising model

$$H = J \sum_{i,j} S_i^z S_j^z + h \sum_i S_i^x \quad \text{with } J=-1, h = h_c = 1.522$$

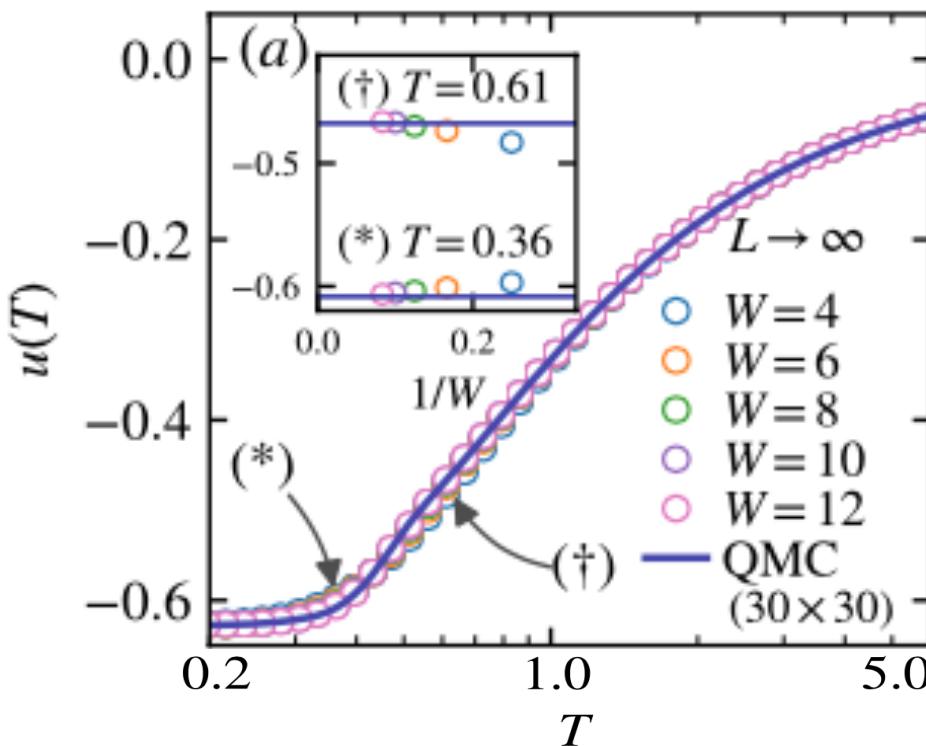
- relative error of free energy



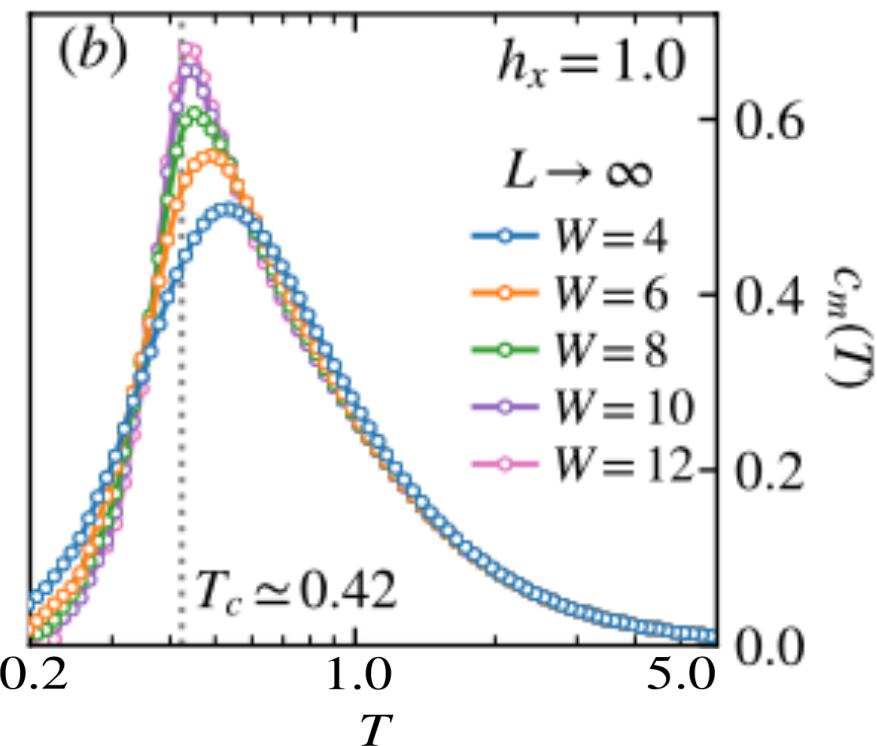
➤ 1.4 transverse-field Ising model

$$H = J \sum_{i,j} S_i^z S_j^z + h \sum_i S_i^x \quad \text{with } J=-1, h=1 \quad T_c = 0.42$$

■ internal energy

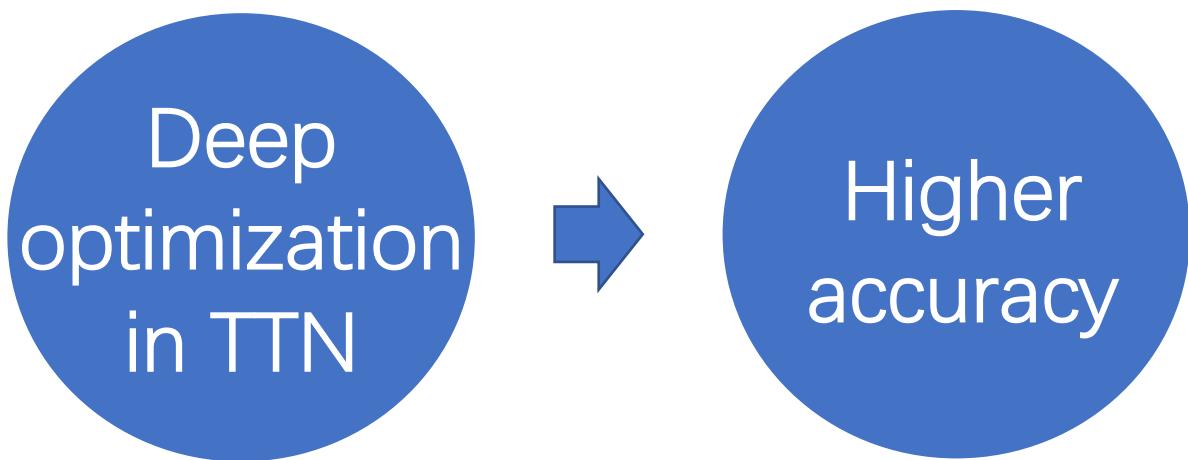


■ specific heat



accurate estimate of transition temperature $\sim 1\%$

➤ Summary of ∂ TRG



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- 1.4 Differentiable tensor renormalization group

□ 2. Application 1

- 2.1 Square-lattice Hubbard model
- 2.2 Triangular-lattice Hubbard model
- 2.3 Magic-angle twisted bilayer graphene model

□ 3. Application 2

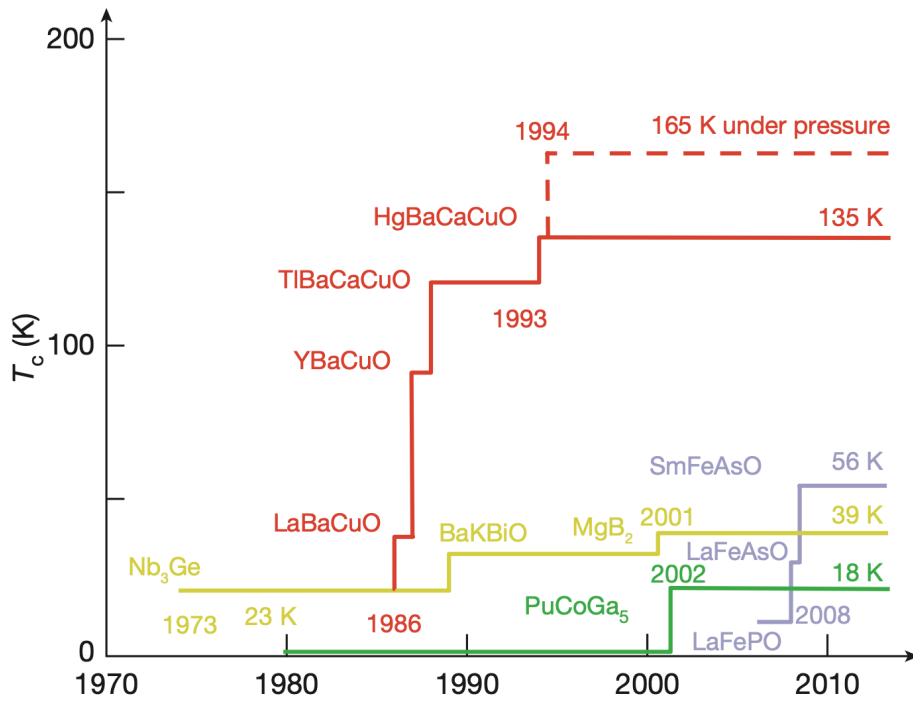
- 3.1 Finite-temperature entanglement spectrum
- 3.2 disorder operator

XTRG study of Finite-T square-lattice Hubbard model

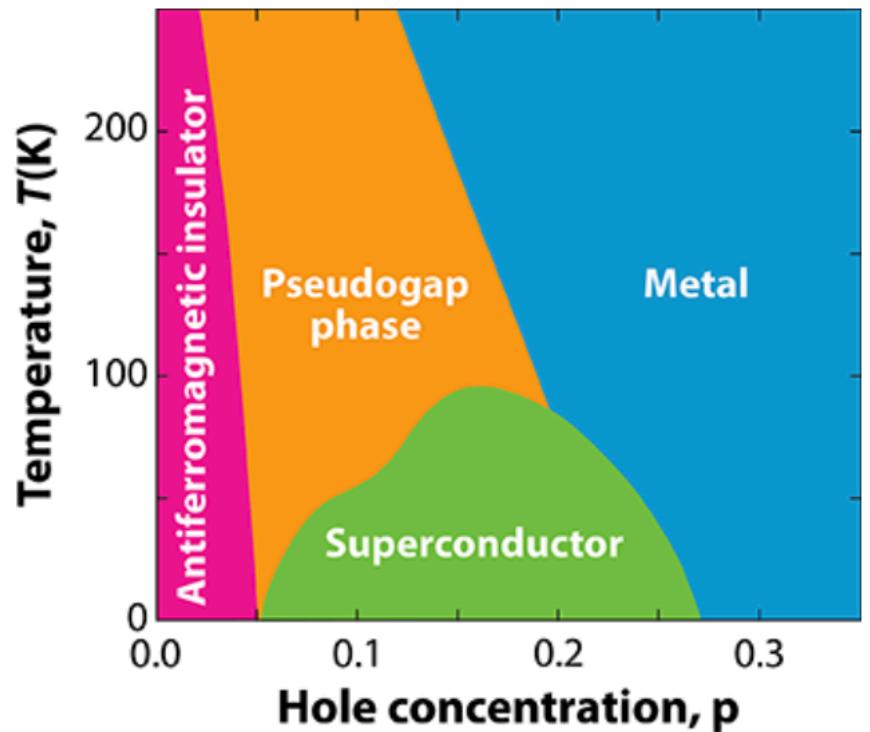
BC, C. Chen, Z. Chen, J. Cui, Y. Zhai, A. Weichselbaum, J. von Delft, Z. Y. Meng, and W. Li. **PRB** 103, L041107

➤ 2.1 High-T_c Superconductivity

- race of highest T_c



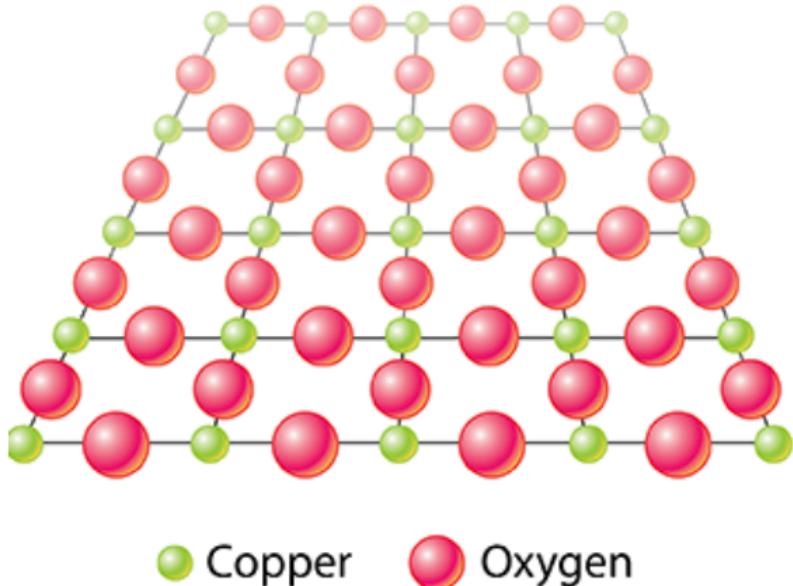
Keimer B, et al. Nature 2015



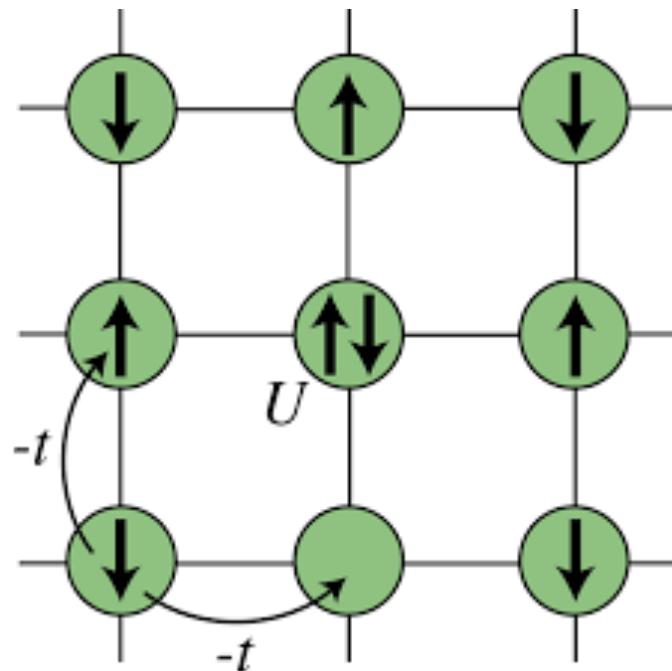
APS/Carin Cain

➤ 2.1 Mechanism of High-Tc ?

- Copper oxide plane



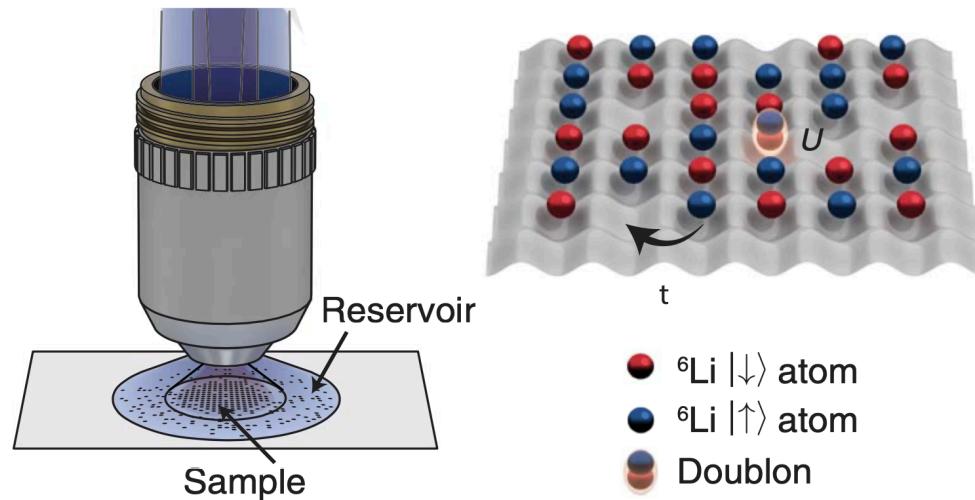
- Single-band Hubbard model



$$H = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

➤ 2.1 Correlated matter: Ultra-cold Fermions

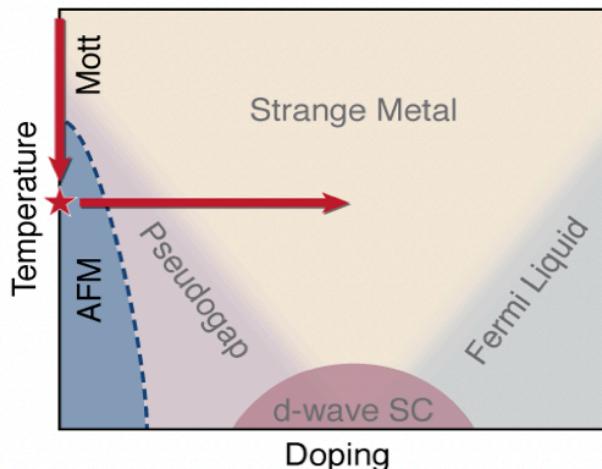
- Realization of Fermi-Hubbard Model



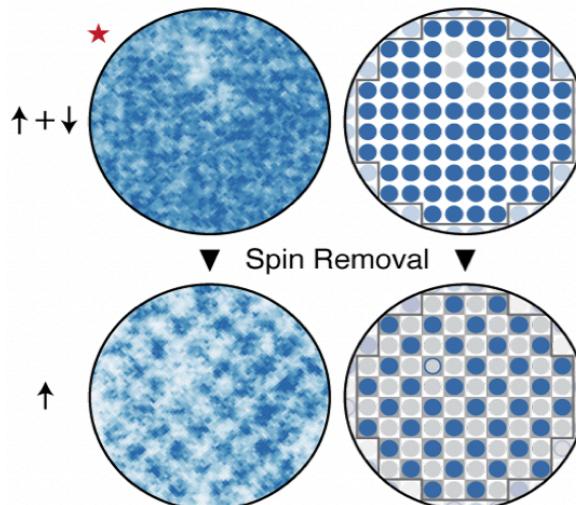
Greiner's group, *Nature* 2017

- ✓ trap fermions in optical lattice
- ✓ tunable interactions
- ✓ single-site resolution

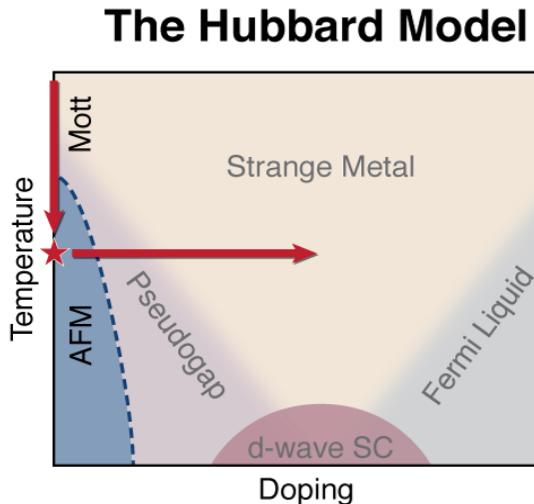
- Current Experimental Scope



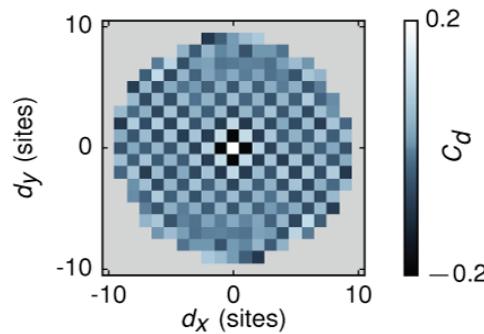
- “Long-range” Antiferromagnet



➤ 2.1 Ultra-cold Fermions in optical lattice

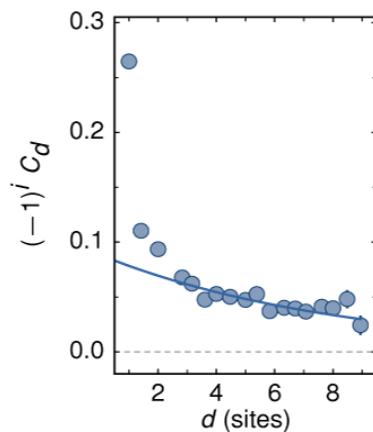
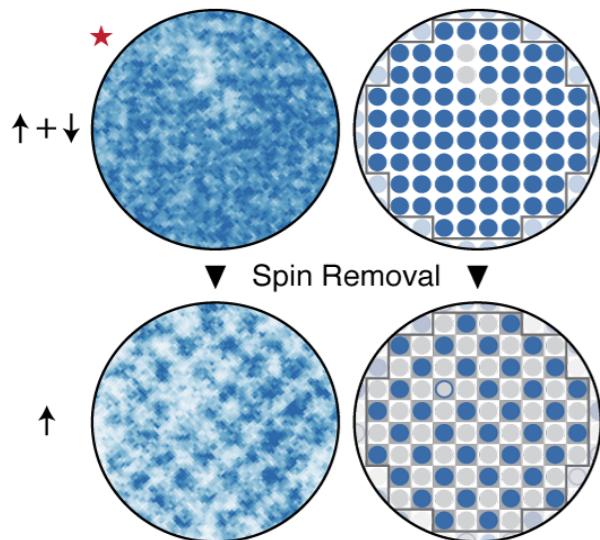


Spin Correlation Function



- fitting & analysis on the raw experimental data
- Determination of temperature and model parameters

Long-range Antiferromagnet



calls for
↓

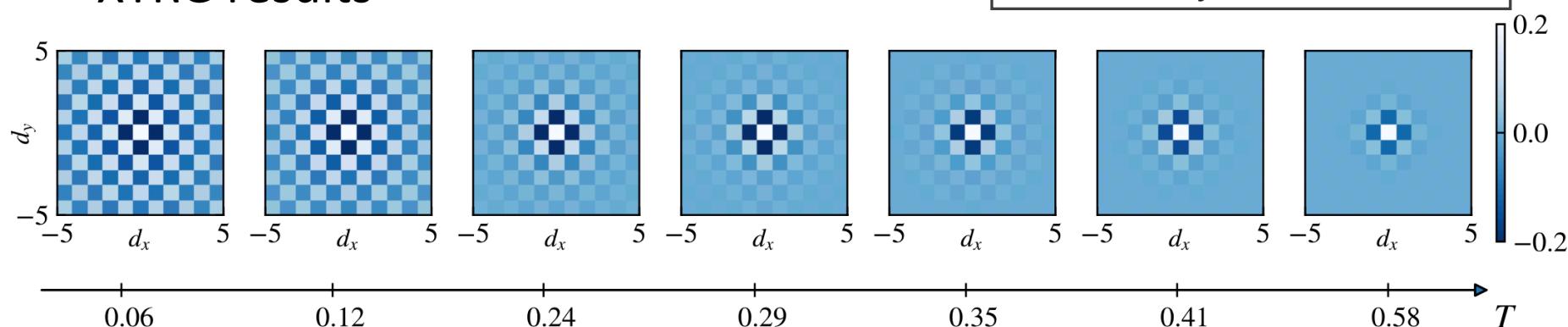
Efficient & accurate
numerical simulations!
(XTRG is promising!)

➤ 2.1 Spin correlation under half-filling

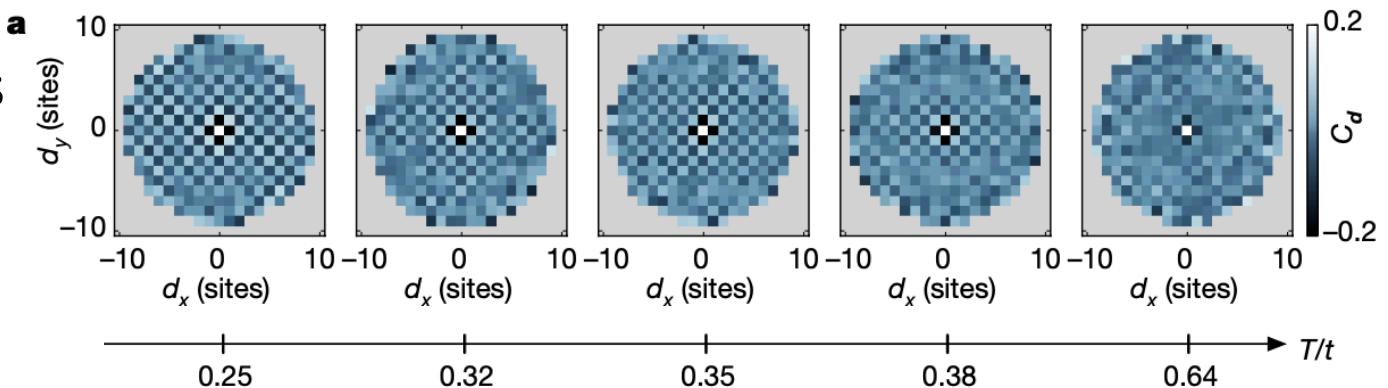
- Definition

$$C_S(d) \equiv \frac{1}{N_d} \sum_{|i-j|=d} \frac{\langle \hat{S}_i \cdot \hat{S}_j \rangle}{S(S+1)}$$

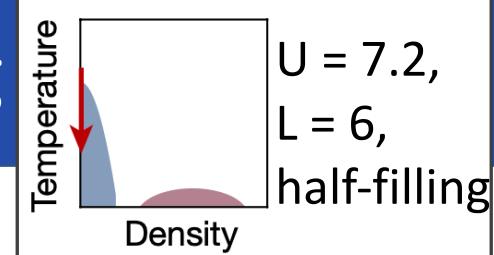
- XTRG results



- Experiments



➤ 2.1 Spin correlation under half-filling

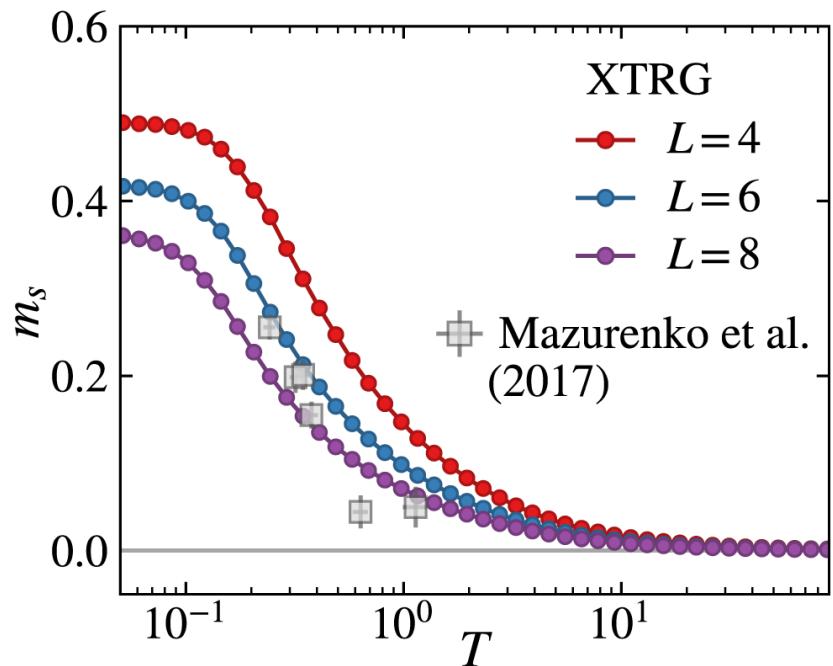
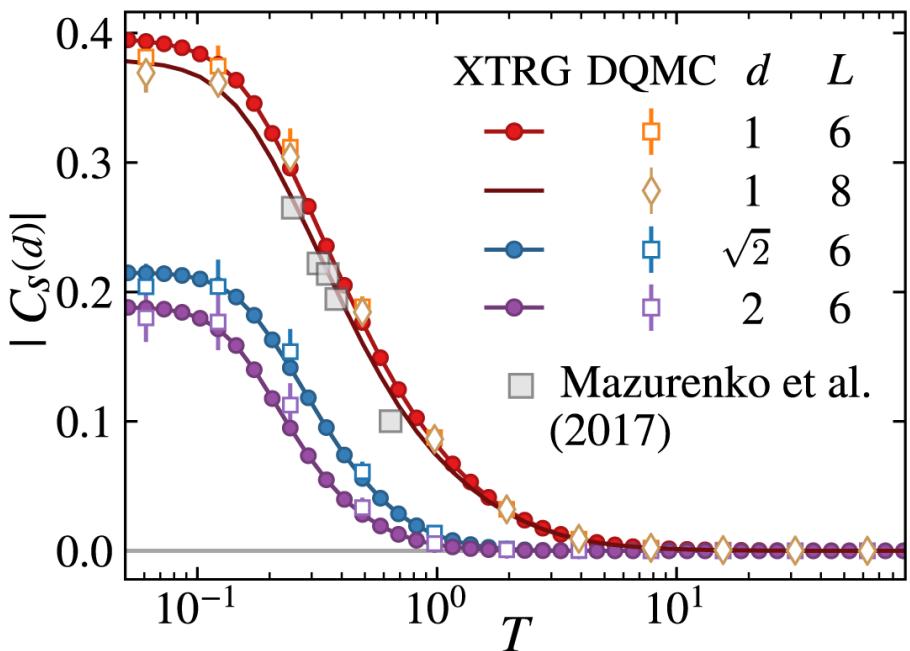


- spin correlation function

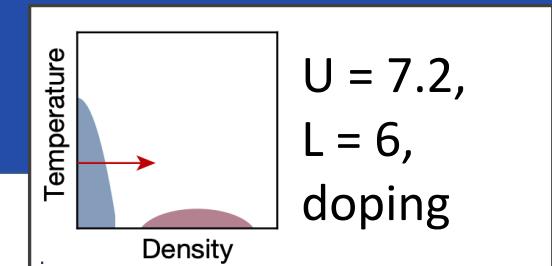
$$C_S(d) \equiv \frac{1}{N_d} \sum_{|i-j|=d} \frac{\langle \hat{S}_i \cdot \hat{S}_j \rangle}{S(S+1)}$$

- spontaneous magnetization

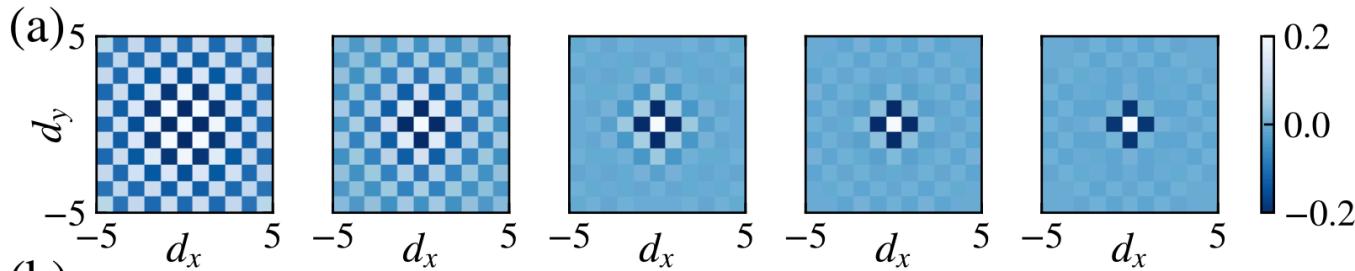
$$m_s \equiv \sqrt{S(\pi, \pi) - S(0)}$$



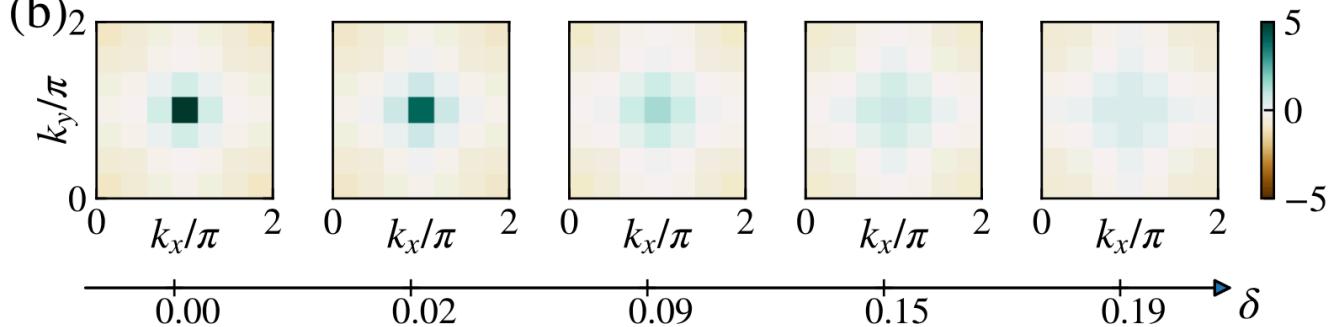
➤ 2.1 Spin correlation upon doping



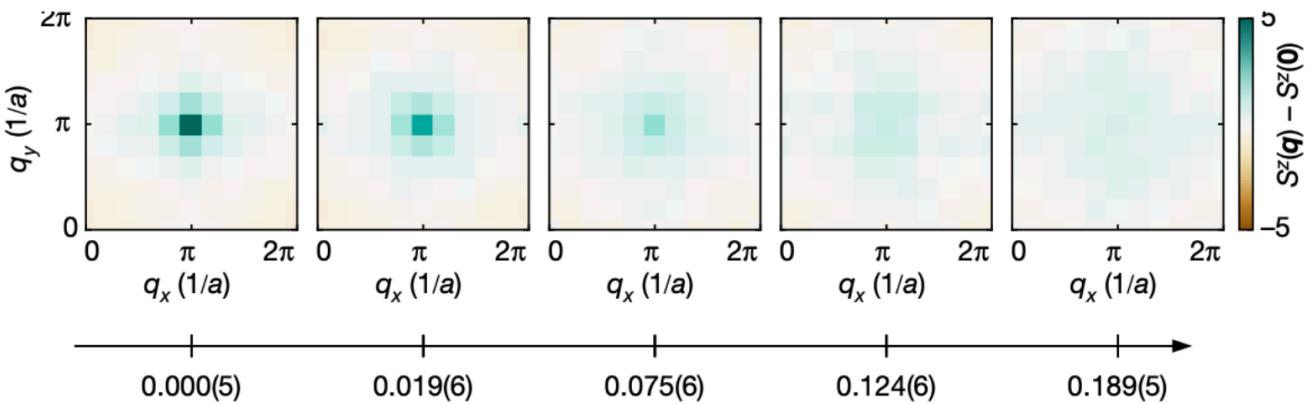
- spin correlation function:



- structure factor:

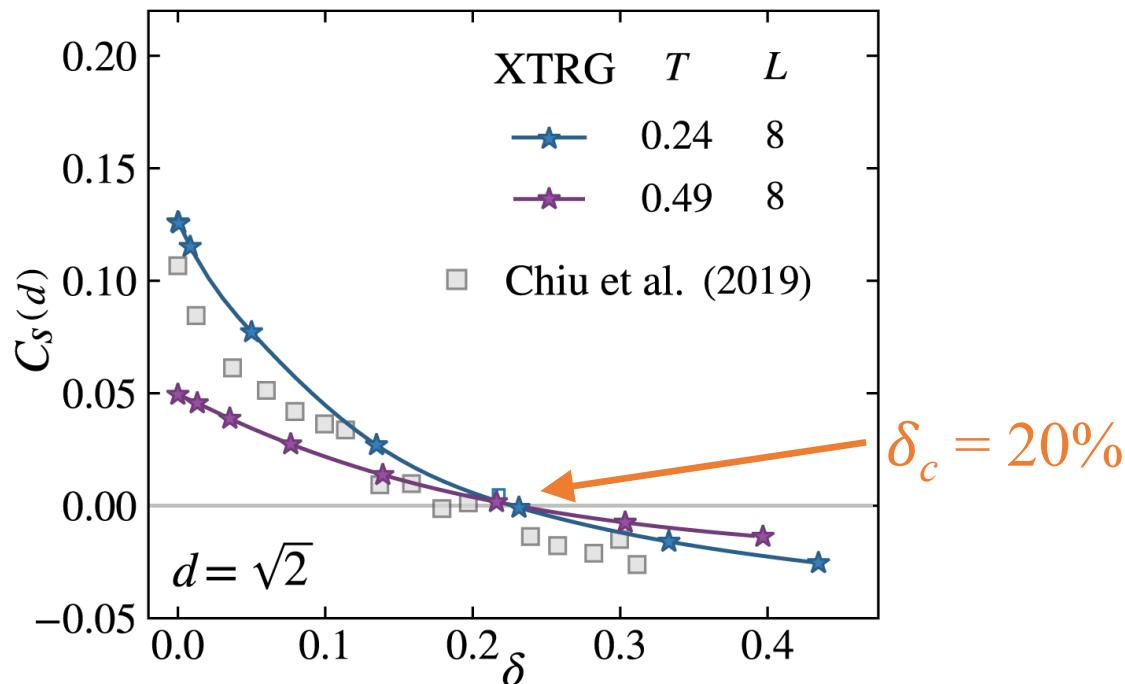
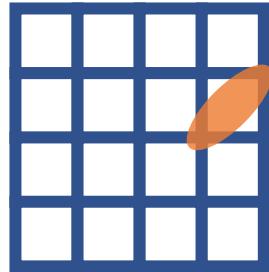


- Experimental results:

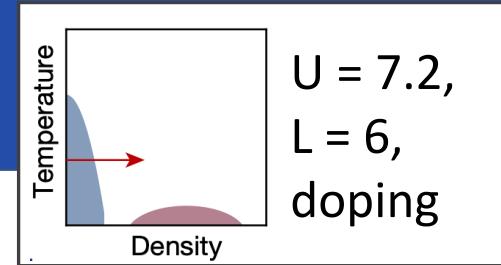
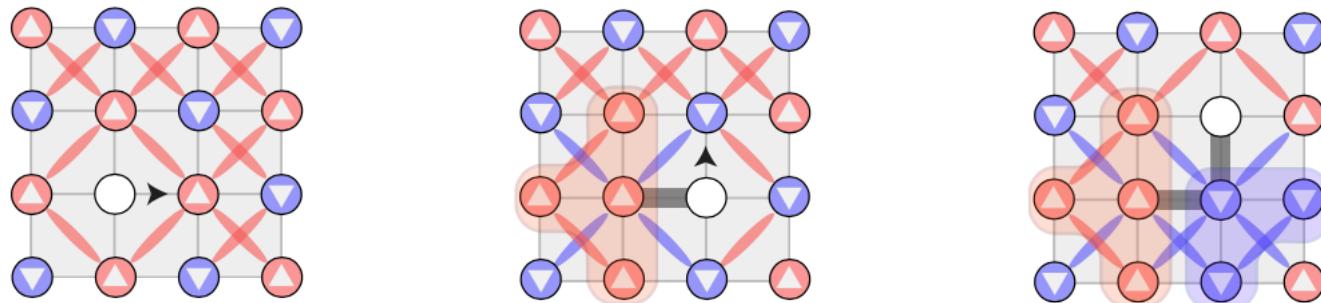


➤ 2.1 Sign-reversal behavior

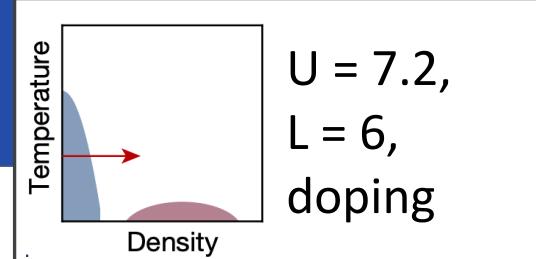
- longer-ranged spin correlation function



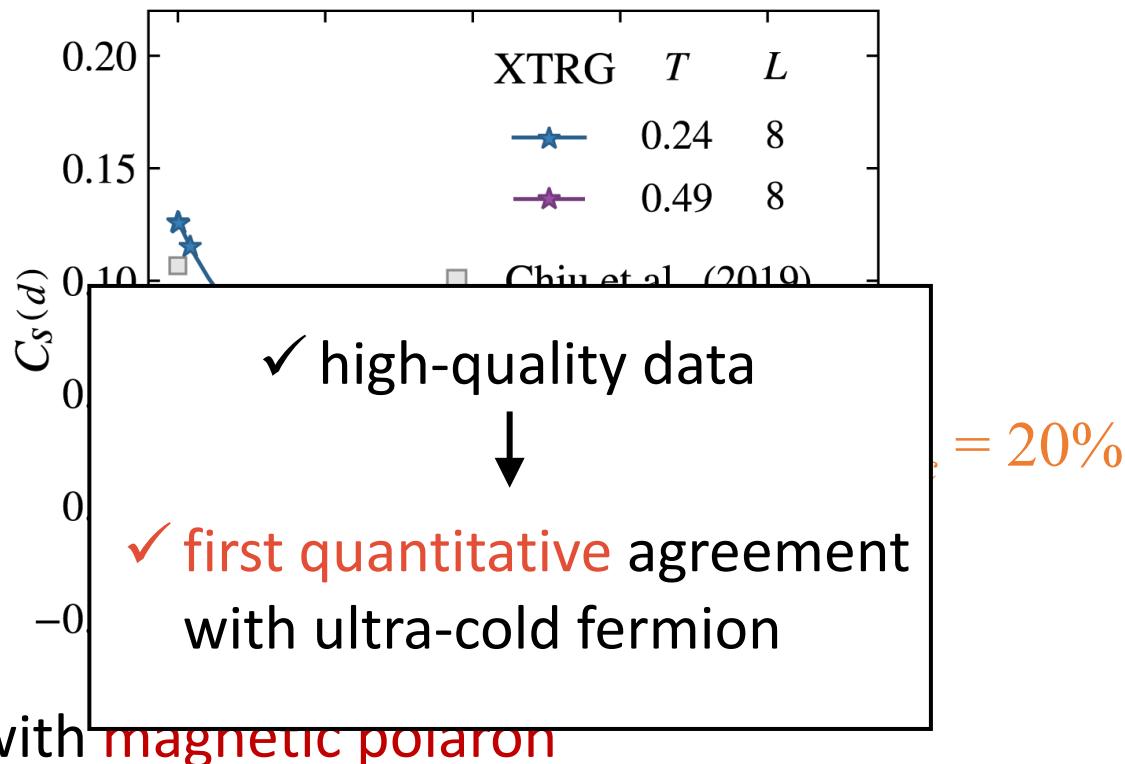
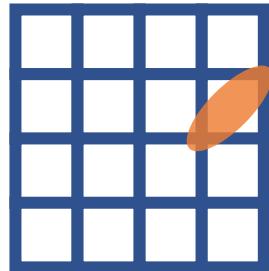
- ✓ consistent with magnetic polaron



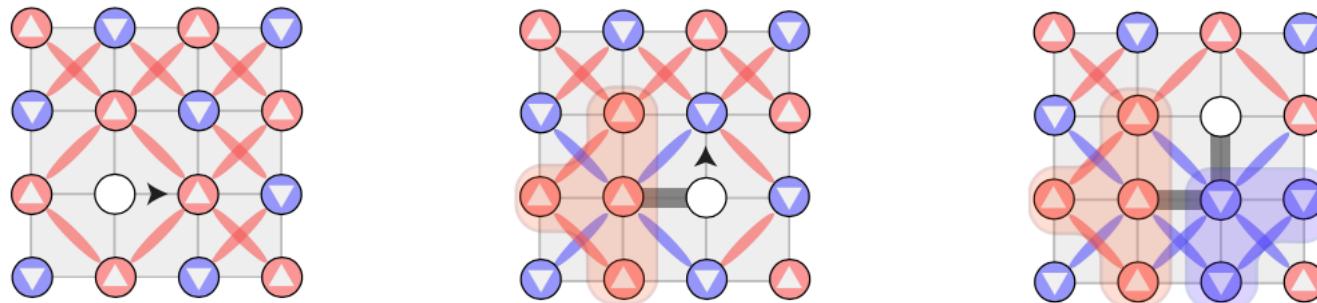
➤ 2.1 Sign-reversal behavior



- longer-ranged spin correlation function



- ✓ consistent with magnetic polaron

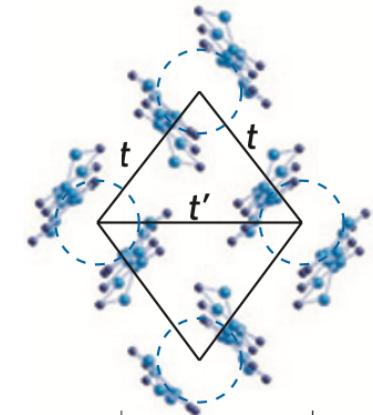
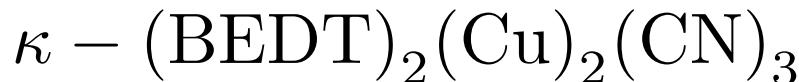


DMRG study of triangular-lattice Hubbard model

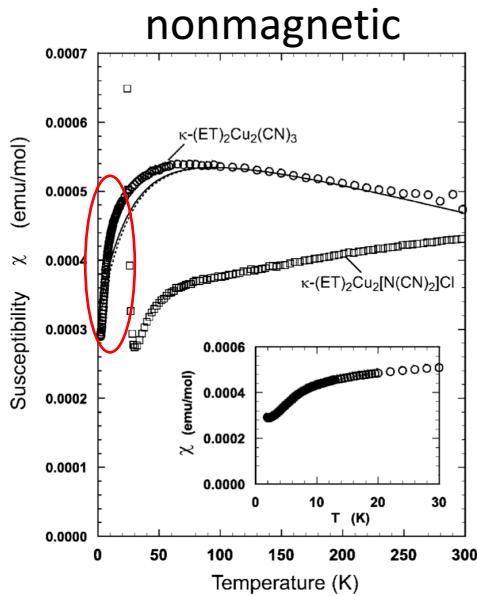
BC, Z. Chen, S.-S. Gong, D. N. Sheng, W. Li, and A. Weichselbaum, **PRB** 106, 094420 (Editors' Suggestion)

➤ 2.2 Motivation from experiments

✓ Organic salt compounds:

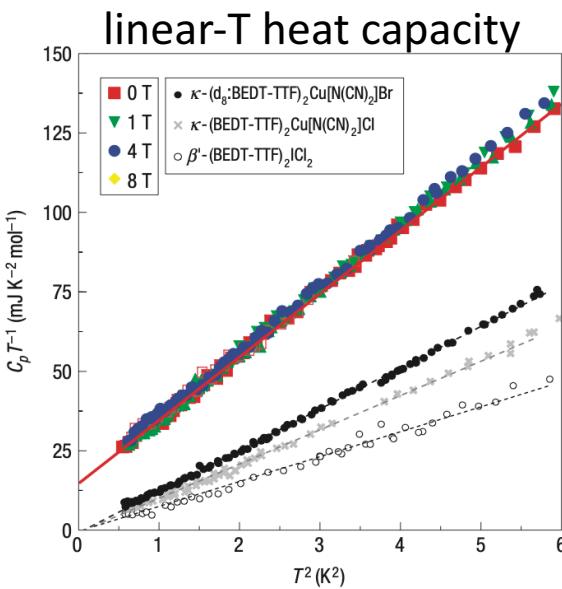


Spin Liquid?



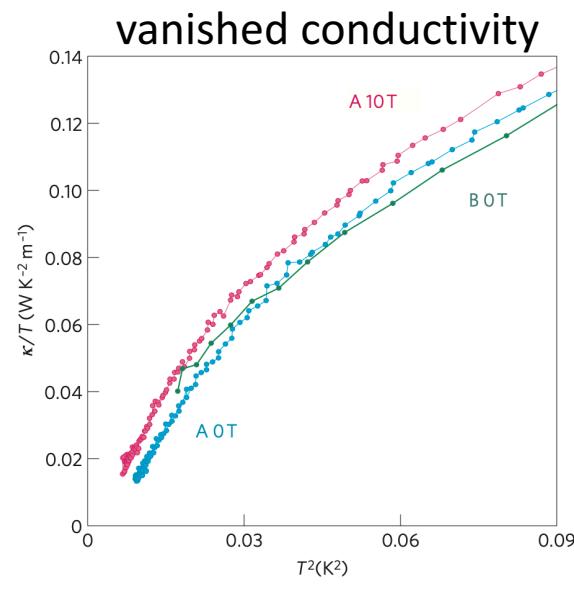
Shimizu et al, PRL 91, 107001 (2003)

Gapless?



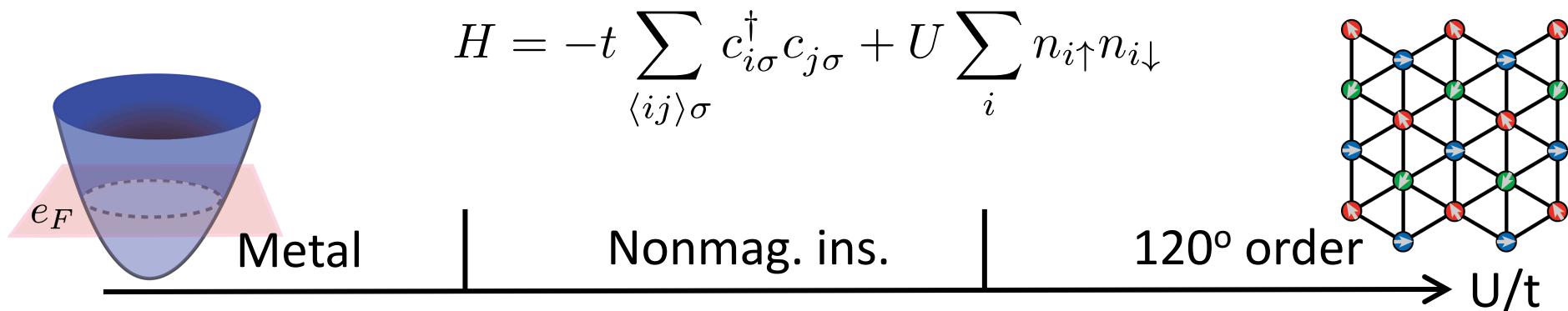
Yamashita et al, Nat. Phys. 4, 459 (2008)

Gapped?



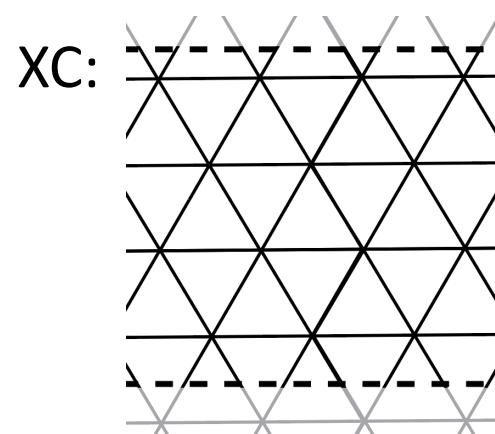
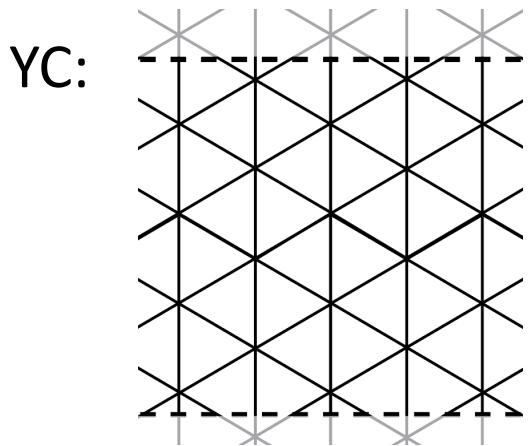
Yamashita et al, Nat. Phys. 5, 44 (2008)

➤ 2.2 Motivation from theoretical studies

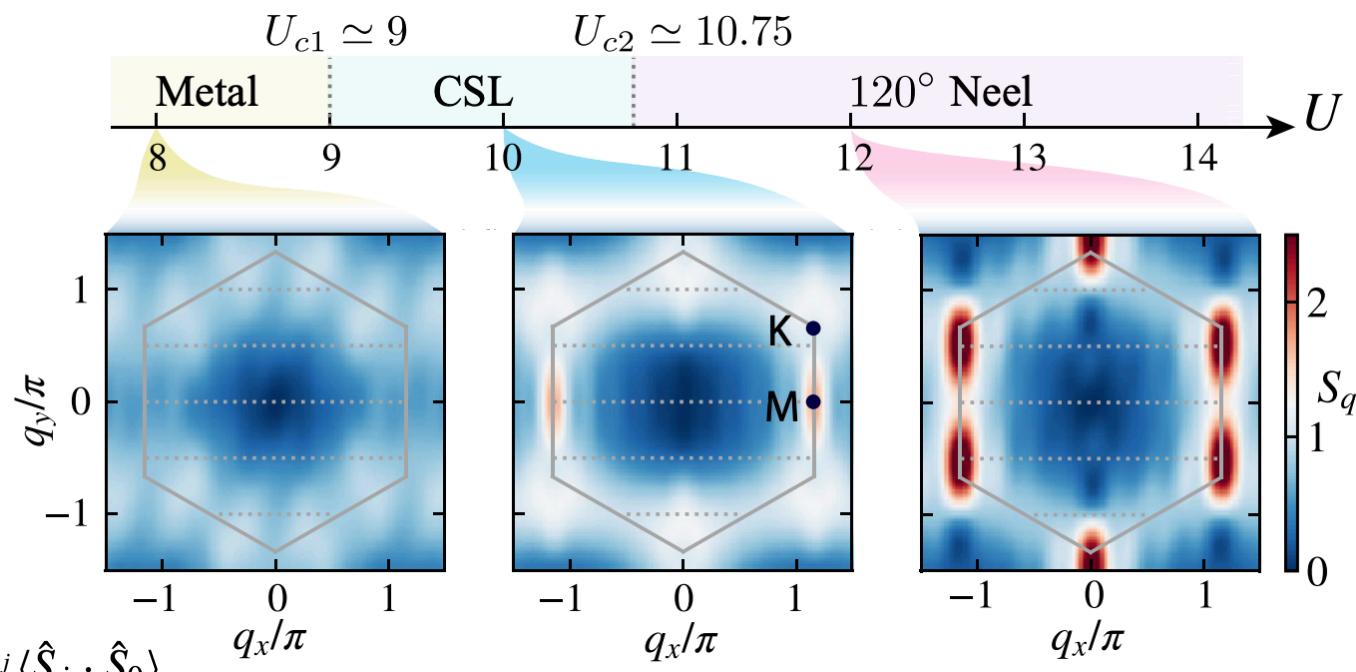


- Nonmagnetic insulator candidates:
 - U(1) spin liquid with spinon Fermi surface --- Motrunich, Phys. Rev. B 72, 045105 (2005)
 - Gapped Z2 spin liquid --- Zhu, Phys. Rev. B 92, 041105 (2015)
 - Nodal gapless spin liquid --- Mishmash, Phys. Rev. Lett. 111, 157203 (2015)
 -
- DMRG Results:
 - 2-leg ladder: U(1) gapless
Mishmash, PRB 91, 235140 (2015)
 - Finite DMRG: gapped
Shirakawa, PRB 96, 205130 (2017)
 - infinite DMRG: chiral spin liquid
Szasz, PRX 10, 021042 (2020)

➤ 2.2 Model and Main Results (Phase diagram)



$$H = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

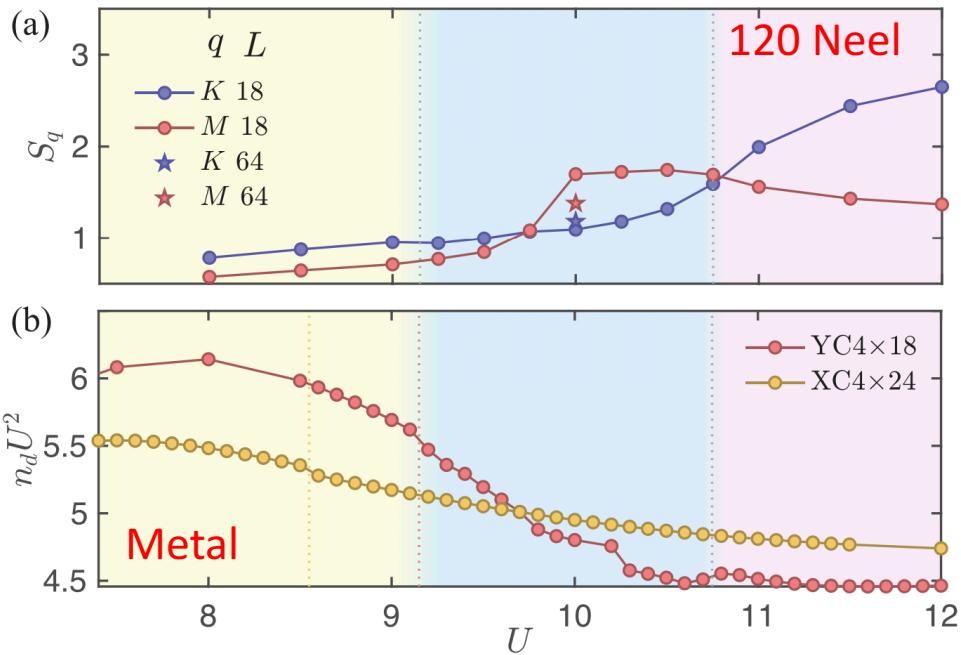


$$S_q \equiv \sum_j e^{iq \cdot R_{0j}} \langle \hat{S}_j \cdot \hat{S}_0 \rangle$$

2.2 Phase diagram

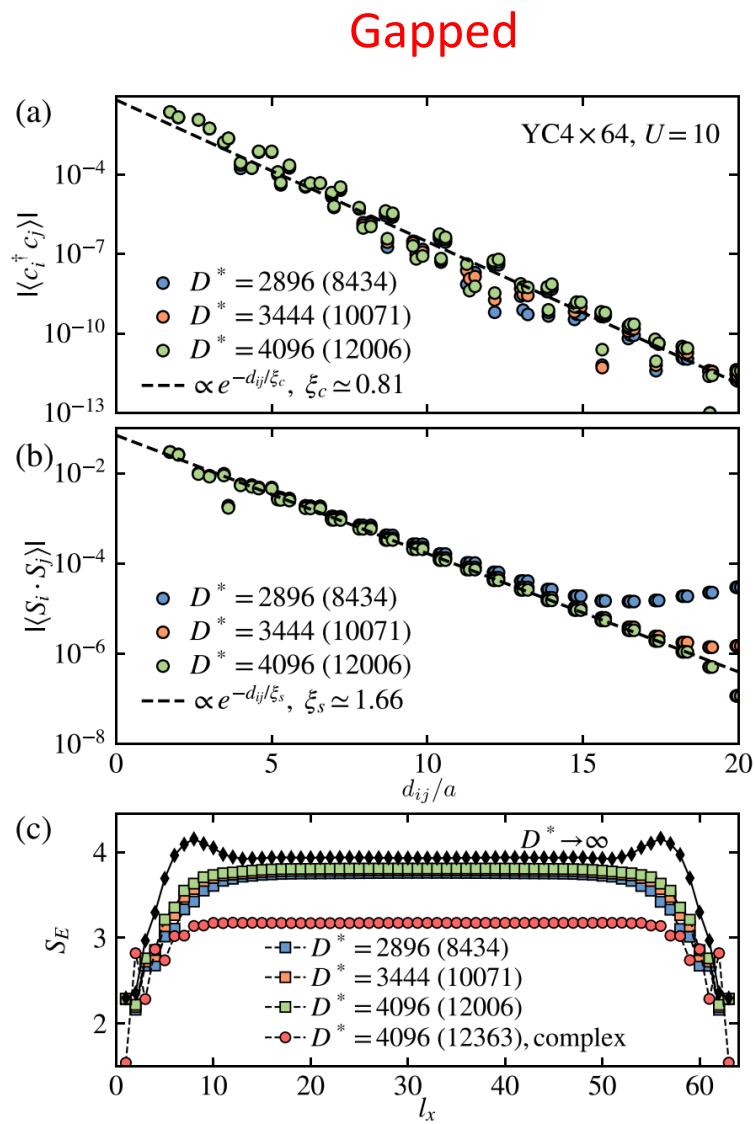
✓ static spin structure factor

$$S_q \equiv \sum_j e^{iq \cdot R_0 j} \langle \hat{S}_j \cdot \hat{S}_0 \rangle$$



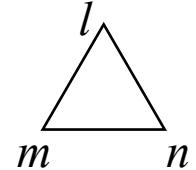
✓ double occupancy

$$n_d \equiv \langle \psi | \hat{n}_{0\uparrow} \hat{n}_{0\downarrow} | \psi \rangle$$

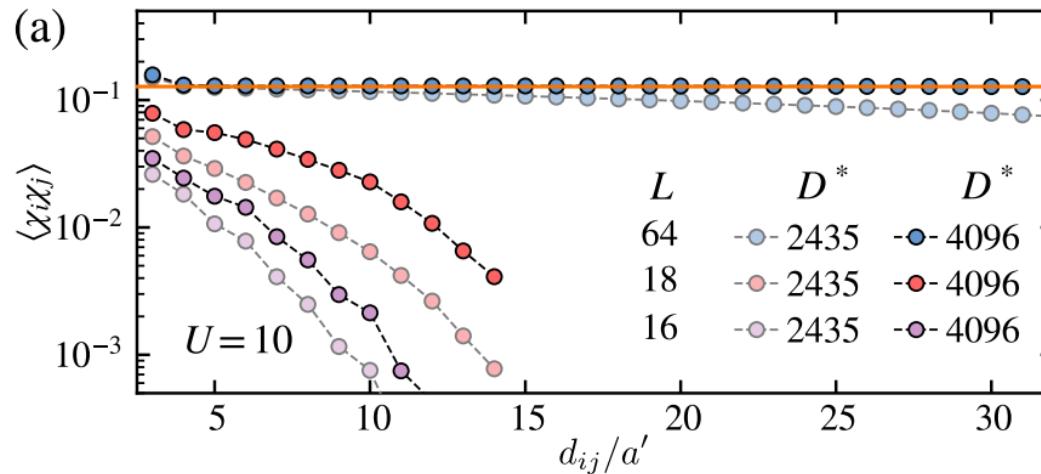


➤ 2.2 Chiral Spin Liquid: TRS-breaking

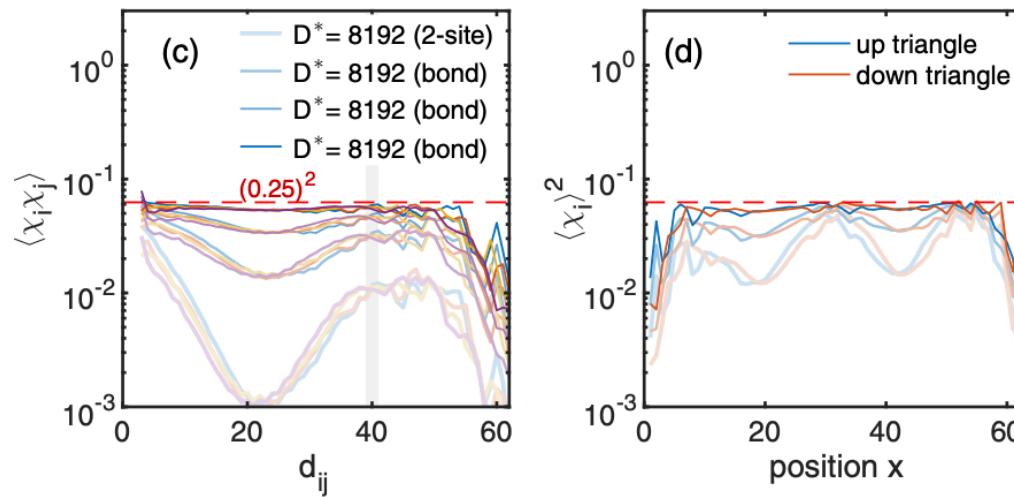
✓ chiral correlation/order: $\langle \chi_i \chi_j \rangle$ and $\langle \chi_i \rangle$ with $\chi_i = \sigma_l \times \sigma_m \cdot \sigma_n$



YC4 Data:

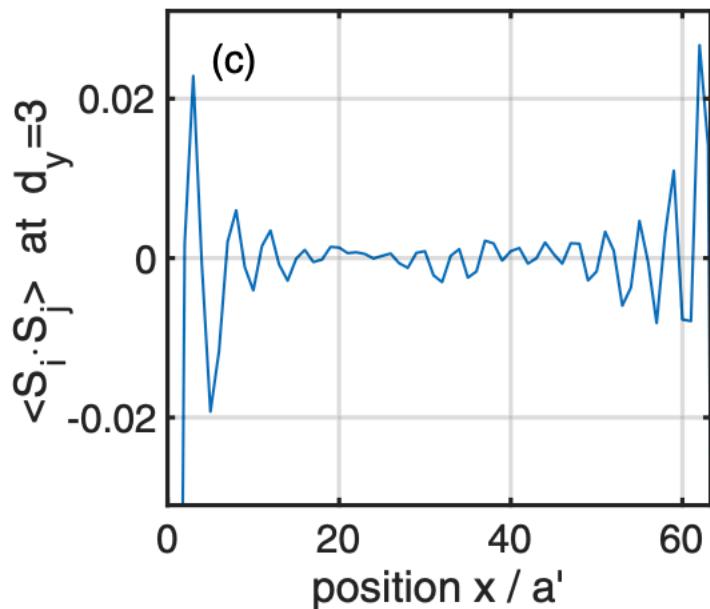


YC6 Data:
($L=64$)

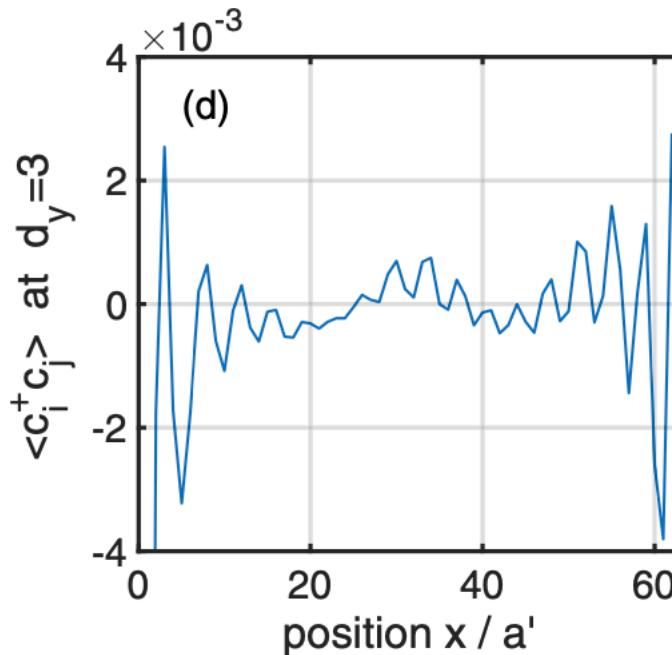


➤ 2.2 CSL: On the need for long cylinder

- spin correlation



- charge correlation

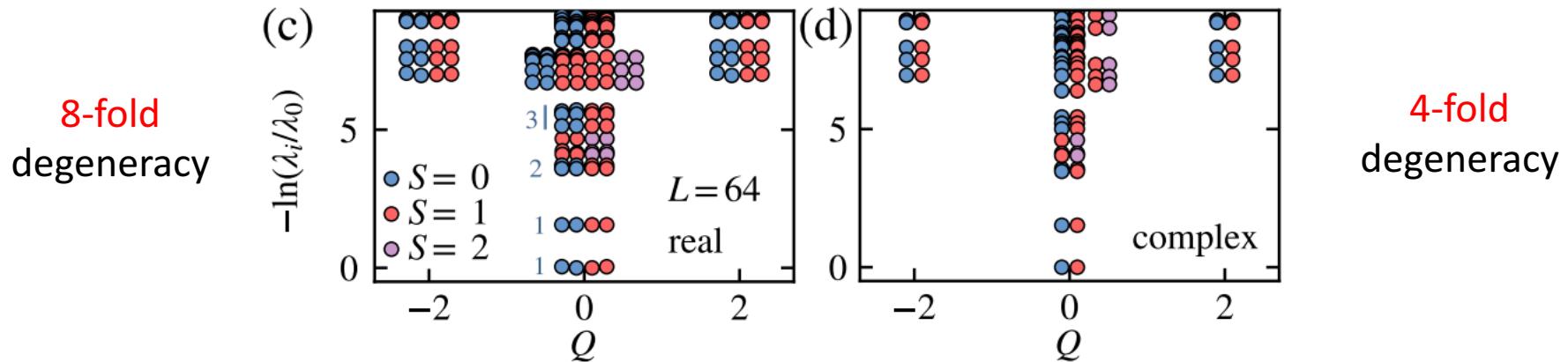


- ✓ Strong boundary oscillation (slow decay ~20 columns)
- ✓ $L > 40$ is needed.

We note that, $L = 8$ in [Shirakawa, PRB 96, 205130 (2017)]

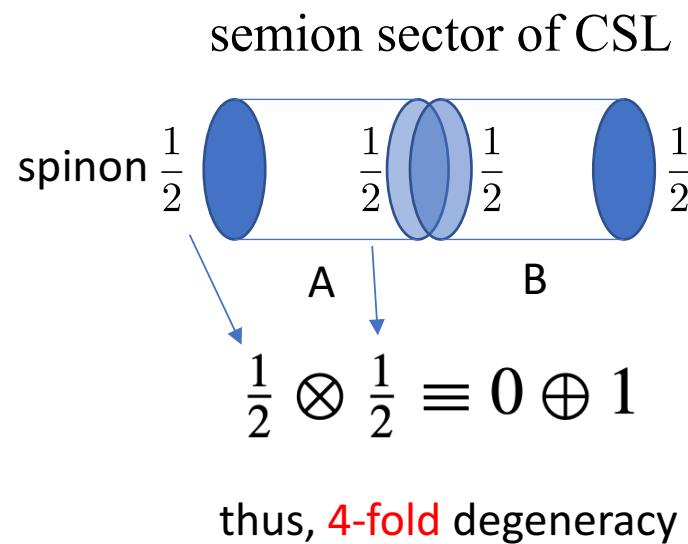
➤ 2.2 Chiral Spin Liquid: chiral edge mode

✓ Entanglement spectrum: SU(2) level-1 WZW CFT (1,1,2,3,5,...)



block diagonal with conserved charge Q
and total spin S

$$\rho^A = \begin{bmatrix} & & \\ (Q_1, S_1) & & \\ & (Q_2, S_2) & \\ & \dots & \\ & (Q_i, S_i) & \end{bmatrix}$$



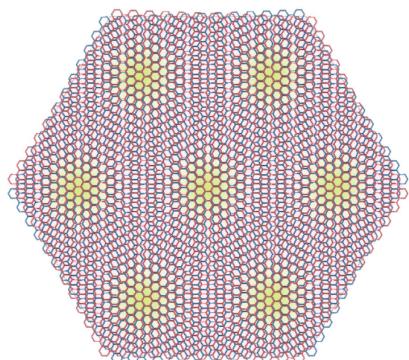
5 min break; will be back at 3 pm

Ground state of twisted bilayer graphene model

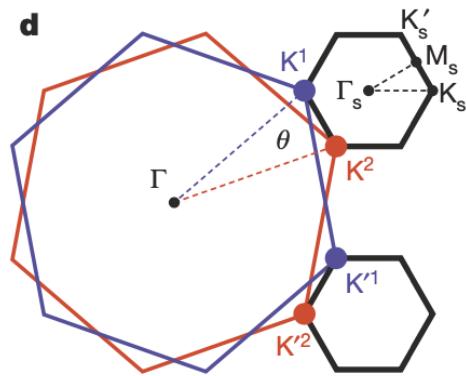
BC, Y. D. Liao, Z.Chen, O. Vafek, J. Kang, W. Li, and Z. Y. Meng. Nat. Commun. 12, 5480 (2021)

➤ 2.3 Magic-Angle Twisted Bilayer Graphene

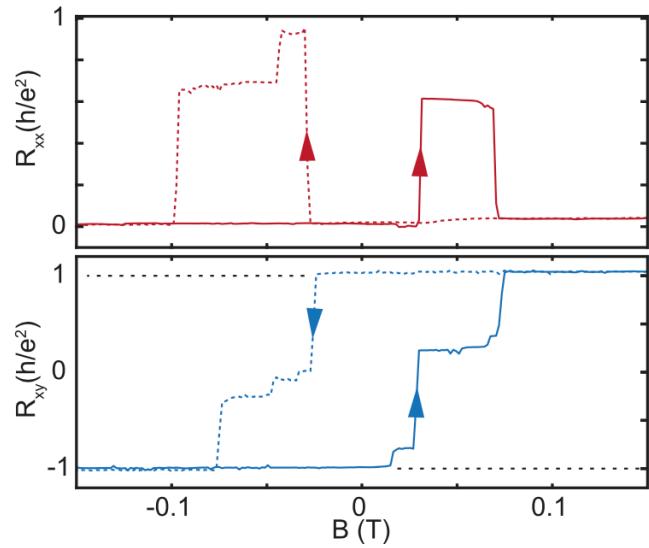
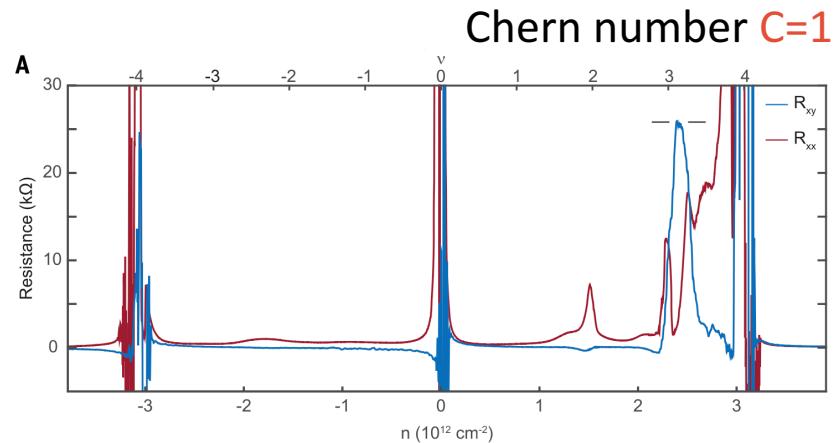
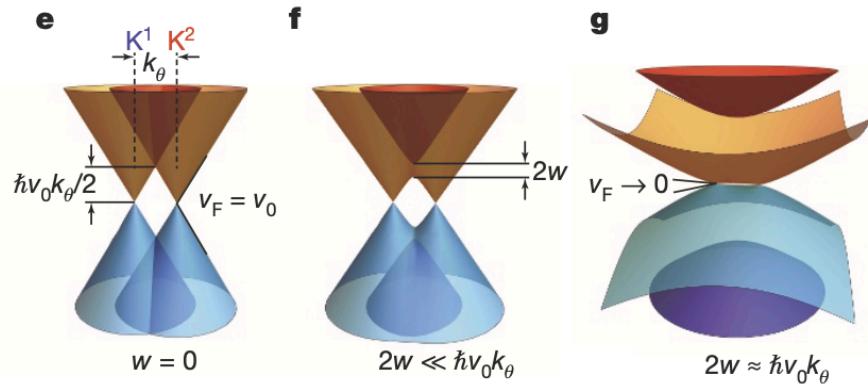
- Twisted Bilayer Graphene (TBG)
- QAH at 3/4-filling



Moiré pattern



- Narrow band at magic angle



New playground for strongly correlated states

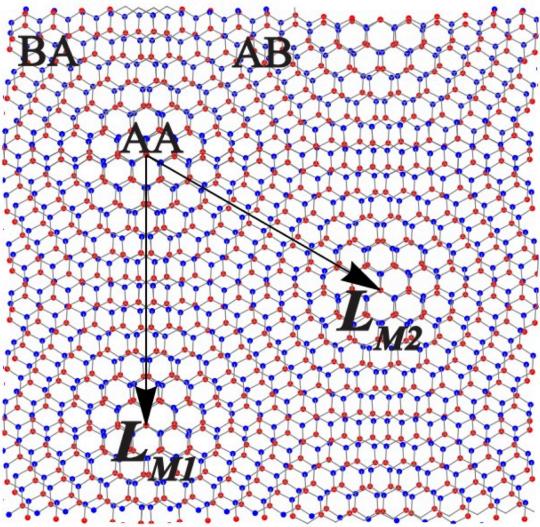
Cao, Nature 556, 43 (2018)

Cao, Nature 556, 80 (2018)

Serlin et al., Science 2020

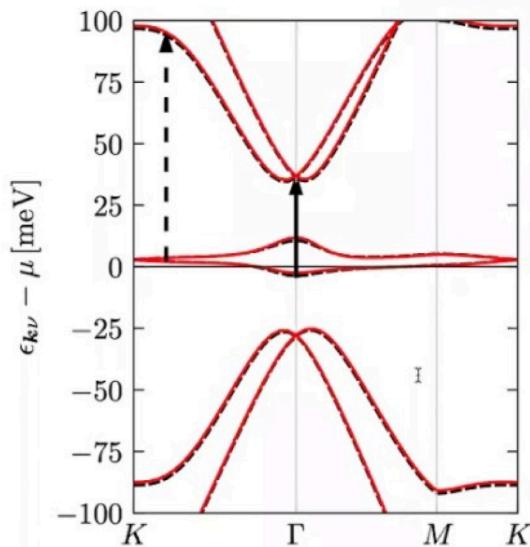
➤ 2.3 Magic-Angle Twisted Bilayer Graphene

✓ Moire Lattice



•Yi Zhang et al, Phys. Rev. B 102, 035136 (2020)

✓ flat bands



8 narrow bands:

2 valleys

x

2 spins

x

2 bands

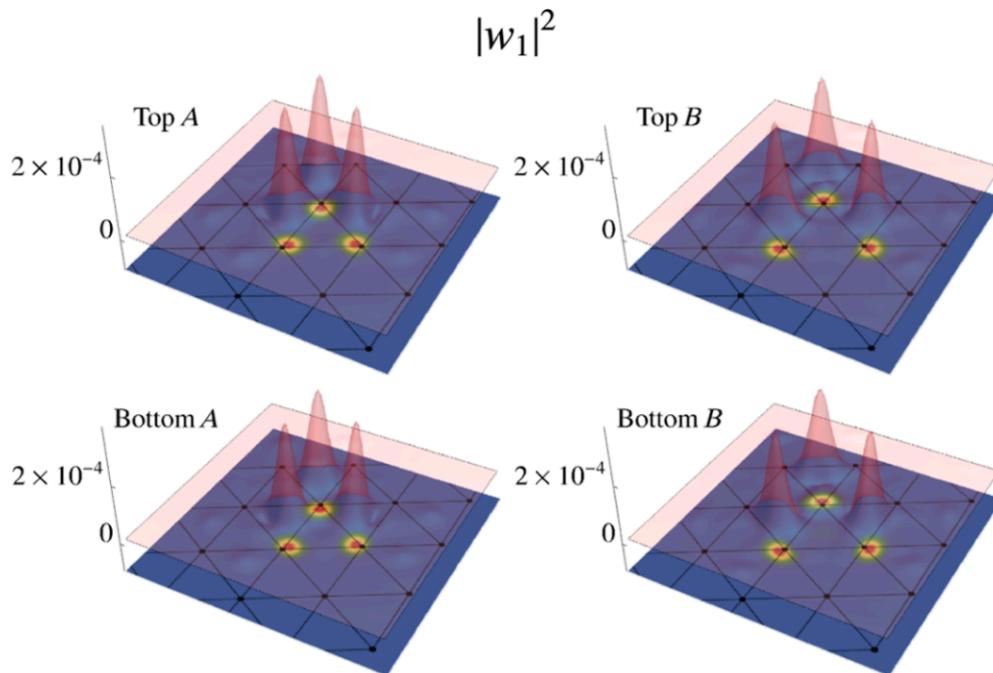


✓ Coulomb interactions between electrons in narrow bands:

$$U = \frac{1}{2} \sum_{r,r'} \sum_{\sigma,\sigma'} \rho_\sigma(r) V(r - r') \rho_{\sigma'}(r')$$

➤ 2.3 Real-space effective model

- ✓ localized wannier function



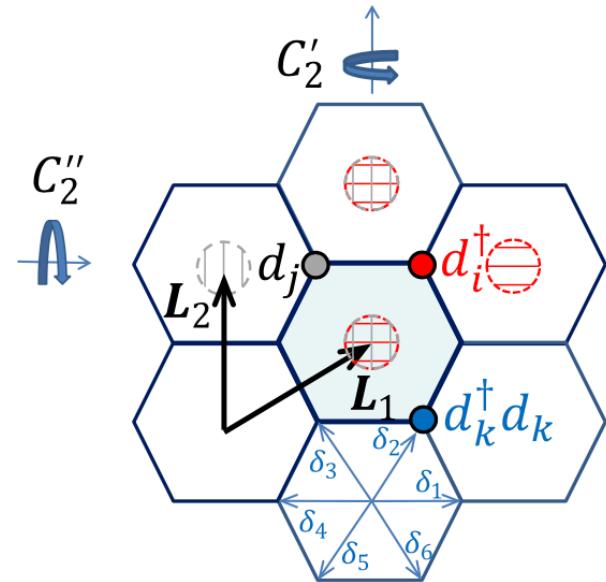
- ✓ 3-peak structure

- ✓ centered at the honeycomb lattice

Projecting onto these Wannier basis:

$$c_\sigma(r) = \frac{1}{3} \sum_R \sum_{p=1}^6 \sum_{j=\pm 1} w_{R+\delta_p, j}(r) d_{j,\sigma}(R + \delta_p)$$

$$U = \frac{1}{2} \sum_{r,r'} \sum_{\sigma,\sigma'} \rho_\sigma(r) V(r - r') \rho_{\sigma'}(r')$$



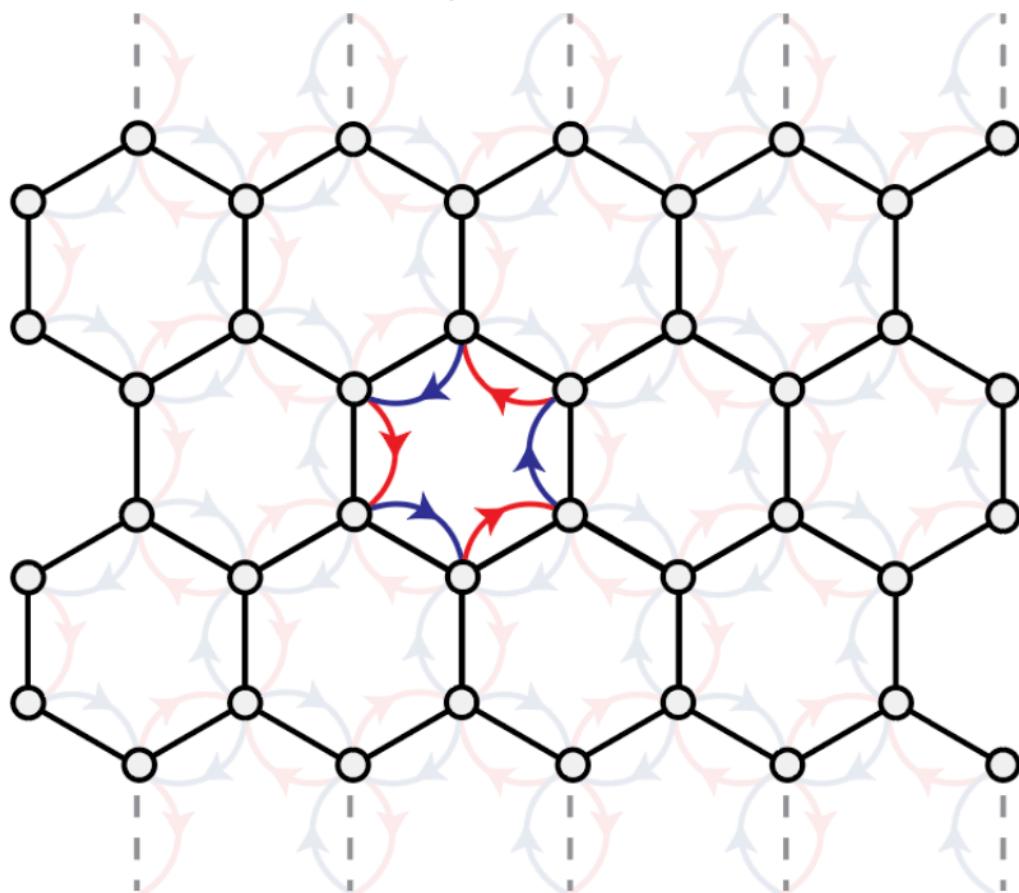
$$U = \frac{V_0}{2} \sum_R \left(\sum_{j,\sigma} O_{j,\sigma}(R) \right)^2$$

$$\text{With } O_{j,\sigma}(R) = Q_{j,\sigma}(R) + \alpha T_{j,\sigma}(R)$$

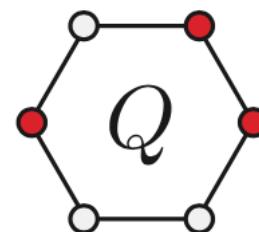
➤ 2.3 TBG Model

- Hubbard-like Model

$$H = U_0 \sum_{\text{hex}} (Q_{\text{hex}} + \alpha T_{\text{hex}} - 1)^2$$

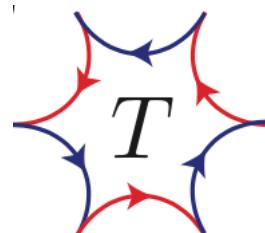


- Cluster charge



$$Q_{\text{hex}} = \sum_{i \in \text{hex}} \hat{n}_i / 3$$

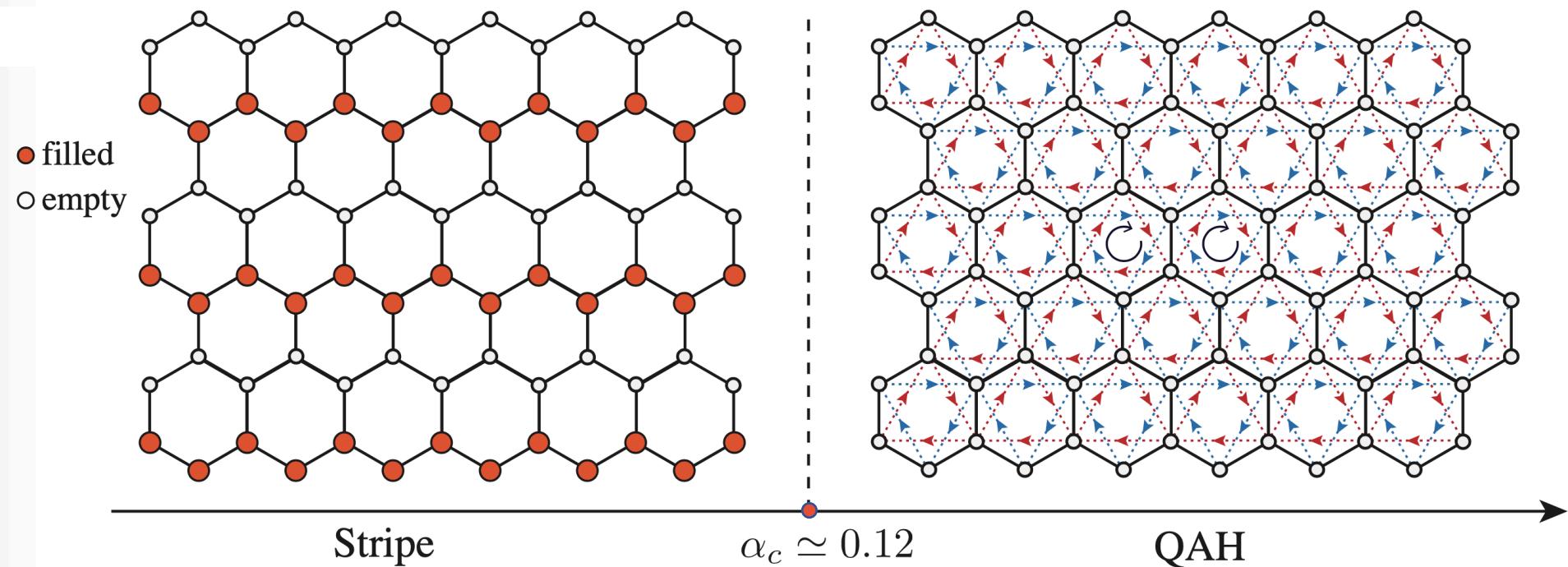
- Hopping Assisted interaction



$$T_{\text{hex}} = \sum_{j \in \text{hex}} [(-1)^j c_{j+1}^\dagger c_j + h.c.]$$

➤ 2.3 Ground-state phase diagram

- TBG Model $H = U_0 \sum_{\text{hex}} (Q_{\text{hex}} + \alpha T_{\text{hex}} - 1)^2$

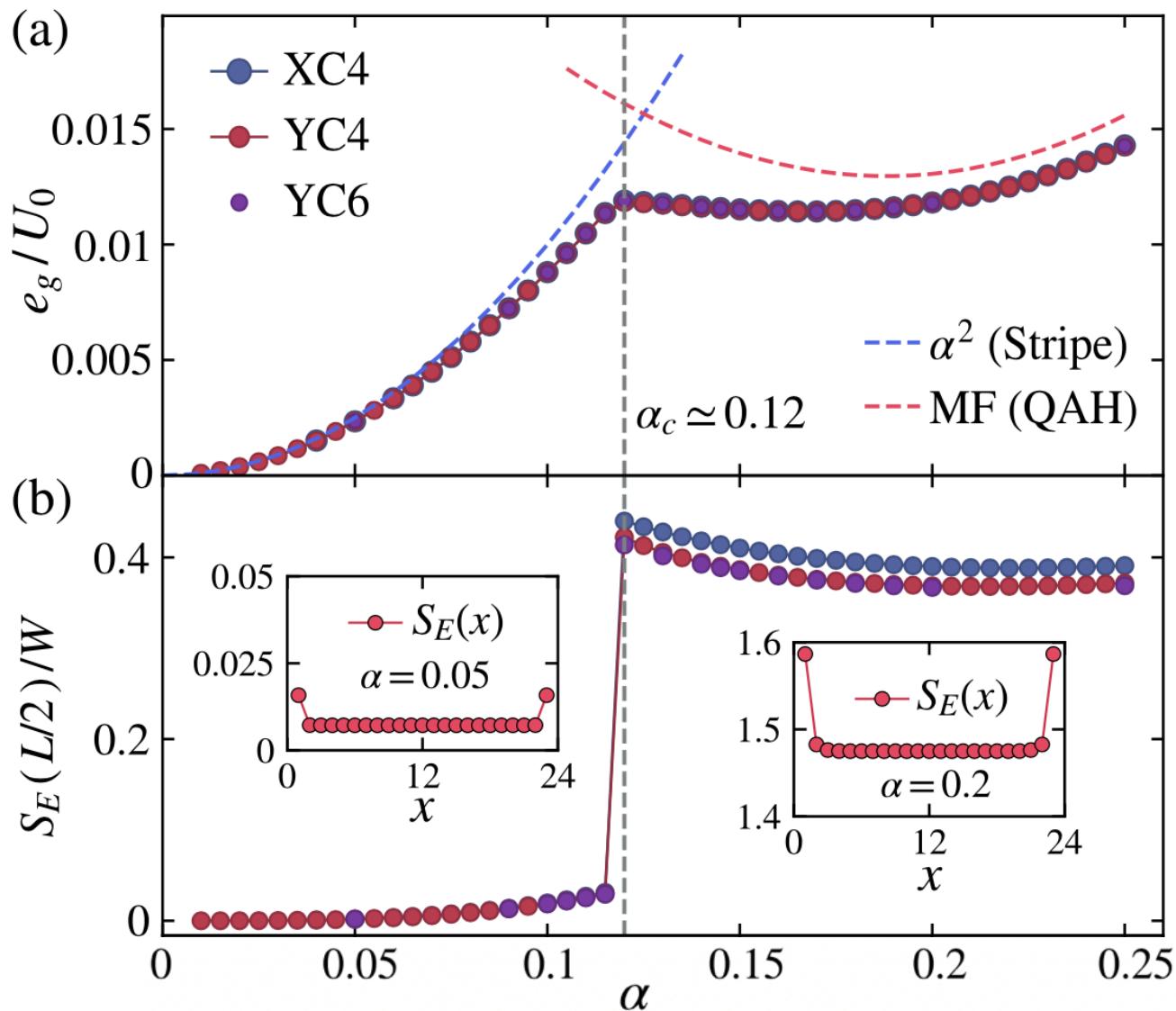


➤ 2.3 Magic-angle twisted bilayer graphene

- Ground-state energy
- Entanglement entropy

first-order transition

Both Gapped

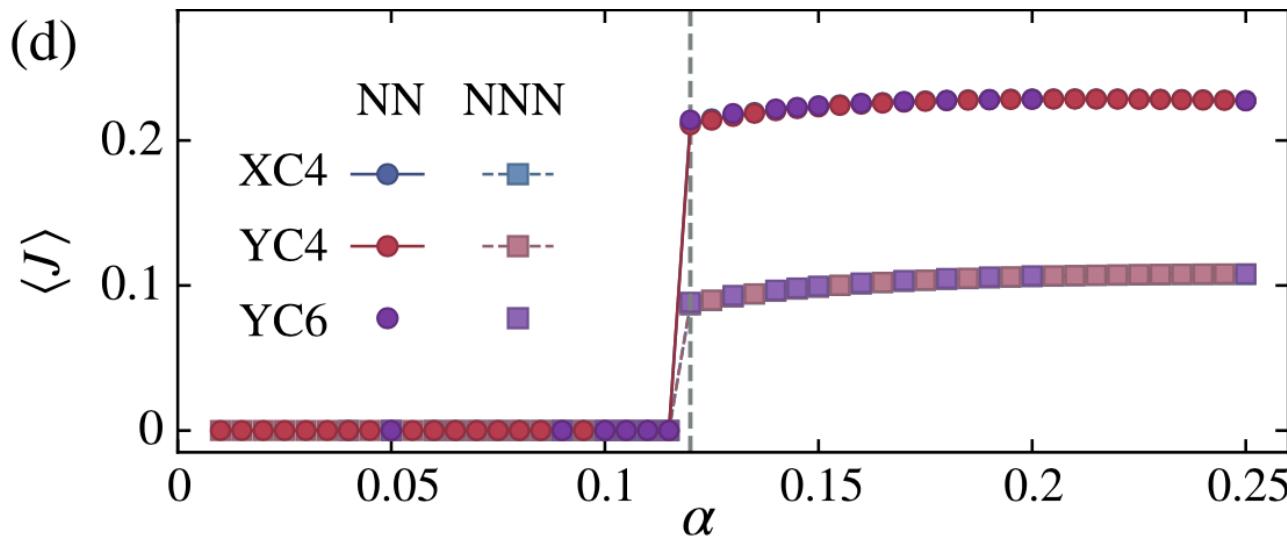


➤ 2.3 Identify QAH

- ❑ Spontaneous time-reversal symmetry (TRS) breaking?
 - ❑ Loop current?
 - ❑ non-zero Chern number?
-

✓ Spontaneous TRS breaking

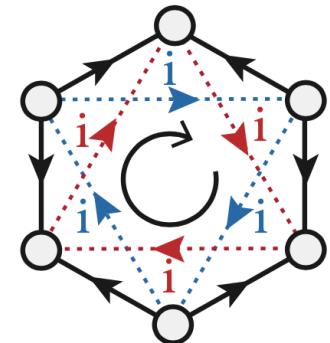
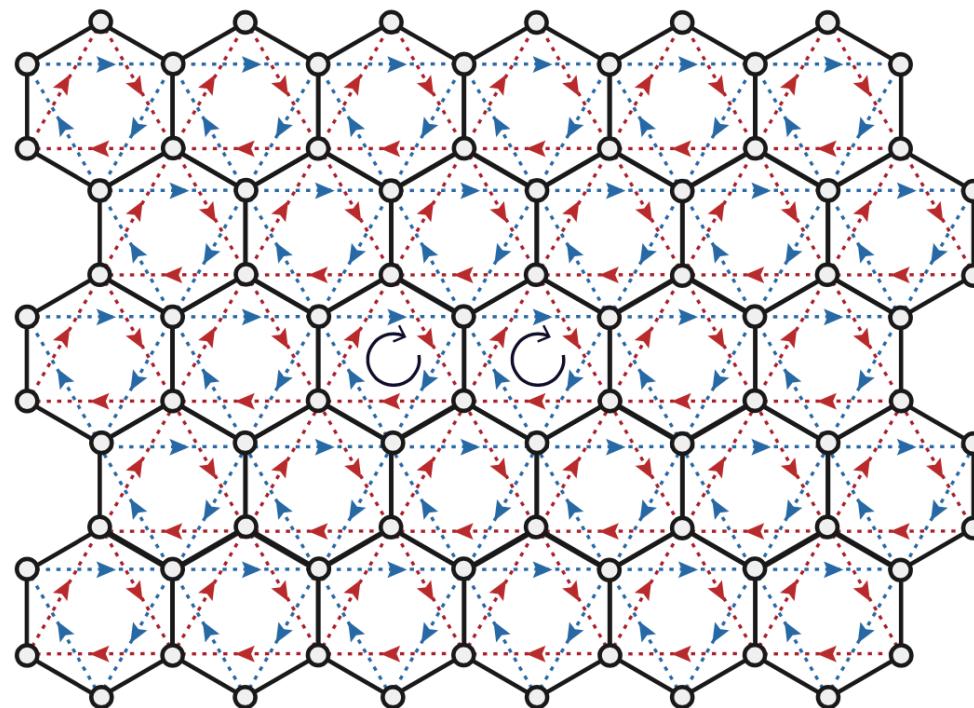
- current operator $J \equiv i(c_i^\dagger c_j - c_j^\dagger c_i)$



➤ 2.3 Identify QAH

- ✓ Spontaneous time-reversal symmetry breaking
 - ☐ Loop current?
 - ☐ non-zero Chern number?
-

- ✓ Loop current $J \equiv i(c_i^\dagger c_j - c_j^\dagger c_i)$



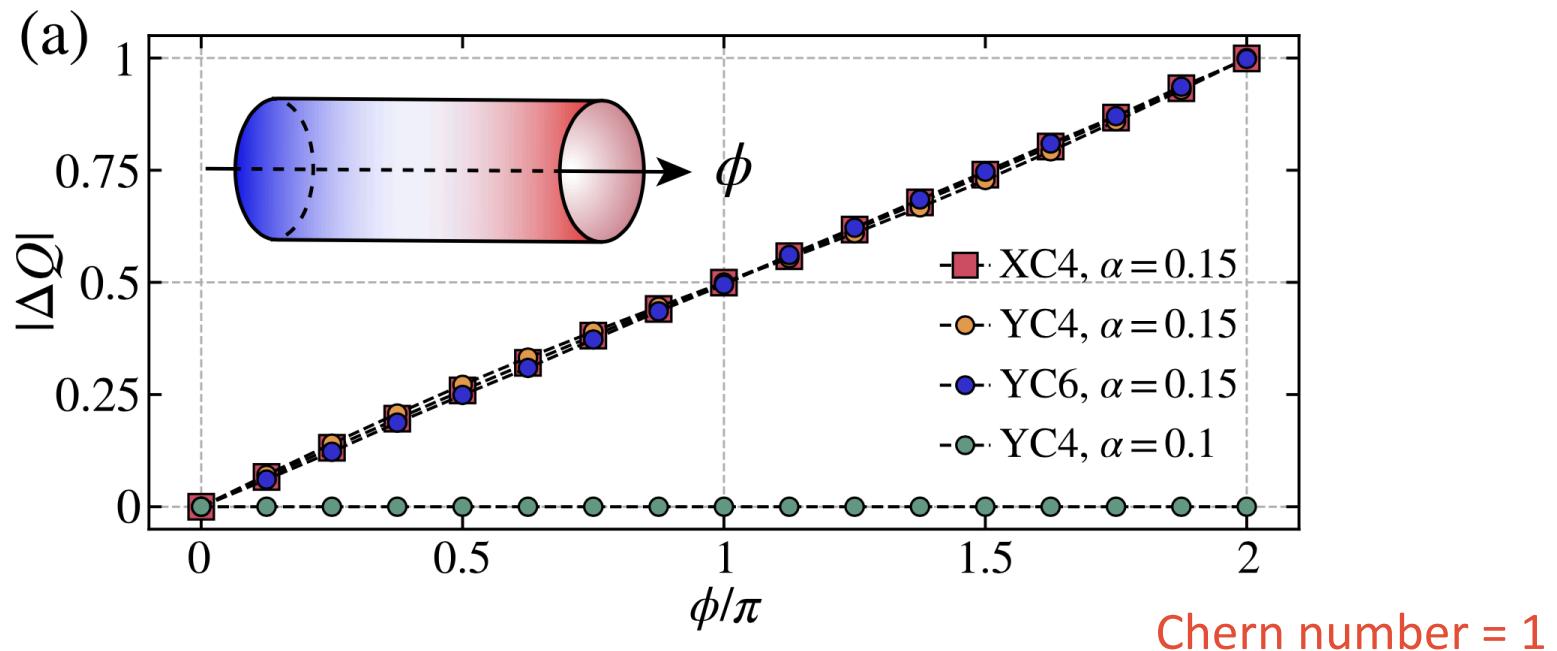
➤ 2.3 Identify QAH

- ✓ Spontaneous time-reversal symmetry breaking
- ✓ Loop current
- ✓ non-zero Chern number

✓ QAH

▪ Flux Insertion (Laughlin's thought experiment)

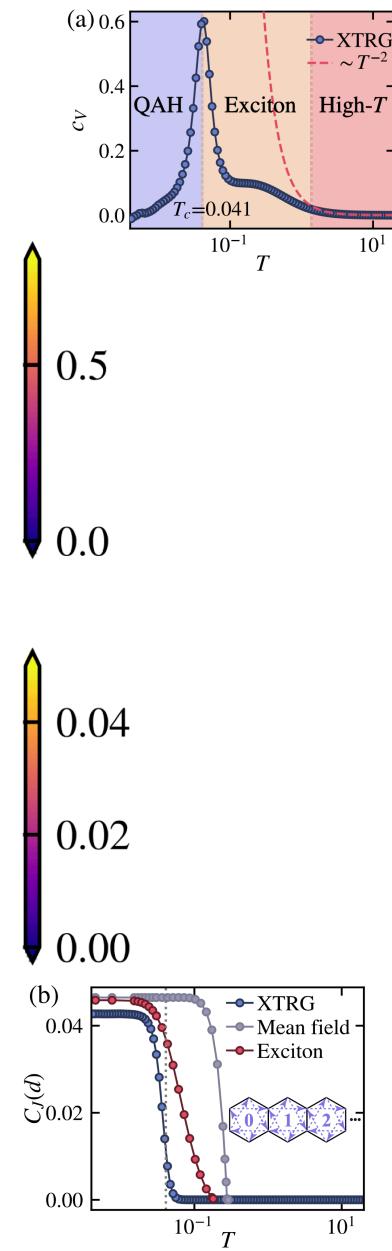
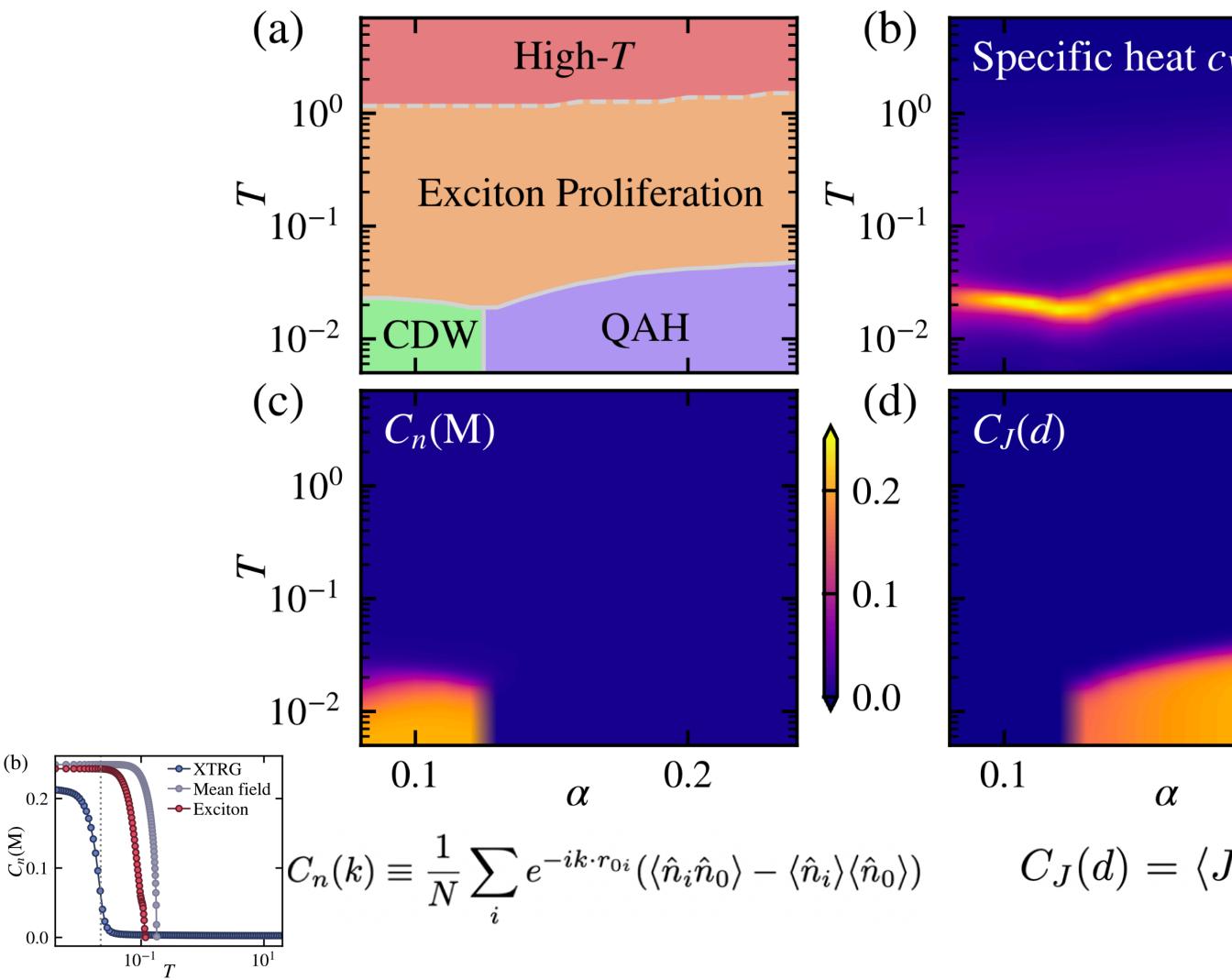
$$c_{\mathbf{i}} \equiv c_{\mathbf{i}+\mathbf{W}} \rightarrow c_{\mathbf{i}} \equiv e^{i\phi} c_{\mathbf{i}+\mathbf{W}}$$



Thermodynamics of twisted bilayer graphene model

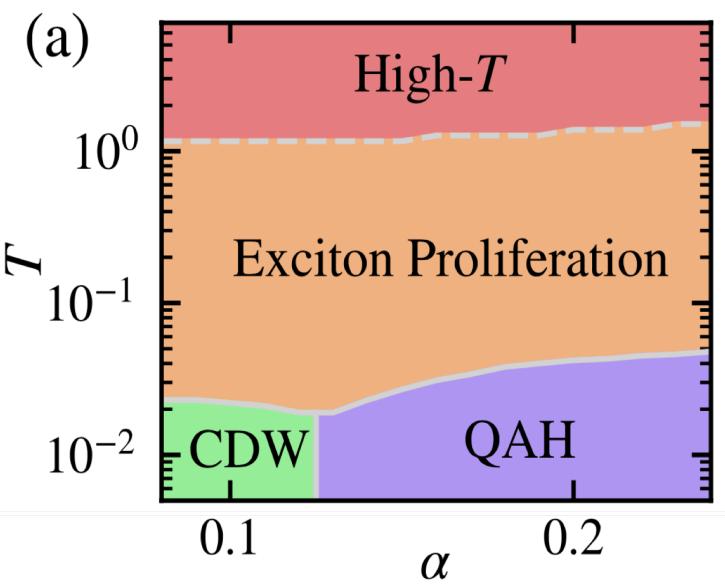
X. Lin, **BC***, Wei Li, Zi Yang Meng*, and Tao Shi*, Phys. Rev. Lett. **128**, 157201

➤ 2.3 Finite-T phase diagram

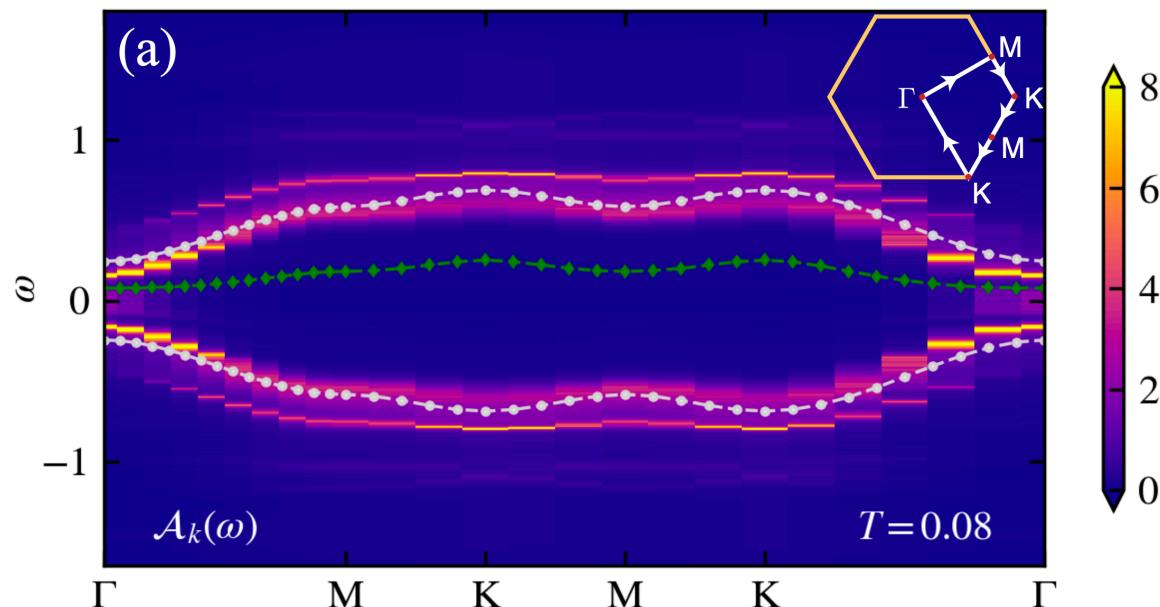


➤ 2.3 Exciton proliferation regime

✓ Finite-T phase diagram

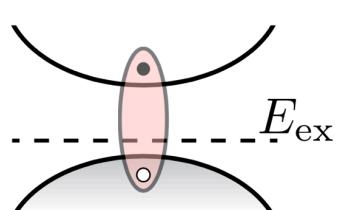


✓ Spectral function



Mean-field estimation of $T_c \sim 100$ K

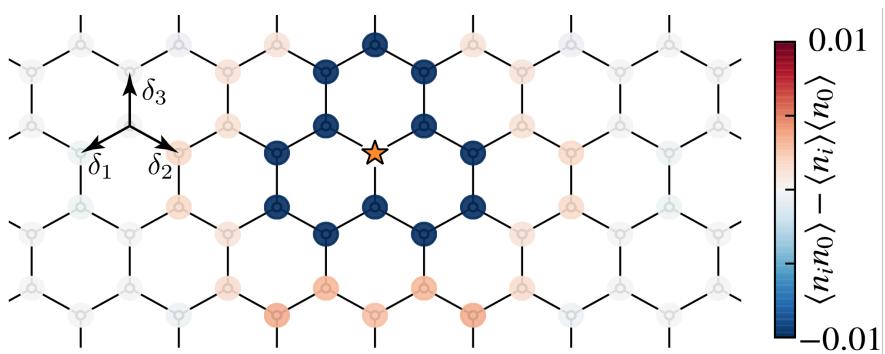
Obtained $T_c \sim 10$ K



Exciton corrects the spectrum!

➤ 2.3 Thermodynamic signature of exciton

✓ Microscopic characteristic



✓ Charge compressibility

